Periodic Replacement When Minimal Repair Costs Vary With Time

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Abstract

A policy of periodic replacement with minimal repair at failure is considered for a complex system. Under such a policy the system is replaced at multiples of some period $T$ while minimal repair is performed at any intervening system failures. The cost of a minimal repair to the system is assumed to be a nondecreasing function of its age. A simple expression is derived for the expected minimal repair cost in an interval in terms of the cost function and the failure rate of the system. Necessary and sufficient conditions for the existence of an optimal replacement interval are exhibited in the case where the system life distribution is strictly IFR.
1. Introduction

A complex system may fail if one of its many components ceases to function. The system is returned to the operating state when the failed component is replaced. As the great majority of components have not been replaced, the remaining life distribution and failure rate of the system are essentially undisturbed. This type of system repair, whereby the failure rate of the system is not altered by the failure and subsequent repair of the system is known as minimal repair.

A policy of periodic replacement with minimal repair at failure is one in which the system is replaced at multiples of some period $T$ while performing minimal repair at any intervening system failures. This type of policy was introduced and investigated by Barlow and Hunter in 1960 (see also Barlow and Proschan (1965)). In their paper, Barlow and Hunter show how to calculate the optimal period $T$ assuming the cost of a minimal repair is constant and using as an optimality criterion the minimization of total expected cost per unit time over an infinite time horizon. Tilquin and Cleroux (1975) modify this model by introducing to the cost analysis a general cost function (to account for depreciation or adjustment costs, interest charges, monitoring costs and the like), which increases continuously with the length of time the system is in use. Boland and Proschan (1981) generalise the Barlow-Hunter model to incorporate the situation when the cost of a minimal repair is an increasing function of the number of previous repairs to the system.
In this paper, it is assumed that the cost of a minimal repair to the system which fails at age $t$ is $C(t)$, where $C(t)$ is a continuous nondecreasing function of $t$. $F$ will denote the life distribution function of the system with density $f$. $R(t)$ and $r(t)$ will denote the hazard function and hazard or failure rate function of the system respectively. It is assumed that $F(0) = 0$ and that $r(t)$ is a positive continuous function of $t$. The problem of finding an optimal period $T_0$ for periodic replacement (where minimal repair is performed according to the cost function $C(t)$ on system failure) is investigated. It is shown that if $C(t)r(t)$ is nondecreasing (in particular if $F$ is IFR), then an optimal period $T_0$ (possibly infinite) exists.

2. Expected Minimal Repair Costs

Now consider the situation where the system is in operation in the time interval $[0,T]$. On failure, minimal repair is made (according to $C(t)$) and we assume that repair time is negligible. If the system is not replaced in this interval, the following Theorem gives two useful expressions for the expected costs of minimal repair in $[0,T]$.

**Theorem 2.1** The expected minimal repair cost of the system in the interval $[0,T]$ is

$$\int_0^{R(T)} C(R^{-1}(t))dt = \int_0^T C(t)r(t)dt.$$  

**Proof** Let $N_T$ be the random variable denoting the number of minimal repairs performed on the system in the age interval $[0,T]$. We know that $N_T$ has a Poisson distribution with parameter $R(T)$. 

Now if $N_T = k$, and $t_1, \ldots, t_k$ are the times of the minimal repairs, then the total minimal repair cost in the interval $[0,T]$ is

$$\sum_{i=1}^{k} C(t_i).$$

Given $N_T = k$, we know that $\tau_1 = R(t_1), \ldots, \tau_k = R(t_k)$ are distributed as the order statistics of a random sample of size $k$ from the uniform distribution on $[0,R(T))]$ (see for example Parzen (1962), pp.139-143 or Thompson (1981)). Hence the expected minimal repair cost given $N_T = k$ is

$$E(C(t_1) + \ldots + C(t_k) | N_T = k)$$

$$= E(C(R^{-1}(\tau_1) + \ldots + C(R^{-1}(\tau_k)) | N_T = k)$$

$$= kE(C(R^{-1}(\tau)) | N_T = k)$$

(where $\tau$ is uniformly distributed on $[0,R(T))]$).

$$= k \int_0^{R(T)} C(R^{-1}(t)) \frac{1}{R(T)} dt$$

$$= \frac{k}{R(T)} \int_0^{R(T)} C(R^{-1}(t)) dt.$$  

Therefore the expected minimal repair cost in the interval $[0,T]$ is

$$E_{N_T}(E(C(t_1) + \ldots + C(t_k) | N_T = k))$$

$$= E_{N_T}(\frac{k}{R(T)} \int_0^{R(T)} C(R^{-1}(t)) dt)$$

$$= \left[ \frac{1}{R(T)} \int_0^{R(T)} C(R^{-1}(t)) dt \right] \left[ E_{N_T}(k) \right]$$

$$= \left[ \frac{1}{R(T)} \int_0^{R(T)} C(R^{-1}(t)) dt \right] \left[ E_{N_T}(k) \right] \left[ R(T) \right]$$
\[
= \int_0^T C(R^{-1}(t)) \, dt \\
= \int_0^T C(w) r(w) \, dw
\]
(using the change of variable \( w = R^{-1}(t) \)).

**Remark 2.2** In the above proof of Theorem 2.1, the number of minimal repairs to the system in \([0,T]\) is a nonhomogeneous (or homogeneous) Poisson process with intensity function \(r(t)\). In particular, given that the system is functioning at age \(t\), the probability of a system failure in the interval \((t, t+h)\) is of the form \(r(t)h + o(h)\). Moreover, the mean number of failures in the interval \([0,T]\) is \(\int_0^T r(t) \, dt = R(T)\). This leads us to interpret \(C(t)r(t)\) in a naive sort of way as the "rate" of spending a dollar on minimal repair at age \(t\). With this interpretation \(\int_0^T C(t)r(t) \, dt\) represents the mean number of dollars spent on minimal repair in \([0,T]\), which we have rigorously demonstrated in Theorem 2.1.

We now consider some examples of minimal repair cost functions and determine the resulting expected minimal repair cost in an age interval \([0,T]\) for a given life distribution. It is reasonable to assume that the cost of a minimal repair to a system should be a nondecreasing function of its age. Perhaps even more appropriate would be a nondecreasing function of the life distribution \(F(t)\) or equivalently \(R(t)\) or \(\frac{1}{F(t)}\).

**Example 2.3** Let \(C(t)\) be of the form \(C(t) = g(R(t))\) where \(g\) is nondecreasing and \(G' = g\). Then
\[
\int_0^T C(t)r(t) \, dt = \int_0^T g(R(t)) \, dR(t) = G(R(t)) - G(0).
\]
We consider the following particular cases:

(a) \( g(y) = c. \) Then \( \int_0^T C(t)r(t)dt = cR(T). \) This is the cost function used by Barlow and Hunter (1960).

(b) \( g(y) = cy^\alpha \ (\alpha > 0). \) Then \( C(t) = c(\log \frac{1}{F(t)})^\alpha \) and

\[
\int_0^T C(t)r(t)dt = c \frac{R^{\alpha+1}(T)}{\alpha+1}.
\]

When \( \alpha = 1, \) i.e. minimal repair cost is proportional to the hazard function, then the expected minimal repair cost in \([0,T]\) is \( c \frac{R^2(T)}{2}. \)

(c) \( g(y) = ce^{gy} \ (\alpha > 0). \) Then \( C(t) = g(R(t)) = ce^{\gamma R(t)} = c \frac{1}{(F(t))^{\gamma}} \)

and

\[
\int_0^T C(t)r(t)dt = \frac{c}{\alpha} (e^{\gamma R(T)} - 1) = \frac{c}{\alpha} \left( \frac{1}{F^{\gamma}(T)} - 1 \right). \]

When \( \alpha = 1, \) i.e. the cost of a minimal repair is inversely proportional to the survival probability, then the expected minimal repair cost in \([0,T]\) is proportional to the odds ratio \( F(T)/F(T). \)

The linear combinations of (a), (b) and (c) constitute a large class of reasonable cost functions for minimal repair.

**Example 2.4** One might consider a cost function of the type \( C(t) = ct^\alpha \) (\( \alpha > 0 \)). In the particular case when \( \alpha = 1 \) this yields an expected minimal repair cost in \([0,T]\) of

\[
\int_0^T cR(t)dt = cR(T)T - \int_0^T cR(t)dt.
\]
If the life distribution is Weibull of the form $F(t) = 1 - e^{-(\lambda t)^\beta}$, then the expected cost is

$$c(\lambda T)^\beta T - \int_0^T c(\lambda t)^\beta dt = c T^\beta \frac{1}{\beta + 1}.$$ 

3. Periodic Replacement with Minimal Repair

Let us now consider the problem of finding a period $T_o$ for replacement which minimizes expected long run cost per unit of time. $c_o$ will denote the cost of a planned system replacement. If $C(T)$ represents the expected long run cost per unit of time when the system is periodically replaced at times $T, 2T, 3T, \ldots$, then $\bar{C}(T)$ has the form

$$\bar{C}(T) = \frac{\int_0^T C(t)r(t)dt + c_o}{T}.$$ 

Therefore $C'(T) = \frac{C(T)r(T)T - \int_0^T C(t)r(t)dt - c_o}{T^2}$.

and this yields

Theorem 3.1 If $C(t)r(t)$ is a nondecreasing function of $t$, then an optimal replacement interval $T_o$ exists. $T_o$ is finite if

$$\lim_{T \to \infty} \int_0^T [C(T)r(T) - C(t)r(t)]dt > c_o$$

or if $C(t)r(t)$ is eventually constant and

$$\lim_{T \to \infty} \int_0^T [C(T)r(T) - C(t)r(t)]dt \geq c_o.$$
otherwise the optimal policy is to never replace.

**Proof** If \( C(t)r(t) \) is nondecreasing then \( C(T)r(T)T - \int_0^T C(t)r(t)dt \) is nondecreasing and hence \( C(T) \) has a root iff

\[
\lim_{T \to \infty} \int_0^T [C(T)r(T) - C(t)r(t)] dt > c_o
\]

(or \( \lim_{T \to \infty} \int_0^T [C(T)r(T) - C(t)r(t)] dt \geq c_o \) if \( C(T)r(T) \) is eventually constant).

**Remark 3.2** If \( F \) is an IFR distribution with strictly increasing failure rate function \( r(t) \), then there exists a unique optimal replacement interval \( T_o \) (possibly infinite).

**Example 3.3** Let us consider \( C(t) \) of the form \( C(t) = ce^{\alpha R(t)} \) (see Example 2.3c). If \( F \) is Weibull of the form \( F(t) = 1 - e^{-(\lambda t)^\beta} \) where \( \beta > 1 \), then the optimal period \( T_o \) must satisfy

\[
ce^{\alpha (\lambda T_o)^\beta} \left[ \beta \lambda T_o^{\beta-1} \frac{1}{\alpha} \right] = c_o - \frac{c}{\alpha}.
\]

If \( \alpha = 0 \), then \( T_o = \left( \frac{c_o}{(\beta-1)c} \right)^{1/\beta} \).

**Example 3.4** Let \( C(t) \) be of the form \( C(t) = c(R(t))^\alpha \) and \( F \) be Weibull where \( F(t) = 1 - e^{-(\lambda t)^\beta} \). Then \( C(t)r(t) = c\lambda (\lambda t)^{\alpha \beta + \beta - 1} \). This is increasing iff \( \alpha \beta + \beta > 1 \), and hence in particular if \( \beta > 1 \) (or for example \( \beta = \frac{1}{2} \) (F is DFR) and \( \alpha = \frac{3}{2} \)) there is a unique optimal replacement interval \( T_o \).
Remark 3.5 We may also wish to consider the problem of finding a replacement interval $T_0$ which minimizes total expected costs over a finite time horizon $[0, t_0)$. For period $T$, where $mT < t_0 < (m+1)T$ for some integer $m$, we see that the total expected cost in $[0, t_0)$ is

$$C_{t_0}(T) = mc_0 + m \int_0^T C(t)r(t)dt + \int_0^{t_0-mT} C(t)r(t)dt.$$

If $C(t)r(t)$ is an increasing function of $t$, it follows that

$$C_{t_0}(T) = m[C(T)r(T) - C(t_0-mT)r(t_0-mT)] \geq 0 \text{ in } (\frac{t_0}{m+1}, \frac{t_0}{m}).$$

Therefore if $C(t)r(t)$ is increasing then the optimal replacement interval $T_0$ is an element in the set \{t_0, t_0/2, t_0/3, \ldots\}. 
References


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