Tests for Properties of the Percentile Residual Life Function

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Abstract

The $\alpha$-percentile ($0 < \alpha < 1$) residual life function at time $x$ is defined to be the $\alpha$-percentile of the remaining life given survival up to time $x$. Tests are developed for testing the null hypothesis of exponentiality against alternatives representing decreasing $\alpha$-percentile residual life and the property "new better than used with respect to the $\alpha$-percentile". These tests are also extended to accommodate randomly censored data. Consistency and asymptotic relative efficiency results are obtained for the proposed tests. Exact critical values are obtained for the test statistics in the complete data case.
1. Introduction

Let \( F \) be a life distribution (a distribution such that \( F(0^-) = 0 \)) with survival function \( \bar{F} = 1 - F \) and right endpoint of support \( F^{-1}(1) = \sup \{ x : F(x) < 1 \} \). Conditional on survival up to time \( x \), the remaining life has survival function

\[
1 - F_X(y) = \bar{F}_X(y) = \frac{\bar{F}(x + y)}{\bar{F}(x)}, \quad y \geq 0, \quad 0 \leq x < F^{-1}(1).
\]

The mean residual life function \( \overline{m}_F \) is

\[
\overline{m}_F(x) = \int_0^\infty \frac{\bar{F}(y)}{\bar{F}(x)} dy = \int_x^\infty \frac{\bar{F}(y)}{\bar{F}(x)} dy/F(x), \quad 0 \leq x < F^{-1}(1),
\]

if \( F \) has a finite mean. The \( \alpha \)-percentile or quantile \((0 < \alpha < 1)\) residual life function of \( F \) is

\[
q_{\alpha, F}(x) = F^{-1}_X(\alpha) = \inf \{ y : F_X(y) > \alpha \}
\]

\[
= \inf \{ y : \bar{F}(x + y) \leq (1 - \alpha)\bar{F}(x) \}
\]

\[
= \inf \{ y : \bar{F}(y) \leq (1 - \alpha)\bar{F}(x) \} - x
\]

\[
= F^{-1}(1 - \overline{a}\bar{F}(x)) - x, \quad 0 \leq x < F^{-1}(1),
\]

where \( \overline{a} = 1 - \alpha \). Note that we are defining \( q_{\alpha, F}(x) \) using the left continuous version of the inverse of a distribution function. Whenever possible, we will suppress the subscript \( F \) and use \( m \) and \( q_{\alpha} \) in place of \( \overline{m}_F \) and \( q_{\alpha, F} \) respectively.

Statisticians find it useful to categorize life distributions according to different aging properties. These categories are useful
for modelling situations where items improve or deteriorate with age. Well-known classes include the following.

1. Increasing failure rate (IFR): $F$ is IFR if $F(0) = 0$ and the conditional distributions $F_x$ are decreasing stochastically in $x \in [0, F^{-1}(1))$, that is, $F_x(y)$ is decreasing in $x \in [0, F^{-1}(1))$ for all $y \geq 0$. If $F$ has a density $f$, then its failure rate function $r(x) = f(x)/F(x)$, $0 \leq x < F^{-1}(1)$, is increasing.

2. New better than used (NBU): $F$ is NBU if $F$ is stochastically larger than $F_x$ for all $0 < x < F^{-1}(1)$, that is, $F(y) \geq F_x(y)$ for all $0 < x < F^{-1}(1)$ and for all $y \geq 0$.

3. Decreasing mean residual life (DMRL): $F$ is DMRL if $F(0) = 0$, $F$ has a finite mean and $m(x)$ is decreasing in $x \in [0, F^{-1}(1))$.

4. New better than used in expectation (NBUE): $F$ is NBUE if $F(0) = 0$, $F$ has a finite mean and $m(0) \geq m(x)$ for all $0 \leq x < F^{-1}(1)$. (Throughout, we use the word decreasing in place of nonincreasing and increasing in place of nondecreasing.)

Classes involving the $\alpha$-percentile residual life function are the following.

5. Decreasing $\alpha$-percentile residual life (DPRL-$\alpha$): $F$ is DPRL-$\alpha$ if $F(0) = 0$ and $q_{\alpha}(x)$ is decreasing in $x \in [0, F^{-1}(1))$.

6. New better than used with respect to the $\alpha$-percentile (NBUP-$\alpha$): $F$ is NBUP-$\alpha$ if $F(0) = 0$ and $q_{\alpha}(0) \geq q_{\alpha}(x)$ for all $0 \leq x < F^{-1}(1)$.

The DPRL-$\alpha$ classes, $0 < \alpha < 1$, are studied in Haines and Singpurwalla (1974) and Joe and Proschan (1982a). The NBUP-$\alpha$
classes, $0 < \alpha < 1$, first appear in Joe (1982) and Joe and Proschan (1982a). Dual classes for the above six classes may be defined by reversing the direction of monotonicity or of equality and deleting the condition that $F(0) = 0$. The boundary members of the first four classes are the exponential distributions but the boundary members of the fifth and sixth classes above form a larger class which includes the exponential distributions. This boundary class is called the constant $\alpha$-percentile residual life class and is denoted by CPRL-$\alpha$. The exponential distributions are the only distribution which are CPRL-$\alpha$ for all $0 < \alpha < 1$. The CPRL-$\alpha$ classes, $0 < \alpha < 1$, are characterized in Joe and Proschan (1982a).

From the above definitions and the fact that IFR and NBU distributions have finite means, we can easily see the following relationships among the classes:

- $F$ is IFR $\Rightarrow$ $F$ is DMRL $\Rightarrow$ $F$ is NBUE,
- $F$ is IFR $\Rightarrow$ $F$ is NBU $\iff$ $F$ is NBUP-$\alpha$ for all $0 < \alpha < 1$ $\Rightarrow$ $F$ is NBUE,
- $F$ is IFR $\iff$ $F$ is DPRL-$\alpha$ for all $0 < \alpha < 1$,
- $F$ is DPRL-$\alpha$ $\Rightarrow$ $F$ is NBUP-$\alpha$, for any $0 < \alpha < 1$.

It is shown in Joe and Proschan (1982a) that the DPRL-$\alpha$ class includes not only IFR distributions but also some distributions with failure rate functions which are eventually increasing.

In the past two decades, tests have been developed for the alternative hypothesis that a life distribution belongs to a specific class of distributions for each of the first four classes above. Tests for
increasing failure rate (IFR) alternatives include those considered by Proschan and Pyke (1967), Barlow (1968), Barlow and Proschan (1969), Bickel (1969), Bickel and Doksum (1969), and Barlow and Doksum (1972). Tests for new better than used (NBU) alternatives are developed by Hollander and Proschan (1972) and Koul (1977, 1978a). Hollander and Proschan (1975) also develop tests for decreasing mean residual life (DMRL) and new better than used in expectation (NBUE) alternatives. Koul (1978b) has considered a class of tests for testing for NBUE alternatives. In all cases, the null hypothesis is that the life distribution is exponential. All these tests can be used for the corresponding dual classes as well, but the adverse aging classes are of more interest in practice.

In Sections 2 and 3 of this paper, we develop tests for decreasing $\alpha$-percentile residual life (DPRL-$\alpha$) alternatives and new better than used with respect to the $\alpha$-percentile (NBU-$\alpha$) alternatives, with the null hypothesis being that the life distribution is exponential. Both test statistics being considered are L-statistics, that is, linear combinations of the order statistics. The approach to obtaining test statistics is that of Hollander and Proschan (1975), which is to find a measure of deviation from exponentiality of the alternative class under consideration. This measure of deviation is zero for exponential distributions and positive for distributions in the alternative class. A test statistic is then the deviation of the sample empirical distribution, using this measure. The tests are extended to accommodate randomly censored data in Section 4.
In Section 5, efficacies of the two proposed tests are calculated for several parametric families. Also, we define the asymptotic relative efficiency for the tests using censored data with respect to the corresponding tests using complete data, and these asymptotic relative efficiencies are calculated for some special cases of the censoring distribution. In Section 6, the exact null distributions of the test statistics (in the complete data situation) are derived and critical values are tabulated for some small sample sizes. An example is presented in Section 7.

Throughout this paper, the dependence of the test statistics on \( \alpha \) is not explicitly shown in order to simplify the notation.

2. Test for DPRL-\( \alpha \) Alternatives.

Let \( F \) be a DPRL-\( \alpha \) \((0 < \alpha < 1)\) distribution with survival function

\[
\tilde{F} = 1 - F \quad \text{and} \quad \alpha \text{-percentile residual life function}
\]

\[
q_\alpha(x) = F^{-1}(1-\tilde{\alpha}F(x)) - x, \quad 0 \leq x < F^{-1}(1), \quad \text{where} \quad \tilde{\alpha} = 1 - \alpha.
\]

A measure of deviation of \( F \) from the constant \( \alpha \)-percentile residual life (CPRL-\( \alpha \)) class is

\[
\Delta_1(F) = \Delta_{1\alpha}(F) = \iint_{x < y} \tilde{F}(x)\widetilde{F}(y)(q_\alpha(x) - q_\alpha(y))dF(x)dF(y).
\]

\( \Delta_1(F) \) equals zero if \( F \) is a CPRL-\( \alpha \) distribution and \( \Delta_1(F) \) is positive if \( F \) is a DPRL-\( \alpha \) distribution whose \( \alpha \)-percentile residual life function is not constant on its support.
Let $X_1 < \ldots < X_n$ be the order statistics (assumed distinct) corresponding to a random sample of size $n$ from $F$, let $F_n$ be the empirical distribution function and let $\tilde{F}_n = 1 - F_n$. It is clear that

$$\Delta_1(F_n) = \iint_{x<y} F_n(x)\tilde{F}_n(y)\left\{[F_n^{-1}(1-\tilde{F}_n(x))-x] - [F_n^{-1}(1-\tilde{F}_n(y))-y]\right\}dF_n(x)dF_n(y)$$

is an L-statistic, that is, a linear combination of the order statistics, $X_1, \ldots, X_n$. For some constants $c_{in}$, $\Delta_1(F_n) = \sum_{i=1}^{n} c_{in} X_i$, but a general formula for $c_{in}$ is difficult to obtain and depends on whether the right continuous version or the left continuous version of $F_n^{-1}$ is used. A better approach to obtaining a test statistic is to write $\Delta_1(F)$ in a different form and then substitute $F_n$ for $F$.

If $F$ is continuous, then after some straightforward calculations, we obtain

$$\Delta_1(F) = \int_0^1 F^{-1}(t)J_1(t)dt = \int_0^\infty x J_1(F(x))dF(x),$$

where

$$J_1(u) = J_{1\alpha}(u) = \begin{cases} \frac{1}{2}(1-u)[2(1-u)^2-1] & \text{if } 0 \leq u \leq \alpha, \\ \frac{1}{2}(1-u)[2(\alpha^{-4}-1)(1-u)^2-(\alpha^{-2}-1)] & \text{if } \alpha < u \leq 1. \end{cases}$$

Let

$$B_1(t) = \int_t^1 J_1(u)du = \begin{cases} \frac{1}{4}(1-t)^2[(1-t)^2-1] & \text{if } 0 \leq t \leq \alpha, \\ \frac{1}{4}(1-t)^2[(\alpha^{-4}-1)(1-t)^2-(\alpha^{-2}-1)] & \text{if } \alpha < t \leq 1. \end{cases}$$
If also $F$ has a finite mean, then

$$
\Delta_1(F) = -\int_0^\infty x B_1(F(x)) dx = -xB_1(F(x)) \bigg|_0^\infty + \int_0^\infty B_1(F(x)) dx = \int_0^\infty B_1(F(x)) dx.
$$

We propose as a test statistic $W_{ln} = \int_0^\infty B_1(F(x)) dx$

$$
= \sum_{i=0}^{n-1} B_1(\frac{i}{n})(X_{i+1} - X_i) = \sum_{i=1}^n X_i (B_1(\frac{i-1}{n}) - B_1(\frac{i}{n}))
$$

$$
= \sum_{i=1}^n \frac{i}{n} \int_{(i-1)/n}^{i/n} J_1(u) du,
$$

where $X_0 = 0$. Since $F_n$ is not continuous, we have only that $\Delta_1(F_n) \sim W_{ln}$.

Notice that $W_{ln}$ is not scale invariant. A scale invariant test statistic for testing

$H_0$: $F$ is exponential against $H_1$: $F$ is DPRL-$\alpha$ (and not CPRL-$\alpha$)

is $W_{ln} = W_{ln} / \bar{X}_n$, where $\bar{X}_n$ is the sample mean of $X_1, \ldots, X_n$. An asymptotic normality theorem of Boos (1979) (see also Serfling (1980), pp. 270-283) can be used to obtain the asymptotic normality of $W_{ln}$ and $W_{ln}^*$ since the condition $\int_0^\infty [F(x)F(x)]^{k-\delta} dx < \infty$ is satisfied for all $0 \leq \delta < \frac{1}{2}$ if $F$ is a DPRL-$\alpha$ distribution (this follows from Theorem 5 of Joe and Proschan (1982a)). The asymptotic distributions of $W_{ln}$ and $W_{ln}^*$ are summarized in the following theorem.

**Theorem 1.** Let $F$ be a continuous DPRL-$\alpha$ distribution such that $F^{-1}$ is continuous at $a$. Then $n^{1/2}(W_{ln} - \Delta_1(F)) \Rightarrow N(0, \sigma^2_1(F))$ as $n \rightarrow \infty$, where

$$
\sigma^2_1(F) = \int_0^\infty \int_0^\infty J_1(F(x)) J_1(F(y)) (F(x\gamma) - F(x)) (F(y) - F(x)) dx dy.
$$

(Here $\wedge$ is the symbol for minimum.)
If \( F \) is exponential with mean \( \nu \), then
\[
n^{1/2} \frac{W_{n}}{\ln} \overset{D}{\rightarrow} N(0, \nu \sigma^2_{1}) \text{ as } n \to \infty, \text{ where } \sigma^2_{1} = \frac{1}{\nu^2} \frac{\sigma^2}{2(3\sigma^2 - 12\sigma + 11)/4205}, \text{ and}
\[
n^{1/2} \frac{W_{n}^*}{\ln} \overset{D}{\rightarrow} N(0, \sigma^2_{1}) \text{ as } n \to \infty.
\]

As a consequence of Theorem 1, an approximate level-\( \gamma \) test is to reject the null hypothesis of exponentiality if \( n^{1/2} \frac{W_{n}^*}{\ln} / \sigma_{1} \geq z_{\gamma} \), where \( z_{\gamma} \) is the upper \( \gamma \)-percentile of the standard normal distribution. A consistency result is given in the following corollary.

**Corollary 2.** Suppose \( F \) is a continuous DPRL-\( \alpha \) distribution such that \( F^{-1} \) is continuous at \( \alpha \) and \( \Delta_{1}(F) > 0 \). Then the test which rejects \( H_{0} \) for significantly large values of \( n^{1/2} \frac{W_{n}^*}{\ln} / \sigma_{1} \) is consistent against \( F \).

**Proof.** By Theorem 1, \( n^{1/2} (W_{n} - \Delta_{1}(F)) \overset{D}{\rightarrow} N(0, \sigma^2_{1}(F)) \) as \( n \to \infty \). Thus, for \( z > 0 \),
\[
P(n^{1/2} \frac{W_{n}^*}{\ln} / \sigma_{1} > z) = P(n^{1/2} (W_{n} - \Delta_{1}(F)) > -n^{1/2} \Delta_{1}(F) + z \sigma_{1} \frac{\ln}{n} > 0) \quad \text{as } n \to \infty.
\]

We can also use the test statistic \( W_{n}^* \) for testing for increasing \( \alpha \)-percentile residual life (IPRL-\( \alpha \)) alternatives. That is, for significantly small values of \( n^{1/2} \frac{W_{n}^*}{\ln} / \sigma_{1} \), we can reject the null hypothesis of exponentiality in favor of IPRL-\( \alpha \) alternatives. This test is consistent against a continuous IPRL-\( \alpha \) distribution \( F \) satisfying the following conditions: (i) \( \Delta_{1}(F) < 0 \), (ii) \( F^{-1} \) is continuous at \( \alpha \), (iii) \( \int_{0}^{\alpha} F(x) \, dx < \infty \) and (iv) \( \int_{0}^{\infty} [F(x) \tilde{F}(x)]^{1-\delta} \, dx < \infty \) for some \( 0 < \delta < 1/2 \). Note that conditions (iii) and (iv) are satisfied by DPRL-\( \alpha \) distributions but need not be satisfied by IPRL-\( \alpha \) distributions.
If $\sigma_1^2(F)/\mu^2(F)$ were the same for all continuous CPRL-$\alpha$ distributions $F$, where $\mu(F) = \int_0^\infty x dF(x)$, then the null hypothesis of the $W_{1n}$ test could be that $F$ is CPRL-$\alpha$. However, this is not the case, as the calculations below show. This is not a serious problem, since Theorem 9 of Joe and Proschan (1982a) characterizes CPRL-$\alpha$ distributions with densities as having periodic failure rate functions, and in some situations it might be unreasonable to assume that the failure rate function is nonconstant and periodic.

For a continuous CPRL-$\alpha$ distribution $F$, detailed calculations show that

$$\sigma_1^2(F) = 2 \sum_{i=0}^\infty \int \int \frac{(i+1)a}{j} \frac{(j+1)a}{i} J_1(F(x)) J_1(F(y)) F(x) F(y) dy dx$$

$$+ \sum_{i=0}^\infty \int \int J_1(F(x)) J_1(F(y)) (F(x \land y) - F(x) F(y)) dy dx$$

$$= a[\alpha(1-\alpha^3)]^{-1} [2(1+\alpha^2+\alpha^2) (A_3 A_4 - B_3) + B_1]$$

$$+ (1-\alpha^2)[\alpha(1-\alpha^3)]^{-1} [(1+\alpha^2) (B_2 A_3 - B_2 A_4) + B_2]$$

$$+ a[4\alpha(1-\alpha^3)]^{-1} [2(1+\alpha)(A_2 A_3 - B_4) + B_1]$$

and $\mu(F) = A_1/\alpha$, where $a = F^{-1}(\alpha)$, $A_i = \int \frac{a}{x} F_i(x) dx$, $i = 1, 2, 3, 4,$

$$B_1 = \int \int F(x) F(y) F(x \land y) dy dx,$$

$$B_2 = \int \int F^3(x) F(y) F(x \land y) dy dx$$

and

$$B_3 = \int \int F^5(x) F^3(y) F(x \land y) dy dx.$$ 

Thus $\sigma_1^2(F)/\mu^2(F)$ depends on $F$ through $A_i$, $i = 1, 2, 3, 4,$ and $B_i$, $i = 1, 2, 3,$ and is not the same for all CPRL-$\alpha$ distributions.
3. Test for NBUP-\(a\) Alternatives

Let \(F\) be an NBUP-\(a\) (\(0 < a < 1\)) distribution with survival function \(\bar{F} = 1 - F\) and \(a\)-percentile residual life function

\[ q_a(x) = F^{-1}(1-\bar{a}\bar{F}(x)) - x, \ 0 \leq x < F^{-1}(1), \text{ where } \bar{a} = 1 - a. \]

A measure of deviation of \(F\) from the CPRL-\(a\) class is

\[ \Delta_2(F) = \Delta_2(a) = \int \bar{F}(x)[q_a(x) - q_a(x)]dF(x). \]

\(\Delta_2(F)\) equals zero if \(F\) is a CPRL-\(a\) distribution and \(\Delta_2(F)\) is positive if \(F\) is an NBUP-\(a\) distribution whose \(a\)-percentile residual life function is not constant on its support.

Let \(X_1 < \ldots < X_n\) be the order statistics (assumed distinct) corresponding to a random sample of size \(n\) from \(F\), let \(\bar{F}_n\) be the empirical distribution function and let \(\bar{F}_n = 1 - \bar{F}_n\). It is clear that

\[ \Delta_2(F_n) = \int \bar{F}_n(x)[F_n^{-1}(1-\bar{a}\bar{F}_n(0^+)) - F_n^{-1}(1-\bar{a}\bar{F}_n(x))]d\bar{F}_n(x) \]

is an L-statistic. As in the DPRL-\(a\) case, \(\Delta_2(F_n)\) is not convenient to work with. Paralleling the approach of Section 2, if \(F\) is continuous, then after some straightforward calculations, we obtain

\[ \Delta_2(F) = \frac{1}{2} F^{-1}(a) - \int_0^{F^{-1}(a)} J_2(t) dt \]

\[ = \frac{1}{2} F^{-1}(a) - \int_0^{\infty} xJ_2(F(x))dx, \]
where

\[ J_2(u) = J_{2a}(u) = \begin{cases} \frac{1}{(\alpha^2-1)(1-u)} & \text{if } 0 < u < 1, \\ -\frac{1}{(1-u)} & \text{if } 0 \leq u \leq a, \end{cases} \]

Let

\[ B_2(t) = \frac{1}{t} \int_{0}^{t} J_2(u) du = \begin{cases} \frac{1}{4} [(1-t)^2 - 1] & \text{if } 0 \leq t \leq a, \\ \frac{1}{4} (\alpha - 1)(1-t)^2 & \text{if } a < t \leq 1. \end{cases} \]

If also $F$ has a finite mean, then

\[ \Delta_2(F) = \frac{1}{2} \int_{0}^{\infty} x dB_2(F(x)) = \frac{1}{2} F^{-1}(a) - \int_{0}^{a} B_2(F(x)) dx. \]

We propose as a test statistic

\[ W_{2n} = \frac{1}{n} \sum_{i=1}^{n} J_2(U_i) = \frac{1}{2} F^{-1}(a) - \sum_{i=1}^{n} \frac{X_i}{n}, \]

where $\frac{1}{n} \sum_{i=1}^{n} X_i$ is the sample mean of $X_1, \ldots, X_n$.

A scale invariant test statistic for testing

\[ H_0: F \text{ is exponential against } H_1: F \text{ is } \text{NBUP-}\alpha \text{ (and not CPRL-}\alpha) \]

is $W_{2n}^* = W_{2n} / \bar{X}_{n}$, where $\bar{X}_{n}$ is the sample mean of $X_1, \ldots, X_n$.

The asymptotic distributions of $W_{2n}$ and $W_{2n}^*$ are summarized in the following theorem.
Theorem 3. Let $F$ be a continuous NBUP-$\alpha$ distribution such that $F'(F^{-1}(\alpha))$ exists and is positive. Then $n^{1/2}(W_{2n} \sim \chi^2_2(F)) \overset{D}{\rightarrow} N(0, \sigma_2^2(F))$ as $n \to \infty$, where

$$
\sigma_2^2(F) = \int_0^\infty \int_0^\infty J_2(F(x)) J_2(F(y)) (F(x) - y) F(x) F(y) \, dx \, dy
$$

$$
+ \frac{1}{4} \left[F'(F^{-1}(\alpha))\right]^{-2} \alpha (1-\alpha)

- \left[F'(F^{-1}(\alpha))\right]^{-1} \int_0^\infty J_2(F(x)) (\alpha F(x) - \alpha F(x)) \, dx.
$$

If $F$ is exponential with mean $\mu$, then $n^{1/2} W_{2n} \overset{D}{\rightarrow} N(0, \mu^2 \sigma_2^2)$ as $n \to \infty$, where $\sigma_2^2 = (3-2\alpha)/12\alpha$, and $n^{1/2} W_{2n}^* \overset{D}{\rightarrow} N(0, \sigma_2^2)$ as $n \to \infty$.

As a consequence of Theorem 3, an approximate level-$$\gamma$$ test is to reject the null hypothesis of exponentiality if $n^{1/2} W_{2n}^*/\sigma_2 \geq z_\gamma$, where $z_\gamma$ is the upper $$\gamma$$-percentile of the standard normal distribution. A consistency result is given in the following corollary.

Corollary 4. Suppose $F$ is a continuous NBUP-$\alpha$ distribution such that $F'(F^{-1}(\alpha)) > 0$ and $\Delta_2(F) > 0$. Then the test which rejects $H_0$ for significantly large values of $n^{1/2} W_{2n}^*/\sigma_2$ is consistent against $F$.

Similarly, for significantly small values of $n^{1/2} W_{2n}^*/\sigma_2$, we can reject the null hypothesis of exponentiality in favor of new worse than used with respect to the $\alpha$-percentile (NBUP-$\alpha$) alternatives.
This test is consistent against a continuous NWOP-α distribution \( F \) satisfying the following conditions: (i) \( \Delta_2(F) < 0 \),

(ii) \( F'(F^{-1}(a)) > 0 \), (iii) \( \int_0^\infty \tilde{F}(x) dx < \infty \) and (iv) \( \int_0^\infty [F(x)\tilde{F}(x)]^{-\delta} dx < \infty \) for some \( 0 < \delta < 1/2 \).

If \( \sigma_2^2(F)/\mu^2(F) \) were the same for all continuous CPRL-α distributions \( F \), where \( \mu(F) = \int_0^\infty x dF(x) \), then the null hypothesis of the \( W_{2n}^* \) test could be that \( F \) is CPRL-α. As in the DPRL-α case, this is not true. For a continuous CPRL-α distribution \( F \) with positive derivative at \( F^{-1}(a) \), direct calculations show that

\[
\sigma_2^2(F) = 2\mu_\tilde{a}[F'(F^{-1}(a))]^{-2} + \tilde{a}(1+\tilde{a}^2)^{-1} \{2(1+\tilde{a})(A_1A_2-B)+B\}
- [F'(F^{-1}(a))]^{-1}(A_2-\tilde{a}A_1)
\]

and \( \mu(F) = A_1/\tilde{a} \), where \( A_i = \int_0^a \tilde{F}^i(x) dx \), \( i = 1, 2 \),

\( B = \int_0^a \int_0^a \tilde{F}(x)\tilde{F}(y)\tilde{F}(x+y) dy dx \) and \( a = F^{-1}(1) \). Thus, \( \sigma_2^2(F)/\mu^2(F) \) depends on \( F \) through \( A_1, A_2, B \) and \( F'(F^{-1}(a)) \), and is not the same for all CPRL-α distributions.
4. Tests for DPRL and NBUP Alternatives Using Randomly Censored Data.

4.1 Introduction. In this section, we develop tests for DPRL-$\alpha$, and NBUP-$\alpha$ alternatives when the data are incomplete due to random censoring. In the random censoring model, instead of observing a complete sample $X_1, \ldots, X_n$ from the life distribution $F$, we are able to observe only the pairs $(Z_i, \delta_i)$, $i = 1, \ldots, n$, where $Z_i = \min(X_i, Y_i)$, $\delta_i = 1$ if $Z_i = X_i$ (the $i^{th}$ observation is uncensored) and $\delta_i = 0$ otherwise (the $i^{th}$ observation is censored), $i = 1, \ldots, n$. We assume that the censoring random variables $Y_1, \ldots, Y_n$ are independently and identically distributed according to the distribution $H$ and that the $X$'s and $Y$'s are mutually independent. Therefore, $Z_1, \ldots, Z_n$ are independently and identically distributed according to the distribution $K$, where $1 - K = \bar{K} = \bar{F} \bar{H} = (1-F)(1-H)$.

Koul and Susarla (1980) propose a statistic for testing for NBUE alternatives using randomly censored data and Chen, Hollander and Langberg (1980a, 1980b) propose statistics for testing for NBU alternatives and DMRL alternatives using randomly censored data. The null hypotheses are that the life distribution $F$ is exponential. Chen, Hollander and Langberg use the Kaplan and Meier (1958) product-limit estimator for the distribution function in place of the empirical distribution function in generalizing the corresponding test statistic using complete data. Kou
and Susarla use a slight variation of the Bayes estimator of Susarla and Van Ryzin (1976) in extending the total time on test statistic.

The Bayes estimator is obtained under a weighted squared error loss function using a Dirichlet process prior; it reduces to the product-limit estimator when an appropriate limit is taken. We will use the product-limit estimator in extending the test statistics proposed in Sections 2 and 3.

The product-limit estimator $F_n$ of $F$, introduced by Kaplan and Meier (1958), is:

$$1 - F_n(x) = \hat{F}_n(x) = \prod_{\{i:Z(i) \leq x\}} [(n-i)/(n-i+1)]^{\delta(i)}, \quad 0 \leq x < Z(n),$$

where $Z(1), \ldots, Z(n)$ are the ordered observations and $\delta(i)$ is the $\delta$ corresponding to $Z(i)$, $i = 1, \ldots, n$. When censored observations are tied with uncensored observations, the convention is to treat uncensored members of the tie as preceding the censored members of the tie. Also, we follow the convention of some authors in treating $Z(n)$ as an uncensored observation whether or not it is uncensored and defining $\hat{F}_n(x) = 0$ for $x \geq Z(n)$. Kaplan and Meier show that $F_n$ is the "nonparametric maximum likelihood" estimator of $F$ when no parametric assumptions about $F$ are made. $F_n$ reduces to the empirical distribution function when all observations are uncensored. Throughout the remainder of this section, $F_n$ will denote the product-limit estimator.
4.2 Test statistics. Consider the statistics
\[ W_n^C = \int_0^\infty B_1(F_n(x))dx \]
and
\[ W_{2n}^C = 4F^{-1}(\alpha) - \int_0^\infty B_2(F_n(x))dx \]
where \( B_1 \) and \( B_2 \) are as defined in Sections 2 and 3 respectively. Joe and Proschan (1982b) prove an asymptotic normality theorem for statistics of this form. Applying this theorem, the following result is easily obtained.

**Theorem 5.** For \( \lambda = 1 \), suppose that \( F \) is a continuous DPRL-\( \alpha \) distribution such that \( F^{-1} \) is continuous at \( \alpha \). For \( \lambda = 2 \), suppose that \( F \) is a continuous NRPUP-\( \alpha \) distribution such that \( F^{-1}(\alpha) \) exists at \( F^{-1}(\alpha) \) and \( F^{-1}(\alpha) \) is positive. Assume that the censoring distribution \( H \) is continuous and satisfies the condition \( [H(x)]^{-1} = O([F(x)]^{-\theta}) \) as \( x \to F^{-1}(1) \) for some \( \theta \in [0, 1] \). Then

\[ n^{1/2} (W_n^C - \Delta_{\lambda}(F)) \approx N(0, \sigma_{\lambda}^2(F, H)) \]
as \( n \to \infty \), \( \lambda = 1, 2 \),

where
\[ \sigma_{1}(F, H) = \int_0^\infty \int_0^\infty \frac{1}{[F^{-H}]^{-1}} df dxdy, \]
\[ \sigma_{2}(F, H) = \int_0^\infty \int_0^\infty \frac{1}{[F^{-2H}]^{-1}} df dxdy \]
\[ + \frac{\beta}{\alpha^{-1}(\alpha)} \int_0^\infty \frac{F^{-1}(\alpha)}{[F^{-2H}]^{-1}} df \]
\[ - \frac{(\alpha/F^{-1}(\alpha))}{\alpha} \int_0^\infty \frac{J_2(F(x))}{J_1(F(x))} \int_0^{F^{-1}(\alpha)} \frac{F^{-1}(\alpha) x}{[F^{-2H}]^{-1}} df dx, \]
and
\[ \Delta_{\lambda}(F) \] and \( J_\lambda(u), \lambda = 1, 2 \), are defined in Sections 2 and 3 respectively.

If \( F \) is exponential with mean \( \mu \) and \( [H(x)]^{-1} = O([F(x)]^{-\theta}) \) as \( x \to \infty \) for some \( \theta \in [0, 1] \), then \( n^{1/2} W_{2n}^C \approx N(0, \nu^2 \tau_{\lambda}^2(F, H)) \) as \( n \to \infty \), \( \lambda = 1, 2 \) where
\[ \tau_{1}(F, H) = \int_0^1 B_1(1 - \tau)^{-1} [K(F^{-1}(\tau))]^{-1} dt \]
and
\[ \tau_{2}(F, H) = \int_0^1 [B_2(t) - B_1(t)]^2 (1 - t)^{-1} [K(F^{-1}(t))]^{-1} dt. \]
The null asymptotic variances \( \tau_k^2(F, H) \), \( k = 1, 2 \), depend on the nuisance parameters \( \nu \) and \( H \). To define test statistics, we must obtain estimators of the null asymptotic variances which are consistent under the null hypothesis and which converge (in probability) under the corresponding alternative hypotheses. The following proposition will be useful for this purpose.

**Proposition 6.** Suppose that \( F \) and \( F \) are continuous distributions and \( H^{-1}(1) \geq F^{-1}(1) \). Define \( h_1(t) = \int_0^t B_1(u)(1 - u)^{-1} \, du \), \( 0 \leq t \leq 1 \), and

\[
h_2(t) = \int_0^t [h_1(0, u)]^{-1} B_2(u)^2(1 - u)^{-1} \, du, \quad 0 \leq t \leq 1.
\]

Let \( 0 < \eta < 1 \). Then

\[
\int_0^{F^{-1}(n)} [\bar{K}_n(x)^-1]^{-1} dh_k(F_n(x)) \rightarrow \int_0^{F^{-1}(n)} [\bar{K}(x)]^{-1} dh_k(F(x)) \quad \text{a.s. as } n \rightarrow \infty, \quad k = 1, 2,
\]

where \( \bar{K}_n \) is the empirical distribution function of the observations \( Z_1, \ldots, Z_n \) and \( \bar{K}_n = 1 - K_n \).

**Proof.** The proof is straightforward and can be found in Joe (1982).

Note that \( \tau_k^2(F, H) = \int_0^{F^{-1}(1)} [\bar{K}(x)]^{-1} dh_k(F(x)), \quad k = 1, 2, \) where \( \tau_k^2(F, H) \) is defined in Theorem 5 and \( h_k(t) \) is defined in Proposition 6. We have been unable to show that

\[
\int_0^{F^{-1}(n)} [\bar{K}_n(x)^-1]^{-1} dh_k(F_n(x)) \text{ converges in probability to } \tau_k^2(F, H) \text{ as } n \rightarrow \infty \text{ for } (F, H) \text{ satisfying the conditions of Theorem 5.}
\]

Thus we use \( \frac{\tau_k^2}{F^{-1}(n)} = \int_0^{F^{-1}(n)} [\bar{K}_n(x)^-1]^{-1} dh_k(F_n(x)) \) as an estimator of \( \tau_k^2(F, H) \) where \( n_k \epsilon (0, 1) \) is chosen so that the limit \( \int_0^{F^{-1}(n)} [\bar{K}(x)]^{-1} dh_k(F(x)) \) is approximately \( \tau_k^2(F, H) \). The next proposition gives sufficient conditions for \( \nu = \int_0^{\infty} x df(x) \) to be a consistent estimator of the mean \( \mu \) of \( F \). The proof follows from the results.
in Joe and Proschan (1982b) (see also Gill (1981)).

Proposition 7. Suppose that $F$ and $H$ are continuous distributions. Let $u_n = \frac{1}{n} \int_0^\infty x dF_n(x)$ and $u = \frac{1}{n} \int_0^\infty x dF(x)$. If $\int_0^\infty \frac{1}{|H(x)|} dF(x) < \infty$ and

$$\int_0^\infty \frac{F^2(x)}{F} \int_0^x \frac{H^2(y)}{H} dy dx < \infty,$$

then $n^\frac{1}{2}(u_n - u) = n^\frac{1}{2} \int_0^\infty \frac{F}{n} (F(x) - F(x)) dx$ converges in distribution to a normal random variable with mean zero and variance $\int_0^\infty \frac{dF(x) dF(y)}{F} \int_0^x \frac{dH^2}{H} dy dx$. Hence $u_n$ converges in probability to $u$.

We propose the following scale invariant test statistics using random censored data:

$$z_{1n}^C = n^\frac{1}{2} \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2}$$

for testing

$H_0$: $F$ is exponential

against

$H_1$: $F$ is DPRL-$\alpha$ (and not CPRL-$\alpha$),

(2) $z_{2n}^C = n^\frac{1}{2} \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i^2}$

for testing

$H_0$: $F$ is exponential

against

$H_2$: $F$ is NBUP-$\alpha$ (and not CPRL-$\alpha$).

Computational formulae for these statistics are given in Subsection 4.3.

Let $i = 1$ or 2. Consider the test which rejects the null hypothesis of exponentiality in favor of the alternative $H_i$ if $z_{2n}^C \geq z_i$. 


where $Z^\gamma$ is the upper $\gamma$-percentile of the standard normal distribution. Under some conditions (given in Theorems 8 and 9 below), this test has approximate $\gamma$-level (for $n$ sufficiently large) under $H_0$ and is consistent against distributions satisfying $H_1$.

**Theorem 8.** For $k = 1, 2$, let $\varepsilon_k > 0$ be a fixed constant which is "very small" and let $0 < \eta_k < 1$ be such that $\int_{\eta_k}^1 (1 - t)^{-2} dh_k(t) < \varepsilon_k$.

Let $F$ be the exponential distribution with mean $\mu$. Suppose that the censoring distribution $H$ is continuous and that $\bar{H}(x) \geq [\bar{F}(x)]^\theta$ for all $x \geq F^{-1}(\eta_k)$, where $\theta \in [0, 1)$. Also suppose that $\varepsilon_k$ is "much smaller" than $\tau_k^2(F, H)$, $k = 1, 2$. Then for $z > 0$, $P(Z_{kn}^\varepsilon > z) \leq 1 - \Phi(z)$ for all $n$ sufficiently large, where $\Phi$ is the standard normal distribution function.

**Proof.** Let $k = 1$ or 2. By Proposition 6, $\tau_{kn}^2 \to k^{-1}(\eta_k), \int_{\eta_k}^1 \bar{K}(x)^{-1} dh_k(F(x))$ a.s. The conditions of Proposition 7 are satisfied so that $\mu_n$ converges in probability to $\mu$. By the definition of $\eta_k$ and $\varepsilon_k$, $\int_{\eta_k}^1 \bar{K}(x)^{-1} dh_k(F(x)) = \int_{\eta_k}^1 \bar{K}(x)^{-1} dh_k(F(x)) \leq \int_{\eta_k}^1 (1 - t)^{-2} dh_k(t) < \varepsilon_k$.

By Slutsky's theorem and Theorem 5, the conclusion of the theorem follows.

**Theorem 9.** Let $\varepsilon_k$ and $\eta_k$ be as defined in Theorem 8, $k = 1, 2$. For $k = 1$ or 2, suppose that $F$ satisfies the conditions of Theorem 5 and that $\delta_k(F) > 0$. Assume that the censoring distribution $H$ is continuous and that $\bar{H}(x) \geq [\bar{F}(x)]^\theta$ for all $x \geq F^{-1}(\eta_k)$, where $\theta \in [0, 1)$. Then the test which rejects $H_0$ for significantly large values of $Z_{kn}^\varepsilon$ is consistent against $(F, H)$.
Proof. By Theorem 5, \( n^{-\frac{1}{2}} (W_{kn}^C - \Delta_k(F)) \) \( \sim \) \( N(0, \sigma_k^2(F, H)) \) as \( n \to \infty \). The conditions of Proposition 7 are satisfied so that \( u_n \) converges in probability to \( u \). By Proposition 6, \( \tau_{kn}^2 \) converges almost surely to a finite limit. Thus, for \( z > 0 \),
\[
P(n^{-\frac{1}{2}} W_{kn}^C / n \tau_{kn}^n > z) = P(n^{-\frac{1}{2}} (W_{kn}^C - \Delta_k(F)) > n^{-\frac{1}{2}} \Delta_k(F) + z n \tau_{kn}^n) \to 1
\]
as \( n \to \infty \). \( \|

4.3 Computational Formulae Let \( \xi_1 < \ldots < \xi_m \) be the ordered distinct uncensored values among \( Z_1, \ldots, Z_n \) and let \( \xi_0 = 0 \). Since we treat \( Z_{(n)} \) as an uncensored observation whether or not it is uncensored, \( \xi_m = Z_{(n)} \).

For computational purposes, formulae for \( W_{1n}^C \), \( W_{2n}^C \), \( \tau_{1n}^2 \), \( \tau_{2n}^2 \) and \( u_n \) are given below.

\[
W_{1n}^C = \sum_{j=1}^{m} \xi_j \left[ B_1(F_n(\xi_{j-1})) - B_1(F_n(\xi_j)) \right]
\]
\[
= \sum_{j=1}^{m} \xi_j \left( \int_{F_n(\xi_{j-1})}^{F_n(\xi_j)} J_1(u) du \right),
\]

\[
W_{2n}^C = \frac{1}{2} F_n^{-1}(a) - \sum_{j=1}^{m} \xi_j \left[ B_2(F_n(\xi_{j-1})) - B_2(F_n(\xi_j)) \right]
\]
\[
= \frac{1}{2} F_n^{-1}(a) - \sum_{j=1}^{m} \xi_j \left( \int_{F_n(\xi_{j-1})}^{F_n(\xi_j)} J_2(u) du \right),
\]

\[
\tau_{kn}^2 = \sum_{j=1}^{m} \left[ \frac{1}{h_k(F_n(\xi_j))} - \frac{1}{h_k(F_n(\xi_{j-1}))} \right],
\]

where the sum is over \( j; \ \xi_j \leq F_n^{-1}(\eta_k) \), \( k = 1, 2 \). (If \( \xi_k \) is of the order \( 10^{-6} \) and \( a \) is between .25 and .75, then \( 1 - \eta_k \) is of the order \( 10^{-5} \) to \( 10^{-3} \) for \( k = 1, 2 \). Thus, for sample sizes of the order \( 10^2 \) to \( 10^3 \), we can assume that \( F_n^{-1}(\eta_k) = Z_{(n)} = \xi_m \).
\[ n_n = \int_0^{2(n)} F_n(x) \, dx = \sum_{j=1}^{m} n_n \left( \xi_j - \xi_{j-1} \right) \left( \xi_{j-1} - \xi_{j-2} \right) = \sum_{j=1}^{m} \xi_j \left[ \bar{F}_n(\xi_{j-1}) - F_n(\xi_j) \right]. \]

Direct calculations yield

\[
h_1(t) = \begin{cases} 
\frac{1}{128} (1-t)^8 + \frac{1}{48} (1-t)^6 - \frac{1}{64} (1-t)^4 + \frac{1}{384} & \text{if } 0 \leq t \leq a, \\
-\frac{1}{256} (\beta^{-4} - 1)^2 (1-t)^6 + \frac{1}{48} (\beta^{-4} - 1) (\beta^{-2} - 1) (1-t)^6 & \text{if } a \leq t \leq 1,
\end{cases}
\]

and

\[
h_2(t) = \begin{cases} 
\frac{1}{16} (1-t)^4 + \frac{1}{16} & \text{if } 0 \leq t \leq a, \\
\frac{1}{16} (\beta^{-2} - 1)^2 (1-t)^4 + \frac{1}{8} (1-\beta^2) & \text{if } a \leq t \leq 1,
\end{cases}
\]

where \( \beta = 1 - a. \)

5. Asymptotic Relative Efficiencies.

Let \( \{F_\theta: \theta \in \Theta\} \) be a parametric family, where \( \Theta = [\theta_0, b] \) and \(-\infty < \theta_0 \leq b \leq \infty\). Suppose \( \theta_0 \) is such that \( F_{\theta_0} \) corresponds to the exponential distribution (with parameter 1, say). It can be shown that the efficacy of the test based on \( W^*_h \), \( h = 1, 2 \), is

\[ K^*_h = \max \left( \frac{d}{d\theta} \Delta_\hat{\theta}(F_{\theta}) \right)_{\theta = \theta_0} \cdot \sigma^{-1}_h \]

where \( \sigma_1^2 \) and \( \sigma_2^2 \) are defined in Theorems 1 and 3, respectively. The efficacies \( K^*_h \) have been calculated for the following four parametric families; formulae which are functions of \( \alpha \) are given in Joe (1982).

1. \( \bar{F}_\theta(x) = \exp(-x-\lambda \theta x^2), \ x \geq 0, \ \theta \geq 0 \) (linear failure rate),

2. \( \bar{F}_\theta(x) = \exp(-x-\theta x^e x^{-1}), \ x \geq 0, \ \theta \geq 0 \) (Makeham),
\begin{align*}
(3) \quad F_\theta(x) &= \exp(-x^\theta), \ x \geq 0, \ \theta \geq 1, \ \text{(Weibull)}, \\
&= e^{-x} \quad \text{if } 0 \leq x < -\frac{1}{2} \log \bar{B}, \\
(4) \quad F_\theta(x) &= \begin{cases} \\
\bar{B}^\theta \exp(-(1-\theta)(x+\theta \log \bar{B})) & \text{if } -\frac{1}{2} \log \bar{B} \leq x < -\log \bar{B}, \\
\bar{B}^{1-\theta/2} \exp(-x-\log \bar{B}) & \text{if } -\log \bar{B} \leq x, \ 0 \leq \theta < 1,
\end{cases}
\end{align*}

that is, \( F_\theta \) has failure rate function

\[
\tau_\theta(x) = \begin{cases} \\
1 & \text{if } 0 \leq x < -\frac{1}{2} \log \bar{B}, \\
1-\theta & \text{if } -\frac{1}{2} \log \bar{B} \leq x < -\log \bar{B}, \\
1 & \text{if } -\log \bar{B} \leq x, \ 0 \leq \theta \leq 1,
\end{cases}
\]

where \( 0 < \beta < 1 \) and \( \bar{B} = 1 - \beta \).

The first three are parametric families of IFR distributions and are used in calculations of asymptotic relative efficiencies in Hollander and Proschan (1975). The fourth is a family of distributions which are DPRL-\( \gamma \) for all \( \beta \leq \gamma \leq 1 \). For families (1), (2) and (4), \( \theta_0 = 0 \), and for family (3), \( \theta_0 = 1 \).

Table 1 gives the squares of the efficacies \( \frac{\hat{\kappa}^2}{W^*_\kappa} \) for \( \kappa = 1, 2 \) and \( \alpha = .25, .5, .75 \), and the squares of the efficacies of the Hollander and Proschan (1975) UMRL and NBUE tests. For families (1) to (3), the DMRL and NBUE tests have higher efficacies. For family (4), the tests based on \( W^*_\ln \) (with \( \alpha = \beta \)) have the highest efficacies.

Chen, Hollander and Langberg (1980a, 1980b) obtain the "asymptotic relative efficiencies" of their NBU and DMRL tests using randomly censored data with respect to the corresponding tests using complete data. We make the same comparison for the tests based on the statistics \( W^C_{\kappa n} \) and \( W^C_{\kappa n}, \ k = 1, 2 \).