Unimodality of Symmetrized Unimodal Laws and Related Results

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UNIMODALITY OF SYMMETRIZED UNIMODAL LAWS AND RELATED RESULTS

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SUMMARY. Hodges and Lehmann (1954) showed that the symmetrization of a unimodal distribution on R is again unimodal. In this paper, Khintchine's representation of unimodal laws is used to give a simpler proof of the Hodges-Lehmann result. When this result is stated in terms of characteristic functions some interesting consequences are observed. These consequences are examined through the concept of "generalized unimodality" introduced by Olshen and Savage (1970).

1. THE HODGES-LEHMANN RESULT. Suppose F is a distribution function (d.f.) on R. Define the d.f. \( \bar{F} \) by \( \bar{F}(-x) = 1 - F(x) \) whenever \( x \) is a continuity point of \( F \). Let \( F^s = F \ast \bar{F} \) be the symmetrized d.f. obtained from \( F \). It was shown by Hodges and Lehmann (1954) that \( F^s \) is unimodal for every unimodal \( F \). Their proof is somewhat complicated. Here we use Khintchine's well known representation of unimodal laws to give a simpler proof. We need the following lemma.
LEMMA 1. Let $c_1, \ldots, c_m$ and $d_1, \ldots, d_n$ be positive numbers. Then

$$\sum_{i<j} c_i c_j + \sum_{i<k<i} d_k d_t > \sum_{i=1}^{m} \sum_{k=1}^{n} c_i d_k$$  \hspace{1cm} (1)$$

PROOF. The left side of (1) equals

$$\frac{1}{2} \left[ (\sum c_i)^2 + \sum c_i^2 + (\sum d_k)^2 + \sum d_k^2 \right]$$

$$= \frac{1}{2} \left[ \sum c_i - \sum d_k \right]^2 + \sum_{i<k} c_i d_k + \frac{1}{2} \left[ \sum c_i^2 + \sum d_k^2 \right],$$

which clearly implies the required result.

THEOREM 1. If $F$ is unimodal, then so is $F^S$.

PROOF. Let $F$ be unimodal. To show that $F^S$ is unimodal we may assume that $F$ is unimodal about zero, because $F^S$ remains the same if $F(x)$ is replaced by $F(x + c)$. Now Khintchine's representation of unimodal laws says that every distribution, which is unimodal about 0, is the weak limit of finite mixtures of uniform distributions on intervals having zero as one of the end points. Therefore we may assume that $F$ is a mixture of the type described.

Denote by $u_a$ (respectively, $v_b$) the density of the uniform distribution on $(0,a)$ (respectively, $(-b,0)$), where $a > 0$ and $b > 0$. Suppose the density $f$ of $F$ is

$$f = \sum_{i=1}^{m} \alpha_i u_a + \sum_{k=1}^{n} \beta_k v_b,$$

where $\alpha_i > 0$, $\beta_k > 0$ and $\sum \alpha_i + \sum \beta_k = 1$. The density $\tilde{f}$ of $F$
is then
\[ \tilde{f} = \sum_{i=1}^{m} \alpha_i v_i + \sum_{k=1}^{n} \beta_k u_k. \]

Therefore $F^S$ has the density
\[
F^S = \sum_{i,j} \alpha_i \alpha_j (u_i \ast v_j) + \sum_{k,t} \beta_k \beta_t (v_k \ast u_t) + \sum_{i,k} \alpha_i \beta_k [(u_i \ast u_k) + (v_i \ast v_k)].
\] (2)

Now all the convolutions appearing on the right side of (2) are either triangular or trapezoidal. It is easy to check that
\[
\begin{align*}
(u \ast v)'(x) &\leq 0 \text{ if } x > 0, \ x \neq a \text{ and } x \neq a - b; \\
(u \ast v)'(x) &= -(ab)^{-1} \text{ if } 0 < x < a \leq b; \\
(u \ast u)'(x) &= (ab)^{-1} \text{ if } 0 < x < \min(a,b) \\
\text{and} \\
(u \ast u)'(x) &\leq 0 \text{ if } x < \min(a,b), \ x \neq a \text{ and } x \neq a + b
\end{align*}
\] (3)

Assume that $a_0 = \ldots = a_m$ and $b_0 = \ldots = b_n$. Write $a^*_0 = b^*_0 = 0$ and $a^*_{m+1} = b^*_{n+1} = \infty$. Suppose $x > 0$ is such that $x$ does not equal any of the $a_i$, $b_k$, $a_i - a_j$, $b_k - b_t$ or $a_i + b_k$.

Then we can find unique integers $r$ and $s$ such that $a_r < x < a_{r+1}$ and $b_s < x < b_{s+1}$. Using (2) and (3) and noting that $(v \ast v)'(x) = 0$, we get
\[
(F^S)'(x) \geq \sum_{r=1}^{m} \frac{\alpha_i \alpha_j}{(a_i - a_j)} / \sum_{r=1}^{m} (a_i \alpha_j) - \sum_{s=1}^{n} \frac{\beta_k \beta_t}{(b_k - b_t)} / \sum_{k=1}^{n} (b_k \beta_t) + \sum_{i=r+1}^{m} \sum_{k=s+1}^{n} \frac{(\alpha_i \beta_k)}{((a_i \beta_k) + (a_i b_k))}.
\]
Lemma 1 can now be applied with $c_i = a_i/a_i$ and $d_k = \beta_k/b_k$. We thus see that $(f^S)'(x) \leq 0$ whenever $x$ is positive and does not assume any of the finite number of values $a_i$, $b_k$, $a_i - a_j$, $b_k - b_t$ or $a_i + b_k$. Since $f^S$ is continuous, it follows that $F^S$ is a unimodal distribution function.

REMARK. If $f$ is a unimodal density on $R$ then Theorem 1 says that the function $g$ defined by

$$g(x) = \int_{-\infty}^{\infty} f(x + y)f(y)dy$$

is again unimodal.

2. FORMULATION IN TERMS OF CHARACTERISTIC FUNCTIONS.
Suppose $\phi$ is the characteristic function (ch.f.) of a d.f. $F$. Khintchine proved that $F$ is unimodal about zero iff there is a ch.f. $\psi$ such that

$$\phi(t) = \int_{0}^{1} \psi(tu)du, \quad t \in R.$$  

(5)

A simple change of variables shows that (5) is the same as

$$\phi(t) = \frac{1}{t} \int_{0}^{t} \psi(v)dv, \quad t \neq 0.$$  

(6)

We see immediately from (6) that

$$\frac{d}{dt}[t\phi(t)] = \psi(t), \quad t \neq 0.$$  

(7)

Thus $\phi$ corresponds to a distribution which is unimodal about 0 iff $\psi$ given by (7) is a ch.f.
Let us formulate the result of Theorem 1 in terms of ch.f.'s. Suppose $F$ is unimodal about zero and has ch.f. $\phi$. Then there is a ch.f. $\psi$ such that (5), (6), and (7) hold. The ch.f. of $\tilde{F}$ is

$$\tilde{\psi}(t) = \frac{1}{t} \int_0^t \tilde{\psi}(v)dv, \ t \neq 0.$$  \hfill (8)

Therefore the ch.f. of $F^S = F \ast \tilde{F}$ is

$$|\tilde{\phi}(t)|^2 = \frac{1}{t^2} \int_0^t \int_0^t \psi(u)\overline{\psi}(v)dudv. \hfill (9)$$

Let $\xi(t) = (d/dt)[t|\phi(t)|^2]$. Then $F^S$ is unimodal iff $\xi$ is a ch.f. We see easily from (6), (8) and (9) that

$$\xi(t) = \phi(t)\overline{\psi}(t) + \psi(t)\overline{\phi}(t) - |\phi(t)|^2. \hfill (10)$$

Thus $\xi$ is a nonconvex linear combination of ch.f.'s. Nevertheless, because of Theorem 1, $\xi$ is a ch.f. We could not find a direct proof that the right side of (10) is a ch.f. whenever $\psi$ is an arbitrary ch.f. and $\phi$ is obtained from $\psi$ by using (6).

3. EXAMINATION OF THEOREM 1 THROUGH GENERALIZED UNIMODALITY. The concept of generalized unimodality was defined by Olshen and Savage (1970). In terms of characteristic functions, a d.f. $F$ with ch.f. $\phi$ is $\alpha$-unimodal about zero iff there is a ch.f. $\zeta$ such that

$$\phi(t) = \int_0^1 \zeta(tu^{1/\alpha})du, \ t \in \mathbb{R}. \hfill (11)$$
A comparison of (11) with (5) shows that the usual unimodality is the same as \(1\)-unimodality. Equations (6) and (7) change as follows:

\[
\phi(t) = \frac{\alpha}{t^\alpha} \int_0^t v^{\alpha-1} \zeta(v) dv, \ t > 0
\]  

(12)

and

\[
d \frac{dt}{dx} [t^\alpha \phi(t)] = ut^{\alpha-1} \zeta(t), \ t > 0
\]  

(13)

Once again, \(\phi\) corresponds to a distribution which is \(\alpha\)-unimodal about zero iff \(\zeta\) defined by (13) is a ch.f. In this case the distribution corresponding to \(\zeta\) is called the \((\alpha\)-index) point distribution for \(F\).

The following theorem recalls two results from Olshen and Savage (1970).

**THEOREM 2.** (a) Let \(F\) be \(\alpha\)-unimodal with point d.f. \(G_\alpha\). Then, for \(\beta > \alpha\), \(F\) is \(\beta\)-unimodal with a \((\beta\)-index\) point d.f. \(G_\beta\), where \(G_\alpha\) and \(G_\beta\) are related by

\[
G_\beta = \frac{\alpha}{\beta} G_\alpha + (1 - \frac{\alpha}{\beta}) F.
\]

(b) If \(F\) is \(\alpha\)-unimodal and \(G\) is \(\beta\)-unimodal then \(F \ast G\) is \((\alpha + \beta)\)-unimodal.

Let us now return to the formulation of section 2. We are given a unimodal ch.f. \(\phi\) with point ch.f. \(\psi\). Since \(\overline{\phi}\) is also unimodal, we see from Theorem 2(b) that \(|\phi|^{2}\) is a
2-unimodal ch.f. The 2-index point ch.f. $\zeta$ for $|\phi|^2$ can be obtained by using (13). We get

$$\zeta(t) = \frac{1}{2t} \frac{d}{dt}(t^2 |\phi(t)|^2), \quad t \neq 0.$$ 

Now (9), (6), and (8) show that

$$\zeta(t) = \frac{1}{2}[\phi(t)\overline{\psi}(t) + \overline{\phi}(t)\psi(t)], \quad t \in \mathbb{R}. \quad (14)$$

From Theorem 1, we know that $|\phi|^2$ is actually 1-unimodal. Its 1-index point ch.f. $\xi$ is given by (10). Therefore by Theorem 2(a), we must have

$$\zeta(t) = \frac{1}{2}[\xi(t) + |\phi(t)|^2]. \quad (15)$$

It is clear from (14) that (15) and (10) are equivalent.

The question considered in Theorem 2(a) can be asked in reverse form as follows. Suppose F is a $\beta$-unimodal d.f. with the $\beta$-index point d.f. $G_\beta$. Let $\alpha < \beta$. What condition must $G_\beta$ satisfy so that F is in fact $\alpha$-unimodal? The answer, from Theorem 2(a), is that the function $G_\alpha$ defined by

$$G_\alpha = \frac{\beta}{\alpha} G_\beta - \left(\frac{\beta}{\alpha} - 1\right) F \quad (16)$$

must be a d.f. This condition can be written in a simple and interesting form if $\alpha = 1$ and $G_\beta$ is smooth. This is exhibited by the following theorem.
THEOREM 3. Suppose $F$ is a $\beta$-unimodal d.f. with the
$\beta$-index point distribution $G_\beta$. Let $\beta > 1$ and let $G_\beta$ be suf-
ficiently smooth. Then $F$ is unimodal iff the density $g_\beta$ of
$G_\beta$ satisfies
$$\int_0^\infty g_\beta'(y)/y^{\beta-1}dy < 0 \quad \text{for all } x > 0$$
and
$$\int_{-\infty}^x g_\beta'(y)/y^{\beta-1}dy \geq 0 \quad \text{for all } x < 0.$$

PROOF. Let $\phi$ and $\zeta$, respectively, denote the c.h.f.'s of
$F$ and $G_\beta$. Then (11) shows that
$$\phi(t) = \int_0^1 \zeta(tu^{1/\beta})du, \ t \in \mathbb{R}. \quad (17)$$
One easily sees from (17) that
$$F(x) = \int_0^1 G_\beta(-xu^{1/\beta})du, \ x \in \mathbb{R}.$$ 
Thus (16) can be written as
$$G_\alpha(x) = \frac{\beta}{\alpha}G_\beta(x) - \left(\frac{\beta}{\alpha} - 1\right) \int_0^1 G_\beta(-xu^{1/\beta})du. \quad (18)$$
We want $G_\alpha$ to be a d.f. It is easy to see that $G_\alpha(\infty) = 1$ and
$G_\alpha(-\infty) = 0$. Since $G_\beta$ is assumed to be smooth, $G_\alpha$ is also
smooth. So $G_\alpha$ will be a d.f. as soon as its derivative $g_\alpha$ is
nonnegative.

Assume now that $\alpha = 1$. Then (18) gives
$$g_1(x) = \beta g_\beta(x) - (\beta - 1) \int_0^1 g_\beta(-xu^{1/\beta})u^{-1/\beta}du.$$
Let $x > 0$. Then a change of variables $y = x u^{-1/β}$ shows that

$$g_1(x) = β g_β(x) - (β - 1) x^{β-1} \int_x^\infty \frac{[g_β(y)/y^β]}{y} dy \quad (19)$$

Note that $(β - 1)/y^β$ is the derivative of $[-1/y^{β-1}]$. Therefore the integral on the right side of (19) can be integrated by parts to give

$$g_1(x) = -β x^{β-1} \int_x^\infty \frac{[g_β'(y)/y^{β-1}]}{y} dy.$$ 

Since $x > 0$, it follows that $g_1(x) \geq 0$ iff

$$\int_x^\infty \frac{[g_β'(y)/y^{β-1}]}{y} dy \leq 0.$$ 

A similar calculation can be done for $x < 0$. The theorem is thus proved.

REMARK. Consider the ch.f. $ζ$ given by (14). We know that $ζ$ is the 2-index point ch.f. of the 2-unimodal ch.f. $|φ|^2$. Since $|φ|^2$ is in fact unimodal, Theorem 3 shows that the density $g$ corresponding to $ζ$ must satisfy

$$\int_x^\infty \frac{[g'(y)/y]}{y} dy \leq 0 \quad \text{for all } x > 0 \quad (20)$$

and

$$\int_{-∞}^X \frac{[g'(y)/y]}{y} dy \leq 0 \quad \text{for all } x < 0.$$ 

If this condition is analyzed in terms of the original d.f. $F$, then one gets back condition (4) after considerable algebra.

REMARK. In the context of the preceding remark, the density $g$ is symmetric. Therefore the second condition in (20)
is the same as the first. But it is possible to consider (20) for densities which are not necessarily symmetric. It is clear that a density which is unimodal about zero satisfies (20). But, as the following example shows, a nonunimodal density may also satisfy (20).

EXAMPLE. Define the function \( g \) by:

\[
g(x) = \begin{cases} 
    e^{-2}(3 + \frac{1}{2}x \cos \pi x - \frac{1}{2\pi} \sin \pi x), & 0 < x \leq 2 \\
    xe^{-x}, & x \geq 2.
\end{cases}
\]

It is easy to see that \( g \) is continuous at \( x = 2 \) and that \( g \) has a positive local minimum at \( x = 1 \). Thus apart from a multiplicative constant, \( g \) is a density. One easily gets

\[
g'(x) = \begin{cases} 
    -\frac{1}{2} e^{-2} \pi x \sin \pi x, & 0 < x \leq 2 \\
    xe^{-x}(2 - x), & x \geq 2.
\end{cases}
\]

Therefore

\[
\int_{2}^{\infty} \left[ g'(y)/y \right] dy = \int_{2}^{\infty} e^{-y}(2 - y) dy = -e^{-2}
\]

and

\[
\int_{1}^{2} \left[ g'(y)/y \right] dy = -\frac{1}{2} e^{-2} \pi \int_{1}^{2} \sin \pi y dy = e^{-2}.
\]

Since \( g'(x) < 0 \) for \( 0 < x < 1 \), it follows that \( g \) satisfies (20).
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Distribution unlimited

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