Large Deviation Local Limit Theorems for
Arbitrary Sequences of Random Variables

by

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FSU Statistics Report M630
USARO Technical Report D-53

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June, 1982

† This report was supported in part by the U.S. Army Research Office
Grant No. DAAG 29-79-C-0158. The United States Government is authorized to reproduce and distribute reprints for governmental purposes.

Key Words: Local Limit Theorems, Large Deviations, Laplace Transform, Nonparametric Inference.

AMS (1980) Subject Classifications: Primary 60F05, 60F10.
1. Introduction.

Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with \( E(X_1) = 0, \ Var(X_1) = 1 \). Let \( \psi(s) \) be the cumulant generating function (c.g.f.) and \( \gamma(u) = \sup_{s \geq 0} [us - \psi(s)] \) be the large deviation rate of \( X_1 \). Let \( S_n = X_1 + \ldots + X_n \). Under some mild conditions on \( \psi \), Richter (1957) obtained an asymptotic expression for the probability density function \( f_n \) of \( S_n/n \) involving the Cramer's series, in the case when \( X_1 \) is non-lattice valued random variable. A close examination of the asymptotic expression reveals that it can be rewritten as

\[
(1.1) \quad f_n(x_n) = \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-n\gamma(x_n)} [1 + o(1|x_n|)]
\]

whenever \( x_n = o(1) \) and \( \sqrt{n}x_n > 1 \). The purpose of this paper is to obtain similar large deviation local limit theorems for arbitrary sequences of random variables, not necessarily sums of i.i.d. random variables, thereby increasing the applicability of Richter's theorem.

Let \( \{T_n, n \geq 1\} \) be an arbitrary sequence of non-lattice random variables with an analytic characteristic function (c.f.) \( \phi_n(z) \), nonvanishing in the region \( \Omega = \{z \in \mathbb{C} \mid \text{Real}(z) < a\} \), where \( \mathbb{C} \) is the set of complex numbers and \( a > 0 \).

Let

\[
(1.2) \quad \psi_n(z) = \frac{1}{n} \log \phi_n(z) \quad \text{for } z \in \mathbb{C} \quad \text{and}
\]

\[
(1.3) \quad \gamma_n(u) = \sup_{s \in (-a,a)} [us - \psi_n(s)] \quad \text{for real } u.
\]
The main Theorem 2.1 in section 2 states that under some standard conditions on \( \psi_n \), which guarantee the existence of the density function, \( k_n \), of \( T_n/n \) and further imply that \( T_n/n \) converges to 0 in probability, we can write

\[
(1.4) \quad k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi} \sqrt{\psi_n''(\tau_n)}} \exp\left\{-n\gamma_n(m_n)[1 + O\left(\frac{1}{n}\right)]\right\},
\]

where \( \{m_n\} \) is any sequence of real numbers and \( \tau_n \) is defined by \( \psi_n'(\tau_n) = m_n \) when \( T_n \) is taken to be the sum of \( n \) i.i.d. random variables the above theorem reduces to Richter's result.

Our extension of Richter's theorem to arbitrary sequences of random variables \( T_n \) may be compared to Steinebach's (1978) extension of the earlier work on large deviations of Chernoff (1952), Bahadur and Ranga Rao (1960) for sums of i.i.d. random variables. Steinebach's (1978) results on large deviations of an arbitrary sequence of random variables \( T_n \), which extended earlier work of Sievers (1969), Plachky and Steinebach (1975) used conditions based solely on the moment generating function of \( T_n \). In this sense Theorem 2.1 is comparable with Steinebach's (1978) results. We also present analogous results for lattice valued random variables.

The proofs of our theorems follow the classical pattern of proofs in this area, including that of Richter's theorem. The p.d.f. of \( T_n/n \) is first expressed in terms of its Laplace transformation. Next, the claimed asymptotic expression for this p.d.f. is extracted leaving a remainder term. The inverse
transformation for Laplace transforms still allows one to pick the value of the real argument in that transform. We pick the value of the real argument in an appropriate way to use a saddle point approximation. Our proof differs from Richter's proof at this stage in two respects. We split the integral in the remainder term into two parts which depend on $n$ (see (2.13)), and we use the Cauchy formula for derivatives of an analytic function (see (2.5)) to obtain sharper estimates of all the quantities involved. This allows us to generalize Richter's theorem.

In section 3 we present various applications of the theorems of Section 2, for statistics occurring in nonparametric inference. Prominent examples are the Wilcoxon signed rank statistic and Kendall's tau statistic for testing independence in a bivariate population. We show that, for example, if $W_n$ is Kendall's tau, then

$$P(W_n = 0) = \frac{6}{\sqrt{2\pi n(n-1)(2n+5)}} \left[ 1 + O\left(\frac{1}{n}\right) \right]$$

as $n$ tends to $\infty$ through the appropriate sequence for which 0 is in the range of $W_n$. 
2. Local Limit Theorems for Arbitrary Sequence of Random Variables.

This section contains the main theorems of this paper, namely Theorems 2.1 and 2.2. We develop some notations before stating these theorems. Let \( \{T_n, n \geq 1\} \) be a sequence of non-lattice random variables with c.f. \( \phi_n(z) \) which is analytic and non-vanishing for \( z \in \Omega = \{z: |\text{Real}(z)| < a\} \) with \( a > 0 \). Let \( \psi_n \) and \( \gamma_n \) be as defined in (1.2) and (1.3). Let \( I = (-a, a) \) and \( I_1 = (-a_1, a_1) \) for some \( a_1 \) with \( 0 < a_1 < a \). Let \( \{m_n\} \) be a sequence of real numbers and let \( G_{n, \tau}(t) = \psi_n(\tau) + itm_n - \psi_n(\tau + it) \), for \( \tau \in I_1 \). The following theorems provide an asymptotic expansion for the density function \( k_n \) of \( T_n/n \) in terms of the large deviation rate \( \gamma_n \).

**Theorem 2.1.** Assume the following conditions for \( T_n \):

(A). There exists \( \beta < \infty \) such that \( |\psi(z)| < \beta \) \( \forall z \in \Omega \), \( \forall n \geq 1 \).

(B). There exists \( \alpha > 0 \) and \( \tau_1 \in I_1 \) such that \( \psi_n(\tau_n) = m_n \) and \( \psi''(\tau) \geq \alpha, \forall \tau \in I_1 \), \( \forall n \geq 1 \).

(C). There exists \( \eta > 0 \) such that for any \( 0 < \delta < \eta \),

\[
\inf \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))],
\]

\( |t| \geq \delta \)

\( \forall n \geq 1 \), where \( G_n(t) = G_{n, \tau_n}(t) \).

(D). There exists \( p, \xi > 0 \) such that

\[
\int_{-\infty}^{\infty} |\phi_n(\tau + it)/\phi_n(\tau)|^{\xi/n} = 0(n^p) \quad \forall \tau \in I.
\]

Then

\[
(2.1) \quad k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi} \sqrt[n]{\psi''(\tau_n)}} e^{-n\gamma_n(m_n)}[1 + O(\frac{1}{n})].
\]
For lattice valued random variables $T_n$ we have the following analogous theorem.

**Theorem 2.2.** Let $T_n$ take values in the set \( \{ a_n + k h_n : k = 0, \pm 1, \pm 2, \ldots \} \).

Let \( \{ m_n = (a_n + k h_n)/n \} \) be a sequence of real numbers, where \( \{ k_n \} \) is a sequence of integers. Assume that conditions (A), (B) of Theorem 2.1 hold and replace conditions (C), (D) by the following:

(C'). There exists \( \eta > 0 \) such that for any \( 0 < \delta < \eta \),

\[
\inf \Real(G_n(t)) = \min \{ \Real(G_n(\delta)), \Real(G_n(-\delta)) \}, \forall n \geq 1.
\]

(D'). There exists \( \varphi, \varphi > 0 \) such that

\[
\frac{\pi}{h_n} \int_{-\pi/h_n}^{\pi/h_n} \left| \phi_n(\tau + it)/\phi(\tau) \right|^{\varphi/n} = o(n^\varphi)
\quad \forall \tau \in \mathbb{I}.
\]

Then

\[
(2.2) \quad \frac{\sqrt{n}}{|h_n|} \Pr(T_n = m_n) = \frac{1}{\sqrt{2\pi} \sqrt{\psi_n'(\tau)}} e^{-n\gamma_n(m_n)} [1 + O(\frac{1}{n})].
\]

We now make a few observations which will explain the implications of the Conditions (A) - (D). The proofs of the above two theorems will be given after Lemma 2.10.

**Remark 2.3.** Condition (A) of Theorem 2.1 implies, as is shown later, that the first and the second derivative of \( \psi_n \) at 0 is bounded in \( n \), which in turn implies that \( (T_n - E(T_n))/n \) is converging in probability to 0.
Remark 2.4. One can easily verify that if $T_n$ satisfies Conditions (A) - (D) of Theorem 2.1 then $T_n^1 = (T_n - E(T_n))$ also satisfies the same four conditions. Hence one can assume that $E(T_n) = 0$, although this assumption is not really needed in the proof of Theorem 2.1.

Remark 2.5. Condition (B) is really a condition on the sequence \{m_n\}. This is trivially satisfied if $m_n$ is equal to $E(T_n)/n$, however in practice we would like to take $m_n$ close to $E(T_n)/n$.

Remark 2.6. Condition (C) is easily verified if $G_n(t)$ is increasing with $|t|$, otherwise it seems a rather difficult condition to check. The following lemma, which holds for any sequence of real valued functions $f_n(t)$, $n \geq 1$, provides an easily verifiable sufficient condition. In Example 3.3 we will be verifying this sufficient condition instead of Condition (C).

Lemma 2.7. Let $f_n(t)$ be a sequence of continuous real valued functions such that $f_n(0) = 0$ and 0 is the unique minimum for all $n \geq 1$. Assume that the following conditions hold for all $n \geq 1$.

(i) There exists $\eta_1 > 0$ such that $f_n(t)$ is increasing with $|t|$ in the interval $(-\eta_1, \eta_1)$.

(ii) There exists $\varepsilon > 0$ such that $\inf_{|t| \geq \eta_1} f_n(t) > \varepsilon$.

(iii) There exists $0 < \eta < \eta_1$ such that $\sup_{|t| \geq \eta} f_n(t) < \varepsilon$.

Then for any $0 < \delta < \eta$
(2.3) \[ \inf_{|t| \geq \delta} f_n(t) = \min[f_n(\delta), f_n(-\delta)] \quad \forall \ n \geq 1. \]

Proof.

Let \( 0 < \delta < n \) be fixed. Conditions (ii) and (iii) imply that

\[ \inf_{|t| \geq \eta_1} f_n(t) > \varepsilon > f_n(\delta). \]

Using Condition (i) we obtain

\[ \inf_{|t| \geq \delta} f_n(t) = \inf_{n \geq |t| \geq \delta} f_n(t) \]

\[ = \min[f_n(\delta), f_n(-\delta)]. \]

To see that this Lemma 2.7 provides a sufficient condition for Condition (C) to be satisfied, set \( f_n(t) = \text{Real}(G_n(t)) \). Clearly \( f_n(0) = 0 \) and 0 is the unique minimum for \( f_n \quad \forall \ n \geq 1. \) Further if \( G_n \) satisfies conditions (i), (ii), (iii) of Lemma 2.7 then there exists \( \eta > 0 \) such that for \( 0 < \delta < \eta, \)

\[ \inf_{|t| \geq \delta} \text{Real}(G_n(t)) = \inf_{|t| \geq \delta} f_n(t) \]

\[ = \min[f_n(\delta), f_n(-\delta)] \]

\[ = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))] \quad \forall \ n \geq 1. \]
Remark 2.8. When \( T_n \) is the sum of i.i.d. random variables with c.g.f. \( \psi \) then \( \text{Real}(\psi_n(\tau + it)) = \text{Real}(\psi(\tau + it)) \quad \forall \ n \geq 1 \). Let \( f(t) = \psi(\tau) - \text{Real}(\psi(\tau + it)) \). Since \( \psi \) is the c.g.f. of a non-lattice valued random variable, \( f(t) \) has a unique minimum at \( t = 0 \) and it satisfies all the three assumptions of Lemma 2.7. Thus Condition (C) is true for sums of i.i.d. non-lattice random variables.

Remark 2.9. Condition (D) of Theorem 2.1 not only guarantees the existence of the density function of \( T_n \), but also permits the use of the inversion formula to get an expression for the p.d.f. of \( T_n \). It is also used to show that the term \( I_n \) appearing in the proof of Theorem 2.1 goes exponentially fast to 0 (see (2.15)).

We will need the following lemma in the proof of Theorem 2.1.

**Lemma 2.10.** Assume that Conditions (A), (B) and (C) of Theorem 2.1 are satisfied. Let \( G_n(t) = G_n, \tau_n(t) = [\psi_n(\tau_n) + \text{it}_n - \psi_n(\tau_n + it)] \). Then there exists \( \delta_1 < \delta \) such that for \( 0 < \delta < \delta_1 \),

\[
\inf_{|t| \geq \delta} \text{Real}(G_n(t)) \geq \alpha \delta^2 / 4 \quad \forall \ n \geq 1.
\]

**Proof.** For \( z \in \mathbb{C} \) and \( r > 0 \) define \( c(z, r) = \{ w \in \mathbb{C} : |z-w| = r \} \).

Since \( \psi_n \) is analytic in \( \Omega \) and \( |\tau_n| < a_1 \), by Cauchy's theorem we get

\[
\psi_n^{(k)}(\tau_n) = \frac{k!}{2\pi i} \int_{c(\tau_n, a_1)} \frac{[\psi_n(w)/(w-\tau_n)^{k+1}]}{c(\tau_n, a_1)} \text{d}w \quad \forall \ k \geq 1.
\]
Using Condition (A), we obtain

\[(2.6) \quad |\psi_n^{(k)}(\tau_n)| \leq \frac{k!}{2^n} \sup_{w \in \mathcal{C}(\tau_n, a-a_1)} |\psi_n(w)| \int \frac{1}{|w-\tau_n|^{k+1}} \, dw \frac{1}{c(\tau_n, a-a_1)} \leq \frac{k! \beta}{(a-a_1)^k} \quad \forall \ k \geq 1.\]

Once again, since \( \psi_n \) is analytic in \( \Omega \) and \( |\tau_n| < a_1 \), we can find a positive number \( \delta_1 < \min(n, (a-a_1)/2) \) such that \( \delta_1 \beta[1+2\delta_1/(a-a_1)] \leq a(a-a_1)^3/4 \) and for \( |t| \leq \delta_1 \) the following expansion is valid for all \( n \geq 1 \).

\[(2.7) \quad \psi_n(\tau_n + it) = \psi_n(\tau_n) + it\psi_n'(\tau_n) + \frac{(it)^2}{2!} \psi_n''(\tau_n) + \frac{(it)^3}{3!} \psi_n'''(\tau_n) + R_n(\tau_n + it),\]

where

\[R_n(\tau_n + it) = \frac{(it)^4}{2\pi i} \int \frac{[\psi_n(w)/(w-\tau_n)^4]dw}{c(\tau_n, a-a_1)} (w-\tau_n - it)]dw.\]

An upper bound for the modulus of the remainder term \( R_n \) can be obtained as follows:

\[(2.8) \quad |R_n(\tau_n + it)| \leq \frac{\beta t^4}{2^n} \sup_{w \in \mathcal{C}(\tau_n, a-a_1)} |\psi_n(w)| \int \frac{1}{|w-\tau_n|^4 |w-\tau_n - it|} \, dw \frac{1}{c(\tau_n, a-a_1)} \leq \frac{\beta t^4}{2^n (a-a_1)^4} \int |w-\tau_n - it| \, dw \frac{1}{c(\tau_n, a-a_1)} \leq \frac{2\beta t^4}{(a-a_1)^4} \quad \forall \ n \geq 1,\]
since \(|w-\tau_n-it| \geq |w-\tau_n| - |t| = (a-a_1) - |t| \geq (a-a_1)/2\). Noting that 
\(\psi'_n(\tau_n) = m_n\), we get from (2.7) that for \(|t| \leq \delta_1\)

\[
(2.9) \quad G_n(t) = \frac{t^2\psi''_n(\tau_n)}{2} + \frac{it^3\psi'''_n(\tau_n)}{3!} - R_n(\tau_n + it),
\]

so that

\[
\left| \frac{G_n(t)}{t^2} - \frac{\psi''_n(\tau_n)}{2} \right| = \left| \frac{it^3\psi'''_n(\tau_n)/6 - R_n(\tau_n + it)}{t^2} \right|
\]

\[
\leq \frac{|t| |\psi'''_n(\tau_n)|}{6} + \frac{|R_n(\tau_n + it)|}{t^2}
\]

\[
\leq \frac{|t| \delta}{(a-a_1)^3} + \frac{2 \delta^2 t^2}{(a-a_1)^4} \quad \text{[by (2.6) and (2.8)]}
\]

\[
\leq \frac{\delta \beta}{(a-a_1)^3} + \frac{2 \delta^2 \beta}{(a-a_1)^4} \leq \frac{\alpha}{4} \quad \forall \ n \geq 1.
\]

Thus for any \(n \geq 1\),

\[
(2.10) \quad |t| \leq \delta_1 \Rightarrow \left| \frac{G_n(t)}{t^2} - \frac{\psi''_n(\tau_n)}{2} \right| < \frac{\alpha}{4}
\]

\[\Rightarrow \left| \text{Real} \left( \frac{G_n(t)}{t^2} \right) \right| \geq \frac{1}{2} [\psi''_n(\tau_n) - \frac{\alpha}{2}]
\]

\[\Rightarrow \text{Real}(G_n(t)) \geq \frac{\alpha t^2}{4} \text{[since } \psi''_n(\tau_n) > \alpha].
\]
Since \( \delta_1 < n \) it then follows from Condition (C) that for \( \delta < \delta_1 \),

\[
\inf_{|t| \geq \delta} \Re(G_n(t)) = \min\{\Re(G_n(\delta)), \Re(G_n(-\delta))\} \geq \frac{\alpha \delta^2}{4} \quad \text{[by (2.10)]}
\]

and the proof of Lemma 2.10 is complete. \\

**Proof of Theorem 2.1.**

Let \( F_n \) be the d.f. of \( T_n \). For \( \tau \in I_1 \) define the conjugate distribution \( H_n \) by

\[
dH_n(x) = \frac{e^{\tau x}}{\phi_n(\tau)} \, dF_n(x).
\]

(2.11)

The c.f. of the d.f. \( H_n \), which is given by \( \phi_n(\tau+it)/\phi_n(\tau) \), is absolutely integrable in view of Condition (D). Thus the p.d.f. of \( H_n \) exists and from the inversion formula it is given by

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \phi_n(\tau+it)/\phi_n(\tau) \right] e^{-itx} \, dt.
\]

Using (2.11) the p.d.f. on \( F_n \) is given by \( \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau+it) e^{-(\tau+it)x} \, dt \).

Thus the p.d.f. of \( T_n/n \) is given by

\[
k_n(x) = \frac{n}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau+it) e^{-n(\tau+it)x} \, dt.
\]

The above expression is valid for any \( \tau \in I_1 \). When \( x = m_n \) the best choice of \( \tau \) is \( \tau_n \) and we arrive at the relation below, which is the starting point of the analysis of the error terms:
\[(2.12)\]
\[k_n(m_n) = \frac{n}{2\pi} \int_{-\infty}^{\infty} \phi_n(\tau_n + it) e^{-n(\tau_n + it)m_n} dt\]

\[= \frac{\sqrt{n}}{\sqrt{2\pi\psi_n^\prime(\tau_n)}} e^{-n\gamma_n(m_n)} I_n,\]

where

\[(2.13)\]
\[I_n = \frac{\sqrt{n}\psi_n^\prime(\tau_n)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{n[\gamma_n(m_n) - (\tau_n + it)m_n]} \phi_n(\tau_n + it) dt\]

\[= \frac{\sqrt{n}\psi_n^\prime(\tau_n)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - itm_n]} dt\]

\[= \frac{\sqrt{n}\psi_n^\prime(\tau_n)}{\sqrt{2\pi}} \left[ \int_{|t|<n^{-\lambda}} e^{-nG_n(t)} dt + \int_{|t|\geq n^{-\lambda}} e^{-nG_n(t)} dt \right]\]

\[= I_{n1} + I_{n2} \text{ (say)}\]

and \(\lambda\) is chosen to be a constant such that \(1/3 < \lambda < 1/2\). The proof is completed by showing that \(I_{n1}\) goes exponentially fast to zero and \(I_{n2} = 1 + O\left(\frac{1}{n}\right)\), as \(n\) goes to \(\infty\). First consider the term \(I_{n1}\).

By Lemma 2.10 we can find an \(N\) such that for \(n \geq N\) we have

\[(2.14)\]
\[\inf_{|t|\geq n^{-\lambda}} \text{Real}(G_n(t)) \geq \frac{an^{-2\lambda}}{4},\]

\[|t|\geq n^{-\lambda}\]
Now for $n \geq N$,

$$(2.15) \quad |I_{n1}| \leq \frac{\sqrt{n} \psi_n'(\tau_n)}{\sqrt{2\pi}} \sup_{|t| \geq n^{-\lambda}} \left| e^{-nG_n(t)} \right| dt$$

$$\leq \frac{\sqrt{n} \psi_n'(\tau_n)}{\sqrt{2\pi}} \sup_{|t| \geq n^{-\lambda}} \left| e^{-n^\lambda G_n(t)} \right| \int |t| \geq n^{-\lambda}$$

$$\leq \frac{\sqrt{n} \psi_n'(\tau_n)}{\sqrt{2\pi}} \sup_{|t| \geq n^{-\lambda}} \left| e^{-n^\lambda G_n(t)} \right| \int \left| \frac{\phi_n'(\tau_n t)}{\phi_n'(\tau_n)} \right|^{\alpha/n} dt$$

$$= O(n^{P+\delta}) \exp[-(n^\lambda) \inf \{\text{Real}(G_n(t))\}] \sup_{|t| \geq n^{-\lambda}}$$

$$= O(n^{P+\delta}) \exp[-(n^\lambda) \alpha/4n^{2\lambda}] \text{ (by } 2.36)$$

$$= O(n^{P+\delta}) \exp[-an^{1-2\lambda/4}],$$

which goes exponentially fast to zero since $1/3 < \lambda < 1/2$. Our next step is to show that $I_{n2} = 1 + O(1/n)$. Recall that

$$I_{n2} = \frac{\sqrt{n} \psi_n'(\tau_n)}{\sqrt{2\pi}} \int \left| e^{-nG_n(t)} \right| dt$$

$$= \frac{\sqrt{n} \psi_n'(\tau_n)}{\sqrt{2\pi}} \int_{|s| < n^{-\lambda}} e^{-nG_n(S)} ds.$$
For \(|s| < \frac{\lambda}{\sqrt{n}}\), \(s/\sqrt{n}\) goes to zero uniformly as \(s \to \infty\), hence we can use (2.9) to get,

\[
I_{n2} = \frac{\psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < \sqrt{n} \lambda} e^{s^2 \psi''(\tau_n)} \left[ -\frac{2}{3} \psi'''(\tau_n) \frac{is^3}{6n\sqrt{n}} \phi''(\tau_n) + R_n(\tau_n + \frac{i s}{\sqrt{n}}) \right] ds
\]

(2.16)

\[
= \frac{\psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < \sqrt{n} \lambda} e^{s^2 \psi''(\tau_n)} \left[ \frac{is^3}{6n\sqrt{n}} \phi''(\tau_n) + R_n(\tau_n + \frac{i s}{\sqrt{n}}) \right] ds
\]

\[
= \frac{\psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < \sqrt{n} \lambda} e^{s^2 \psi''(\tau_n)} \left[ \frac{1 - \frac{is^3}{6n\sqrt{n}} \phi''(\tau_n) + R_n(\tau_n + \frac{i s}{\sqrt{n}})}{6n\sqrt{n}} \right] ds,
\]

where

(2.17) \( L_n(s) = [e^{s^2 \psi''(\tau_n)} - 1 - z_n] \) and \( z_n = \left[ \frac{is^3}{6n\sqrt{n}} \phi''(\tau_n) + R_n(\tau_n + \frac{i s}{\sqrt{n}}) \right] \).

Thus,

\[
I_{n2} = \frac{\psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < \sqrt{n} \lambda} e^{s^2 \psi''(\tau_n)} ds
\]

(2.18)

\[-\frac{i \psi''(\tau_n)}{6n2m} \phi''(\tau_n) \int_{|s| < \sqrt{n} \lambda} s^3 e^{-s^2 \psi''(\tau_n)} ds\]

\[+ \frac{\psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < \sqrt{n} \lambda} e^{s^2 \psi''(\tau_n)} R_n(\tau_n + \frac{i s}{\sqrt{n}}) ds\]
\[ \frac{s^2}{2} \psi''(\tau_n) L_n(s) ds + \frac{\sqrt{\psi''(\tau_n)}}{\sqrt{2\pi}} \int_{|s| < n^{\frac{1}{2} - \lambda}} e^{-s^2/2} \psi''(\tau_n) L_n(s) ds. \]

The first term equals 1 - \(2\Phi(-n^{\frac{1}{2} - \lambda}\sqrt{\psi''(\tau_n)})\) and the second term is zero.

It is easy to show that \(\Phi(-n^{\frac{1}{2} - \lambda}\sqrt{\psi''(\tau_n)}) = o\left(\frac{1}{n}\right)\). Thus we get

\[ (2.19) \quad I_{n2} = 1 + o\left(\frac{1}{n}\right) + \frac{n \sqrt{\psi''(\tau_n)}}{\sqrt{2\pi}} \int_{|s| < n^{\frac{1}{2} - \lambda}} e^{-s^2/2} \psi''(\tau_n) R_n(\tau_n + i^s \sqrt{n}) ds \]

\[ + \frac{\sqrt{\psi''(\tau_n)}}{\sqrt{2\pi}} \int_{|s| < n^{\frac{1}{2} - \lambda}} e^{-s^2/2} \psi''(\tau_n) L_n(s) ds. \]

We now show that the last two terms on the r.h.s. are \(O\left(\frac{1}{n}\right)\).

Consider the third term. The second inequality below follows from (2.8) since \(s/\sqrt{n}\) goes uniformly to zero for \(|s| n^{\frac{1}{2} - \lambda}\).

\[ \left| \frac{n \sqrt{\psi''(\tau_n)}}{\sqrt{2\pi}} \int_{|s| < n^{\frac{1}{2} - \lambda}} e^{-s^2/2} \psi''(\tau_n) R_n(\tau_n + i^s \sqrt{n}) ds \right| \]
\[ \leq \frac{n \sqrt{\psi''(\tau_n)}}{\sqrt{2\pi}} \int_{|s| < n^{\frac{1}{2} - \lambda}} e^{-s^2/2} \psi''(\tau_n) |R_n(\tau_n + i^s \sqrt{n})| ds \]
\[ \leq \frac{\sqrt{\psi''(\tau_n)}}{\sqrt{n \sqrt{2\pi}}} \frac{2\beta}{(a-a_1)^4} \int_{s < n^{\frac{1}{2} - \lambda}} 4s^4 e^{-s^2/2} \psi''(\tau_n) ds \]
\[ = O\left(\frac{1}{n}\right). \]
Thus,

$$\frac{n^{2} \psi''(\tau_n)}{\sqrt{2\pi}} \int_{|s| < n^{1/2-\lambda}} e^{i s \psi''(\tau_n) \mathbb{R}_n(\tau_n + i s / \sqrt{n})} ds = O \left( \frac{1}{n} \right).$$

We now get an upper bound for \( L_n(s) \) which will be used to show that the last term on the r.h.s. of (2.19) is \( O \left( \frac{1}{n} \right) \). For \( |s| < n^{1/2-\lambda} \), since \( 1/3 < \lambda < 1/2 \), \( |s|/\sqrt{n} \), \( |s|^3/\sqrt{n} \) and \( s^4/n \) converges to zero uniformly in \( s \) as \( n \to \infty \). Hence there exists \( N_1 \), not depending on \( s \), such that for \( n \geq N_1 \) the following inequalities are valid. The second inequality follows from (2.6) and (2.8). Recall that

$$|z_n| = \left| \frac{is^3 \psi''(\tau_n) + n \mathbb{R}_n(\tau_n + i s / \sqrt{n})}{6 \sqrt{n}} \right|$$

$$\leq \frac{|s|^3}{6 \sqrt{n}} |\psi''(\tau_n)| + n |\mathbb{R}_n(\tau_n + i s / \sqrt{n})|$$

$$\leq \frac{|s|^3}{\sqrt{n}} \frac{\beta}{(a-a_1)^3} + \frac{28 s^4}{n (a-a_1)^4} < 1/2.$$ 

It is easy to check that \( |z| < 1/2 \) implies \( |e^{z} - 1 - z| < 2e |z|^2 \). Thus for \( n \geq N_1 \) and \( |s| \leq n^{1/2-\lambda} \) we get

$$|L_n(s)| \leq 2e \left[ \frac{|s|^3}{\sqrt{n}} \frac{\beta}{(a-a_1)^3} + \frac{28 s^4}{n (a-a_1)^4} \right]^2$$

$$= \frac{N_1}{n} \left[ |s|^3 + 2s^4/\sqrt{n(a-a_1)} \right]^2.$$
where

\[ M = \frac{2e\delta^2}{(a-a_1)^6}. \]

Therefore for \( n \geq N_1 \),

\[
\left| \sqrt{\frac{\psi_n'(\tau_n)}{\sqrt{2\pi}}} \int_{|s| \leq n^{\frac{1}{2}-\lambda}} e^{-\frac{s^2}{2}} \psi_n'(\tau_n) L_n(s) ds \right| \leq \frac{1}{\sqrt{n}} \frac{\beta}{(a-a_1)} \int_{|s| \leq n^{\frac{1}{2}-\lambda}} e^{-\frac{as^2}{2}} |L_n(s)| ds \quad [\text{using (2.6) }] \]

\[
\leq \frac{M}{\sqrt{n}} \frac{\sqrt{\beta}}{(a-a_1)^{\sqrt{\pi}}} \int_{|s| \leq n^{\frac{1}{2}-\lambda}} e^{-\frac{as^2}{2}} \left[ |s|^3 + 2s^4 / \sqrt{n(a-a_1)} \right]^2 ds
\]

\[ = O\left( \frac{1}{n} \right). \]

From (2.19), (2.20) and (2.23) we get \( I_n = 1 + O\left( \frac{1}{n} \right) \) and the proof is complete. ||

Remark 2.11. In the above proof we only need the weaker condition that the second derivative of \( \psi_n \) at the point \( \tau_n \) is bounded below by \( \alpha \ \forall \ n \geq 1 \). The stronger condition \( \psi_n''(\tau) \geq \alpha \ \forall \ \tau \in I_1, \forall \ n \geq 1 \) will be used to obtain further refinements of the expression (2.1).

Corollary 2.12. Let \( \{T_n, n \geq 1\} \) be a sequence of random variables satisfying the conditions of Theorem 2.1. Let \( m \) be a real number such that for each \( n \geq 1 \) there exists \( \xi_n \in I_1 \) satisfying \( \psi_n'\left( \xi_n \right) = m_n \). Suppose that \( m_n + m \) and \( n^\delta |m_n - m| > 1 \) for fixed \( 0 < \delta < 1 \). Then
\[ K_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi n^2}} e^{-\gamma n m_n} \left[ 1 + O(1) \right]. \]

**Proof.**

The Mean Value Theorem for real valued functions yields

\[ |m_n - m| = |\psi_n'(\tau_n) - \psi_n'\xi_n| = |\tau_n - \xi_n| |\psi_n''(\lambda_n)| \geq |\tau_n - \xi_n| \alpha, \]

where \( \lambda_n \in I_1 \) for each \( n \geq 1 \). Since \( \alpha > 0 \) then by taking limits as \( n \to \infty \), we get \( \lim_{n \to \infty} |\tau_n - \xi_n| = 0 \). Also (2.25) shows that

\[ |\tau_n - \xi_n| = O(|m_n - m|). \]

By Condition (B), (2.6) and the MVT we get

\[ \left| \frac{\psi_n'(\xi_n) - \psi_n'(\tau_n)}{\psi_n'(\tau_n)} \right| \leq \frac{|\xi_n - \tau_n| \psi_n'''(\delta_n)}{2 \psi_n'(\tau_n) \psi_n'(\delta_n)} \]

\[ \leq |\xi_n - \tau_n| \frac{3\delta^3}{(a-a_1)^3 \alpha} \leq O(|m_n - m|), \]

where \( \delta_n \in I_1 \), so that

\[ |m_n - m| = O(|m_n - m|). \]

From Theorem 2.1 we obtain

\[ k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi n^2}} e^{-\gamma n m_n} \left[ 1 + O\left(\frac{1}{n}\right) \right]. \]
\[
= \frac{\sqrt{n}}{\sqrt{2\pi \sqrt{\psi''(\xi_n)}}} e^{-n \gamma_n(m_n)} \left[ 1 + \frac{\sqrt{\psi''(\xi_n)} - \sqrt{\psi''(\tau_n)}}{\sqrt{\psi''(\tau_n)}} \right] [1 + O\left(\frac{1}{n}\right)]
\]

\[
= \frac{\sqrt{n}}{\sqrt{2\pi \sqrt{\psi''(\xi_n)}}} e^{-n \gamma_n(m_n)} [1 + O(|m_n - m|)] [1 + O\left(\frac{1}{n}\right)]
\]

\[
= \frac{\sqrt{n}}{\sqrt{2\pi \sqrt{\psi''(\xi_n)}}} e^{-n \gamma_n(m_n)} [1 + O(|m_n - m|)],
\]

since \( n^\delta |m_n - m| > 1 \).

**Remark 2.13.** Theorem 2.3.1 is still true if Condition (C) is replaced by the weaker condition below:

(C1). There exists \( \eta > 0 \) and \( 0 < k \leq 1 \) such that for any \( 0 < \delta < \eta \),

\[
\inf_{|t| \geq \delta} \operatorname{Real}(G_n(t)) \geq k \min[\operatorname{Real}(G_n(\delta)), \operatorname{Real}(G_n(-\delta))] \forall n \geq 1.
\]

Condition (C) was used mainly in the inequalities in relation (2.15). It can be easily checked that Condition (C1) will provide a similar inequality.

**Remark 2.14.** We can omit Condition (C) in Theorem 2.1 and obtain the same conclusion (2.1) if there exists functions \( H_n(t) \) satisfying the following two properties.

(i) \( nH_n(zn^{-\lambda}) \to \infty \) as \( n \to \infty \), for some \( 1/3 < \lambda < 1/2 \).
(ii) There exists $\eta > 0$ such that for $0 < \delta < \eta$

$$\inf_{|t| \geq \delta} \text{Real}(G_n(t)) \geq H_n(\delta) \quad \forall \ n \geq 1.$$

The only modification in the proof is in the inequalities (2.15) where we will use (i) and (ii). In Example 3.4 we will choose

$$H_n(t) = n^{-(1-\lambda_1)} \text{Real}(G_{\lambda_1}^{-1}(t))$$

where $\lambda_1$ is such that $2\lambda < \lambda_1 < 1$, in which case (i) is satisfied trivially by Lemma 2.10.

**Proof of Theorem 2.2.**

The proof of this theorem parallels the proof of Theorem 2.1 and the only major change is that the range of integration is from $-\pi/h_n$ to $\pi/h_n$ instead of the whole real line.

Let $P_n(k) = \Pr(T_n = a_n + kh_n)$. Then by definition,

$$\phi_n(z) = \sum_{k=-\infty}^{\infty} P_n(k) e^{z(a_n + kh_n)}.$$

(2.28)

Multiplying both sides by $e^{-z(a_n + k h_n)}$ and integrating from $\tau_n - i\pi/h_n$ to $\tau_n + i\pi/h_n$ along the imaginary axis, we obtain

$$P_n(k) = \frac{1}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \frac{e^{-(\tau_n+i\tau)(a_n + k h_n)}}{\phi_n(\tau_n + it)} \ dt.$$
\[ (2.30) \quad \frac{\sqrt{n}}{h_n} P_n(k_n) = \frac{\sqrt{n}}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \phi_n(t) \phi_n(t+it)e^{-(t+it)n} dt \]

where

\[ I_n = \frac{\sqrt{n}/\phi_n(t_n)}{\sqrt{2\pi}} \frac{\pi/h_n}{-\pi/h_n} \int \phi_n(t+it)e^{-(t+it)n} dt \]

and

\[ G_n(t) = [\psi_n(t_n) + itm_n - \psi_n(t_n + it)]. \]

Imitating the proof of Theorem 2.1 we can show that

\[ (2.31) \quad I_n = 1 + O(\frac{1}{n}). \]

We thus obtain

\[ (2.32) \quad \sqrt{n} Pr(T_n=m_n) = \frac{\sqrt{n}}{h_n} P_n(k_n) = \frac{1}{\sqrt{2\pi/\phi_n(t_n)}} e^{-\gamma_n(m_n)} [1 + O(\frac{1}{n})]. \]

Remark 2.15. Let \( m \) be a real number such that there exists \( \varepsilon_n \in I_1 \) and \( \psi_n(\varepsilon_n) = m \). Under the further assumptions that \( m_n \to m \) and \( n^\delta|m_n-m| > 1 \) for some \( 0 < \delta < 1 \) we can show that
\[ (2.33) \quad \frac{\sqrt{n}}{|h_n|} \Pr(\frac{T_n}{n} = m_n) = \frac{1}{\sqrt{2\pi} \sqrt{\psi''(\xi_n)}} e^{-\gamma_n(m_n)} [1 + o(\sqrt{m_n - m})] \]

**Remark 2.16.** Condition (D') of Theorem 2.2 can be replaced by a stronger but more easily verifiable Condition (D1):

(D1). There exists \( p > 0 \) such that \( 1/|h_n| = O(n^p) \).

3. **Applications.**

In this section we present some examples to illustrate the theorems proved in Section 2.

**Example 3.1.**

Let \( T_n \) have a Binomial distribution with parameters \( n \) and \( p \). Then \( T_n \) is a lattice random variable with \( a_n = 0 \) and \( h_n = 1 \). The c.f. \( \phi_n(z) \) of \( T_n \) is given by \( (pe^z + q)^n \) and hence \( \psi_n(z) = \log (pe^z + q) \). Here \( q = (1-p) \). Let \( \varepsilon \) be a positive number less than \( p \). Let \( m_n = (p - \varepsilon) \) for all \( n \geq 1 \). We now verify all the conditions of Theorem 2.2.

**Condition (A).** For \( a = \log[(p-\varepsilon)q/p(q+\varepsilon)] \) and \( \beta = pe^{3|a|} + q \) we have \(|z| < :|a| \) implies \(|\psi_n(z)| < :\beta|z| \).

**Condition (B).** Let \( a_1 = |a|/2 \), then \( \tau_n = a/3 \) satisfies the following:

\[ |\psi_n(\tau_n) = m_n, \psi''(\tau_n) = (p-\varepsilon)(q+\varepsilon)^2 > 0. \]

**Condition (C).** Note that

\[ \frac{1}{n} \log |\phi_n(\tau + it)| = \log|pe^{\tau + it}| + q = \frac{1}{2} \log[(pe^\tau)^2 + q^2 + 2pq \cos t], \]
which is decreasing with $|t|$ in the interval $(-\pi, \pi)$ and hence
Condition (C) is satisfied.
Condition (D1) is trivially satisfied since $h_n = 1$. A little calculation shows that

$$
\gamma_n \left( \frac{m}{n} \right) = [m \tau_n - \psi_n (\tau_n)] = \log \left[ \left( \frac{p-\varepsilon}{p} \right)^{p-\varepsilon} \left( \frac{q+\varepsilon}{q} \right)^{q+\varepsilon} \right].
$$

Substituting the values of $\gamma_n \left( \frac{m}{n} \right)$ and $\psi_n (\tau_n)$ in (2.28) we obtain

$$
\text{Pr} \left( \frac{T_n}{n} = (p-\varepsilon) \right) = \frac{1}{\sqrt{2\pi n}} \frac{p^n(p-\varepsilon)^{\frac{q}{m}}n(q+\varepsilon)}{(p-\varepsilon)^{n(p-\varepsilon)+\frac{q}{m}(q+\varepsilon)n(q+\varepsilon)+\frac{1}{n}} [1+O\left( \frac{1}{n} \right)].
$$

The above expression can also be obtained using Stirling's formula.

In this example $T_n$ is the sum of $n$ i.i.d. Bernoulli random variables; however, Richter's theorem for lattice valued random variables is not applicable since $m_n = (p-\varepsilon)$ does not converge to $E(T_n)/n = p$.

Example 3.2.

Let $X_1, X_2, \ldots$ be a sequence of independent normal random variables with mean $\mu$ and variance $\sigma^2$. Let $T_n = \frac{n}{n-1} \sum_{i=1}^{n-1} (X_i - \bar{X})^2$ for $n \geq 2$. Here $T_n$ is of the form $\frac{\sigma^2}{n-1} \sum_{i=1}^{n-1} Y_{ni}$, where $Y_{ni}$'s are i.i.d. chi-squared with 1 degree of freedom. The c.f. of $T_n$ is given by

$$
\phi_n(z) = \left[ 1 - \frac{2\sigma^2}{n-1} z \right]^{n-1} \text{ for } |z| < \frac{2\sigma^2}{2n-1}
$$
Hence

\[ \psi(z) = \frac{n-1}{2n} \log \left[ 1 - \frac{2n\sigma^2}{n-1} z \right]. \]

Let \( m_n \) be a sequence of real numbers such that \( m_n + \sigma^2 \) and \( n^\delta |m_n - \sigma^2| > 1 \). We now verify conditions (A)-(D) of the Theorem 2.1.

**Condition (A).** Take \( a = 1/4\sigma^2 \), \( \beta = \log[\sqrt{5}/2] \). It is easy to check that \( |z| < a \) implies \( |\psi_n(z)| < \beta \).

**Condition (B).** Note that for \( \tau \) real, \( \psi'_n(\tau) = \frac{(n-1)\sigma^2}{(n-1-2n\sigma^2\tau)} \). Solving \( \psi'_n(\tau_n) = m_n \) for \( \tau_n \) we get

\[ \tau_n = \frac{n-1}{2n} \left[ \frac{1}{\sigma^2} - \frac{1}{m_n} \right], \tag{3.3} \]

which is less than \( a_1 = 1/8\sigma^2 \) for \( n \) sufficiently large. A simple computation shows that

\[ \psi'_n(\tau_n) = \frac{2n(n-1)\sigma^2}{[n-1-2n\sigma^2\tau_n]^2} = \frac{2nm_n^2}{(n-1)^2} > \sigma^2, \tag{3.4} \]

**Condition (C).** We get from (3.2)

\[ \text{Real}(\psi_n(\tau+it)) = \frac{1}{n} \log |\phi_n(\tau+it)| = \frac{(n-1)}{4n} \log \left[ 1 - \frac{2n\sigma^2\tau^2}{n-1} + \left( \frac{2n\sigma^2}{n-1} \right)^2 t^2 \right]. \]
This is clearly a decreasing function of $|t|$ on the whole real line
$orall n \geq 1, \forall \tau$. Thus Condition (C) is verified.

**Condition (D).** Take $\xi = 5, p = 1$. Consider

\[
(3.5) \quad \frac{\phi_n(\tau + it)}{\phi_n(\tau)} = \frac{1 - \frac{2n\sigma^2}{n-1} (\tau + it) \frac{5}{2} \frac{1 - \frac{1}{n}}{n}}{\left(1 - \frac{2n\sigma^2}{n-1}\right) \frac{5}{2} \frac{1 - \frac{1}{n}}{n}}
\]

\[
= \left[1 + \left(1 - \frac{2n\sigma^2}{n-1}\right) \frac{-2 \left(\frac{2n\sigma^2}{n-1} \tau\right)}{}\right] \frac{5}{4} \left(1 - \frac{1}{n}\right)
\]

\[
\leq \left[1 + \left(1 - \frac{2n\sigma^2}{n-1}\right) \frac{-2 \left(\frac{2n\sigma^2}{n-1} \tau\right)}{}\right] \frac{5}{8}
\quad \forall n \geq 2.
\]

Thus,

\[
(3.6) \quad \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right| \frac{5}{n} \, dt \leq \int_{-\infty}^{\infty} \left[1 + \left(1 - \frac{2n\sigma^2}{n-1}\right) \frac{-2 \left(\frac{2n\sigma^2}{n-1} \tau\right)}{}\right] \frac{5}{8} \, dt = O(n).
\]

Since $E(T_n)/n = \sigma^2, \xi_n = 0$ satisfies the equality $\psi_n'(\xi_n) = \sigma^2$

for $n \geq 1$. A little calculation shows that $\psi_n''(\xi_n) = 2n\sigma^2/(n-1)$ and

\[
\gamma_n(m) = \frac{n-1}{2n} \left[\frac{m^2 - \sigma^2}{\sigma^2} - \log \left(\frac{\sigma^2}{m^2 n^2}\right)\right].
\]

Thus all the conditions of the Corollary 2.12 are satisfied. Let $k_n$ denote the p.d.f. of $T_n/n$,

then by Corollary 2.12 we obtain
$$k_n(m_n) = \frac{\sqrt{n}}{\sqrt{2\pi}\sqrt{\psi_n'(\xi_n)}} \exp^{-n\gamma_n(m_n)} [1+O(\{m_n-\sigma^2\})].$$

Substituting the values of $\psi_n'(\xi_n)$ and $\gamma_n(m_n)$ and simplifying the above expression we get the following asymptotic expansion for $k_n$.

$$k_n(m_n) = c_n m_n^\frac{(n-1)}{2} e^{-\frac{(n-1)m_n}{2\sigma^2}} [1+O(\{m_n-\sigma^2\})],$$

(3.7)

where

$$c_n = \frac{(n-1)}{2n-1} e^{\frac{1}{2n-1} \sigma^2 n + 1}.$$

This agrees with the exact expression for $k_n$ except for the normalizing constant. ||

Example 3.3. Wilcoxon Signed-Rank Test.

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common absolutely continuous d.f. $F$, which is symmetric about the median $m$. Arrange $|X_1|, |X_2|, \ldots, |X_n|$ in increasing order of magnitude and assign ranks 1, 2, $\ldots, n$.

Let

$$z_i = \begin{cases} 
1 & \text{if the } X_j \text{ having rank } i \text{ is positive}.
\end{cases}$$

$$= \begin{cases} 
0 & \text{otherwise}.
\end{cases}$$

Define

$$W_n = \text{sum of the ranks of positive } X_i's$$

$$= \sum_{i=1}^{n} i z_i.$$
A is non singular. The class of all linear $\xi$-unbiased predictors of $\lambda' y$ is denoted by $U_{\lambda}$. Finally we write

$$B = [X' I_S V^{-1} I_S X]^{-1} X' I_S V^{-1} I_S,$$

THEOREM 1 - Among all elements of $U_{\lambda}$, $h'I_S y$ has the property that $E_{\xi} [h'I_S y - \lambda' y]^2 = \min_{g \in U_{\lambda}} E_{\xi} [g(y) - \lambda' y]^2 \text{ if}$

$$h'I_S y = \lambda'I_S y + \lambda'(I - I_S) \hat{\beta},$$

(1)

where $\hat{\beta} = B I_S y = By$.

This result may be proved by recalling that $I_S y = (I_S X) \beta + I_S e$, by taking $(I_S V)^{-1} = I_S V^{-1}$, and by using the results of the general theory of Linear Models [see for example Rao (1971) pg. 301].

Note that the predictor $h'I_S y$ satisfying equation (1) relates itself to the $\xi$-model only through the matrix $X$ and the function $f$. In that way, we write $T(X, f)$ for the linear function $h'I_S y$ that satisfies the conditions of Theorem 1; that is, $T(X, f)$ is a linear $\xi$-unbiased predictor of minimum $\xi$-mean square error.

III - ROBUST LINEAR PREDICTION

In this section general results of the theory of Linear Models are used to obtain a characterization of Robustness in Linear Prediction. We start by restating the Robustness problem.

Let us denote by $X^*$ a matrix of order $N \times (M+J)$ where the first $M$ columns are the columns of $X$ and the last $J$ columns are $Z_1, \ldots, Z_J$ which form a matrix $Z$ of order $(N \times J)$. As usual we write $X^* = [X, Z]$ to indicate this construction. Analogously we write

$$\beta^* = \begin{bmatrix} \beta \\ Y \end{bmatrix}$$
to indicate that the first $M$ components of $\beta^*$ are those of $\beta$ and the last $J$ are the components of another column vector $\gamma$. In order to state the robustness problem we consider an alternative $\zeta^*$-model which is defined as the $\zeta$-model having $X^*$ (known) replacing $X, \beta^*$ (unknown) replacing $\beta$, and a function $f^*$ (known) replacing $f$. It is clear that (i) $f^*$ is a function of $M+J$ arguments [it is evaluate in each row $(X^*_k)$ of $X^*$]; (ii) $V$ must be replaced by $V^*$, a diagonal matrix whose $k$-th $(k=1, \ldots, N)$ diagonal element is $f^*(X^*_k)$; and (iii) $e$ must be replaced by $e^*$, another column random vector of $N$ elements. By recalling the "$\zeta$-best" linear predictor $T(X,f)$ defined at the end of Section II and using the above notation we introduce the following definition which describes the problem of Robustness in our context.

**DEFINITION 2** - The $\zeta$-best predictor $T(X,f)$ of $\lambda'y$ is said to be Robust for the $\zeta^*$-model if it is $\zeta^*$-best; that is, $T(X,f) = T(X^*,f^*)$.

It is clear that if $T(X,f)$ satisfies Definition 2, then besides being $\zeta$-unbiased it is $\zeta^*$-unbiased. The following result introduces a necessary and sufficient condition to have both kinds of unbiasedness for $T(X,f)$.

**THEOREM 2** - The linear predictor $T(X,f)$ is $\zeta^*$-unbiased if and only if the following equation holds:

\[(2) \quad \lambda'((I-I_S)X_{\beta Z} = \lambda'(I-I_S)Z)\]

**PROOF** - By the definition of $T(X,f)$ we have

$T(X,f) - \lambda'y = \lambda'(I-I_S)\hat{\beta} - \lambda'(I-I_S)y$

where $\hat{\beta}$ is defined in Theorem 1. That is,
The statistic $W_n$ is known as the Wilcoxon statistic and is used to test the hypothesis

$$H_0: m = 0 \text{ vs } H_1: m \neq 0.$$ 

Let $T_n = W_n/n$. The c.f. of $T_n$, $\phi_n(z)$, under the null hypothesis is given by

$$\phi_n(z) = \sum_{k=1}^{n} \left( e^{kz/n} + 1 \right)/2, \quad (3.8)$$

and

$$\psi_n(z) = \frac{1}{n} \sum_{k=1}^{n} \log \left[ \left( e^{kz/n} + 1 \right)/2 \right], \quad z \in \mathbb{C}. \quad (3.9)$$

Here $\{T_n, n \geq 1\}$ is a sequence of lattice random variables with $a_n = 0, b_n = 1/n$. The range of $\psi'(\tau)$ for $\tau \in \mathbb{R}$ contains the open interval $(0, 1/2)$, for $n \geq 1$.

Thus if $\{m_n, n \geq 1\}$ is a sequence of real numbers contained in a proper subinterval of $(0, 1/2)$, we can find a constant $a_1 > 0$ and $\tau_n \in (-a_1, a_1)$ such that $\psi'(\tau_n) = m_n$ $\forall n \geq 1$. Let us now check all the conditions of Theorem 2.2.

Condition (A). Let $a = 2a_1$, then it is easy to check that there exists $\beta > 0$ such that $|z|$ implies $|\psi_n(z)| < \beta$.

Condition (B). The existence of $\{\tau_n, n \geq 1\}$ is already discussed above. Straightforward calculations show that $\psi''(\tau)$ is bounded below by a positive number for $|\tau| < a$. 
**Condition (C').** Let

\[(3.10) \quad f_n(t) = \text{Real}(\psi_n(\tau) - \psi_n(\tau + it))\]

\[= -\frac{1}{2n} \sum_{k=1}^{n} \log \left[ 1 - \frac{2(1 - \cos kt/n)}{(e^{kt/2n} + e^{-kt/2n})^2} \right].\]

Note that \(f_n(0) = 0 \quad \forall \quad n \geq 1\). Condition (C') is verified by showing that \(f_n\) satisfies the three assumptions of Lemma 2.7 \(\forall \quad n \geq n_0\).

(i) Take \(\eta_1 = \pi/2\). Since cosine is decreasing with \(|t|\) in the interval \((-\eta_1, \eta_1)\), \(f_n(t)\) is decreasing with \(|t|\) in that interval \(\forall \quad n \geq 1\).

(ii) Since \(1/h_n = n\), all we need to show is that there exist \(\varepsilon > 0\) and \(n_0 \geq 1\) such that

\[(3.11) \quad \inf_{\pi/2 \leq |t| \leq n\pi} f_n(t) > \varepsilon \quad \forall \quad n \geq n_0.\]

From (3.10) we have

\[(3.12) \quad -f_n(t) = \frac{1}{2n} \sum_{k=1}^{n} \log \left[ 1 - \frac{2(1 - \cos kt/n)}{(e^{kt/2n} + e^{-kt/2n})^2} \right]

\leq \frac{1}{n} \sum_{k=1}^{n} \left[ \frac{(1 - \cos kt/n)}{(e^{kt/2n} + e^{-kt/2n})^2} \right]

\leq \frac{1}{n} \sum_{k=1}^{n} \left[ \frac{(1 - \cos kt/n)}{4e^{a}} \right] \quad [\text{since } |\tau| < a]
\[
\leq \frac{1}{4e^a} + \frac{1}{4ne^n} \sum_{k=1}^{n} \cos \frac{kt}{n}
= \frac{1}{4e^a} + \frac{1}{4e^n} \left[ \frac{\sin(n+1)t/n + \sin t}{2n \sin t/n} - \frac{1}{2n} \right].
\]

The last equality follows from the relation

\[
\cos kx = \frac{\sin(k+1)x - \sin(k-1)x}{2 \sin x}.
\]

Substituting \( t = ns \) in (3.12) we can see that (3.11) is verified once we show that there exists \( \epsilon_1 > 0 \) and \( n_0 \geq 1 \) such that

(3.13) \[ \sup_{\pi/2n \leq |s| \leq \pi} \frac{\sin(n+1)s + \sin ns}{2n \sin s} < 1 - \epsilon_1 \quad \forall \ n \geq n_0. \]

Take \( \epsilon_1 = 5/33 \). There exists \( n_1 \) such that

\[ n \sin \pi/2n > 3\pi/8 \quad \forall \ n \geq n_1. \]

Thus for \( n \geq n_1 \)

(3.14) \[ \sup_{\pi/2n \leq |s| \leq \pi - \pi/2n} \left| \frac{\sin(n+1)s + \sin ns}{2n \sin s} \right| \leq \sup_{\pi/2n \leq |s| \leq \pi - \pi/2n} \left| \frac{1}{n \sin s} \right| \leq \frac{1}{n \sin \pi/2n} \leq \frac{8/3\pi}{2} = 1 - \epsilon_1. \]

Now,

(3.15) \[ \sup_{\pi - \pi/2n \leq |s| \leq \pi} \left| \frac{\sin(n+1)s + \sin ns}{2n \sin s} \right| = \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\sin(n+1)(\pi - \theta) + \sin n(\pi - \theta)}{2n \sin(\pi - \theta)}. \right| \]
\[
= \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\sin((n+1)\theta) - \sin n\theta}{2n \sin \theta} \right|
\]

\[
= \sup_{0 \leq |\theta| \leq \pi/2n} \left| \frac{\cos((2n+1)\theta/2) \sin \theta/2}{n \sin \theta} \right|,
\]

which goes to 0 as \( n \to \infty \). Thus there exists \( n_2 \) such that \( n \geq n_2 \)
implies the r.h.s. of (3.15) is less than \( \epsilon_1 \). Then \( n_0 = \max(n_1, n_2) \)
satisfies (3.13) and the proof is complete.

(iii) Let \( \epsilon \) be as in (ii). We want to show that there exists
\( n < \pi/2 \) such that \( \forall \ n \geq 1 \)

(3.16) \[
\sup_{0 \leq |t| \leq n} f_n(t) < \epsilon.
\]

Since \( \log(1-x) > -x/1-x \) for \( 0 < x < 1 \), we get

(3.17) \[
-f_n(t) = \frac{1}{2n} \sum_{k=1}^{n} \log \left[ 1 - \frac{2(1-\cos k\tau/n)}{(k\tau/2n+e^{-k\tau/2n})^2} \right]
\]

\[
\geq -\frac{1}{2n} \sum_{k=1}^{n} \frac{2(1-\cos k\tau/n)}{(e^{k\tau/2n}+e^{-k\tau/2n})^2(1-\cos k\tau/n)}
\]

\[
\geq -\frac{1}{2n} \sum_{k=1}^{n} \frac{(1-\cos k\tau/n)}{(1+\cos k\tau/n)} \ [\text{since } (e^x + e^{-x}) \geq 2].
\]

Let \( \delta = 4\epsilon/1+2\epsilon \). Choose \( n \) such that \( \cos n > 1-\delta \), then it follows
from (3.17) for \( |t| < \pi/n \)

\[
f_n(t) \leq \frac{1}{2n} \sum_{k=1}^{n} \frac{(1-\cos k\tau/n)}{(1+\cos k\tau/n)}
\]

\[
\leq \frac{\delta}{2(2-\delta)} = \epsilon.
\]
Thus,

$$\sup_{|t| \leq n} f_n(t) < \epsilon \quad \forall n \geq 1.$$ 

Condition (D1). Since $1/h_n = n$, this condition is trivially satisfied for $p = 1$.

Thus from Theorem 2.2 we can get an asymptotic expression for $\Pr(T_n/n = m_n)$. In this example $T_n$ is the sum of $n$ independent but not identically distributed random variables. ||

Example 3.4. Kendall's Distribution-Free Test for Independence.

Let $(X_1, Y_1), (X_n, Y_n), \ldots, (X_n, Y_n)$ have a common bivariate distribution with continuous c.d.f. $F(x, y)$, and marginals $G(x)$ and $H(y)$. We wish to test the hypothesis $H_0: F(x, y) = G(x)H(y)$ for all $x, y$.

Define

$$sgn(x) = 1 \text{ if } x > 0$$

$$= -1 \text{ if } x < 0.$$ 

Let $V_{ij} = sgn(X_i - X_j)sgn(Y_i - Y_j), 1 \leq i < j \leq n$.

Let $Q_n = \sum_{1 \leq i < j \leq n} (1-V_{ij})/2$ and $W_n = 1 - 4Q_n/n(n-1)$.

Suppose the ranks of $Y$'s are arranged in the natural order 1, 2, ..., $n$ and let the corresponding ranks of $X$'s be $x_1, x_2, \ldots, x_n$. $Q_n$ measures the extent of departure of the $x$'s from the natural order (1, 2, 3, ..., $n$) by counting the number of inversions among them.
The statistic $W_n$ was proposed by Kendall (1938) as a nonparametric test statistic for testing the null hypothesis $H_0$.

Let $T_n = nW_n = n - 4Q_n/(n-1)$.

It is clear that $T_n$ is a lattice random variable with $a_n = n$ and $h_n = 4/(n-1)$. The c.f. of $T_n$ under the assumption of independence of $X$ and $Y$ is given by (Kendall and Stuart (1979), pp. 505-506),

\[
\phi_n(z) = \frac{e^{nz}}{n!} \prod_{k=1}^{\lfloor z \rfloor} \left[ \frac{e^{-4kz/n-1}}{e^{-4z/n-1}} \right]
\]

\[
(3.18)
\]

\[
(3.19) \psi_n(z) = z - \frac{1}{n} \log n! + \frac{1}{n} \sum_{k=1}^{\lfloor z \rfloor} \log \left[ \frac{1-e^{-4kz/n-1}}{1-e^{-4z/n-1}} \right]
\]

\[- \log[n(1-e^{-4z/n-1})].\]

The range of $\psi_n(\tau)$ for $\tau \in \mathbb{R}$ is the interval $(-1, 1)$ and the random variable $T_n/n$ takes values in this interval. Thus if 

$\{\alpha_n, n \geq 1\}$ is a sequence of real numbers contained in $(-M, M)$,

$0 < M < 1$, then for sufficiently large $\alpha_1$, we can find $\{\tau_n, n \geq 1\}$ such that $\psi_n(\tau_n) = \alpha_n$ with $|\tau_n| < \alpha_1$. For simplicity let us choose $\alpha_n = 0$ then $\tau_n = 0 \ \forall \ n \geq 1$ and verify the conditions of the Theorem 2.2. The verification for general sequence $\alpha_n$ is similar but tedious.
Condition (A). Let \( a = 2a_1 \). It follows from (3.19) that

\[ |z| < a \text{ implies } |\psi_0(z)| < 2[1+a+\log[1+ e^{4a}]+\log(4a)] = \beta. \]

Condition (B). By Remark 2.11 we only need to verify that \( \psi''(\tau_n) \geq a \) \( \forall \ n \geq 1 \). Let \( a = 2/9 \) then since \( \tau_n = 0 \) we have

\[ \psi''(0) = \text{Var}(T_n)/n = 2(2n+5)/(n-1) \geq a \ \forall \ n \geq 2. \]

Condition (C). As was mentioned in Remark 2.14, we set

\[ H_n(t) = n^{-(1-\lambda_1)} \text{Re}(G_{\lambda_1}(t)), 2/3 < \lambda_1 < 1 \text{ and verify (ii) of } \]

Remark 2.14. Note that since \( \tau_n = 0 \),

\[ (3.20) \quad \text{Re}(G_n(t)) = \frac{1}{n} \log|\phi_n(it)| \]

\[ = \frac{1}{2n} \sum_{k=1}^{n} \log \left[ \frac{1-\cos(4kt/(n-1))}{k(1-\cos(4t/(n-1)))} \right] \]

\[ = \frac{1}{4n} \sum_{k=1}^{n} \log \left[ \frac{\sin(2kt/(n-1))}{k \sin(2t/(n-1))} \right] \]

let \( n \) be a small positive number and \( \delta > 0 \) be less than \( n \). Since \( T_n \) is a lattice random variable with span \( h_n = 4/(n-1) \) to verify (ii) we need to show that for some \( n_0 \geq 1 \),

\[ (3.21) \quad \inf_{\delta \leq |t| \leq (n-1)\pi/4} \text{Re}(G_n(t)) \geq H_n(\pm \delta) \ \forall \ n \geq n_0. \]

Let \( g_n(t) = \left| \frac{\sin kt}{k \sin t} \right|^{1/n} \). Since \( \text{Re}(G_n(t)) = -k \log g_n(2t/(n-1)) \)

if suffices to show that
\[(3.22) \quad \sup_{\delta/n \leq |t| \leq \pi/2} g_n(t) \leq \left[ g_{\frac{\lambda_1}{\lambda}}(\pm\delta/n) \right]^{1/n^{1-\lambda_1}} \quad \forall \; n \geq n_0.\]

We first show that supremum of \( g_n(t) \) in the interval \([\delta/n^{\lambda_1}, \pi/2]\) goes to 0 as \( n \to \infty \). By the arithmetic-geometric inequality we get

\[
\sup_{\delta/n^{\lambda_1} \leq |t| \leq \pi/2} g_n(t) \leq \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k \sin(\delta/n^{\lambda_1})}
\]

\[
\leq \frac{2}{\delta n^{1-\lambda_1}} \sum_{k=1}^{n} \frac{1}{k} = \frac{2 \log n}{\delta n^{1-\lambda_1}}
\]

\(+\) 0 as \( n \to \infty \) since \( \lambda_1 < 1 \).

Since \( g_{\frac{\lambda_1}{\lambda}}(\pm\delta/n) \) is converging to 1 as \( n \to \infty \), we can find \( n_0 \) such that \( \forall \; n \geq n_0 \)

\[
\sup_{\delta/n^{\lambda_1} \leq |t| \leq \pi/2} g_n(t) \leq \left[ g_{\frac{\lambda_1}{\lambda}}(\pm\delta/n) \right]^{1/n^{1-\lambda_1}}.
\]

For \( n \geq n_0 \)

\[
\sup_{\delta/n \leq |t| < \delta/n^{\lambda_1}} g_n(t) = \sup_{\delta/n \leq |t| < \delta/n^{\lambda_1}} \left( \frac{\sin kt}{k \sin t} \right)^{1/n}
\]

\[
\leq \sup_{\delta/n \leq |t| < \delta/n^{\lambda_1}} \left( \frac{\lambda_1^{\lambda_1} \sin kt}{k \sin t} \right)^{1/n}
\]
\[ \lambda_1^n \prod_{k=1}^{n} \left| \frac{\sin(k\delta/n)}{k \sin(\delta/n)} \right|^{1/n} \]

\[ = [g_{\lambda_1}^{(\pm\delta/n)}]^{1/n^{1-\lambda_1}}. \]

This verifies (3.22) and hence (3.21).

Condition (D1). Since \( h_n = 4/(n-1) \), clearly \( 1/h_n = O(n^p) \) for \( p = 1 \).

Thus we have verified all the conditions of Theorem 2.2.

Since \( E(T_n') = 0 = m_n \), by (2.32) we get

(3.23) \[ \Pr(T_n'/n = 0) = \frac{6}{\sqrt{n}(2n+5)(n-1)} \left[ 1 + O(1/n) \right] \text{ as } n \to \infty \]

through the subsequence of \( \{n\} \) for which 0 is in the range of \( T_n'/n \). ||
REFERENCES


The results of W. Richter (Theory Prob. Appl. (1957) 2 206-219) on sums of independent, identically distributed random variables are generalized to arbitrary sequences of random variables $T_n$. Under simple conditions on the cumulant generating function of $T_n$, which imply that $T_n/n$ converges to 0, it is shown, for arbitrary sequences $\{m_n\}$ converging to 0, that $k_n(m_n)$, the probability density function of $T_n/n$ at $m_n$, is asymptotic to an expression involving the large deviation rate of $T_n/n$. Analogous results for lattice random variables are also given. Applications of these results to statistics appearing in nonparametric inference are presented.