The Gauss-Tchebyshev Inequality
for Unimodal Distributions

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Key Words and Phrases. Gauss' inequality, Tchebyshev's inequality,
Markov's inequality, unimodal distributions, convex structure.

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Summary

Let $X$ be a random variable whose distribution is unimodal with mean $\mu$. For $r > 0$, let $\lambda_r = \{E|X - \mu|^r\}^{1/r}$. In this paper, we determine a value $k_r$ such that

$$P(|X - \mu| \geq k\lambda_r) \leq \left[\frac{r}{(r + 1)}\right]^r k^{-r},$$

for all $k \geq k_r$. This improves and extends a recent result of Vysochanskii and Petunin (1979) who have only considered the case $r = 2$ with a higher value for $k_2$. Our proof is also considerably simpler because it uses the convex structure of the class of unimodal distributions.

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1. Introduction.

Let $X$ be a real random variable with mean $\mu$ and let $r > 0$. Markov's inequality states that, for every given $a$ and every $k > 0$,

$$P(|X - a| \geq k) \leq \frac{E(|X - a|^r)}{k^r}.$$  

(1.1)

If $a = \mu$ and $r = 2$, (1.1) reduces to the usual Tchebyshev inequality. Suppose now that the distribution of $X$ is unimodal with a mode $M$. A result attributed to Gauss (1821) states that

$$P(|X - M| \geq k) \leq \frac{(4/9) E(|X - M|^2)}{k^2},$$

(1.2)

for all $k > 0$. In other words, if $a = M$, the bound on the right side of (1.1) can be reduced by a factor $(4/9)$ when $r = 2$. As a consequence, if the distribution of $X$ is both symmetric and unimodal, then $M = \mu$ and (1.2) gives

$$P(|X - \mu| \geq k) \leq \frac{4\sigma^2}{(9k^2)},$$

(1.3)

where $\sigma^2 = \text{Var}(X)$. Recently, Vysochanskii and Petunin (1979) showed that (1.3) is valid without the assumption of symmetry as long as $k \geq \sqrt{8/3}$. In this paper, we first obtain the factor by which the bound in (1.1) can be improved if the distribution is unimodal and $a = M$. We then show that the improved bound is valid even if $a = \mu$ as long as $k$ is suitably large. For $r = 2$, we need $k \geq \sqrt{19/3}$, which is better than the value $\sqrt{8/3}$ obtained by Vysochanskii and Petunin.

2. Preliminaries.

In this section we give some results on certain convex sets of distributions.

**DEFINITION 2.1.** A distribution function $F$ is said to be **unimodal** about a mode $M$ if $F$ is convex on $(-\infty, M)$ and concave on $(M, \infty)$. 
Let $CM$ denote the set of all distributions on $R$ that are unimodal about $M$. Then $CM$ is clearly convex (under mixtures). It is also closed under weak convergence; see Gnedenko and Kolmogorev (1968), Section 32. Let $UM$ denote the set of all uniform distributions on intervals with $M$ as one end point. Then $CM$ is the closed convex hull of $UM$. Another equivalent statement of this result is as follows; [see Feller (1971), p. 158].

**THEOREM 2.1.** A random variable $X$ has a unimodal distribution with mode $M$ if, and only if, $X$ is distributed as $M + UZ$, where $U$ is uniform on $(0, 1)$ and $U, Z$ are independent.

This theorem enables one to reduce many problems involving unimodal distributions to those involving uniform distributions.

Let $D_μ$ denote the set of all distributions on $R$ which have mean $μ$ and finite support. The following lemma is possibly known.

**LEMMA 2.1.** Every distribution in $D_μ$ is a finite convex mixture of one or two point distributions with mean $μ$.

**Proof.** Without loss of generality, let $μ = 0$. Let $P \in D_0$ and let $ν$ be the size of the support of $P$. The lemma holds if $ν ≤ 2$. Suppose the lemma holds for $ν ≤ n$, where $n ≥ 2$. Let $Y$ be a random variable with distribution $P$ and suppose $Y$ takes exactly $(n + 1)$ values. Since $Y$ is not degenerate and $E(Y) = 0$, we can find $a > 0$ such that $ξ = P(Y = -a) > 0$ and $η = P(Y = b) > 0$.

Without loss of generality, assume that $aξ ≥ bη$. Consider the two-point distribution $P_0$ which puts mass $a/(a + b)$ at the point $b$ mass $b/(a + b)$ at the point $(-a)$. Then $P_0$ has zero mean and
\begin{equation}
(2.1) \quad P = \alpha P_0 + (1 - \alpha)P_1
\end{equation}

where \( \alpha = \eta(a + b)/a \). Note that \( \alpha P_0 \) accounts for all the mass at \( b \). It is clear that \( \alpha > 0 \). On the other hand, since \( Y \) takes at least 3 values, we must have \( \xi + \eta < 1 \). Therefore

\[ \eta(a + b) = a\eta + b\eta \leq a\eta + a\xi = a(\xi + \eta) < a. \] Thus \( \alpha < 1 \).

The quantity \( P_1 \) in (2.1) is a distribution which puts positive mass at \( \leq n \) points, since the mass at \( b \) is accounted for by \( \alpha P_0 \). By the induction hypothesis, \( P_1 \) is expressible as a mixture of one or two point distributions with zero mean. Therefore, by (2.1), \( P \) also can be expressed as a mixture of the required type.

The proof of the lemma is now complete.

The following lemma is standard.

\textbf{Lemma 2.2.} Let \( r > 0 \) and let \( X \) be a real random variable with \( E(\|X\|^r) < \infty \). Then we find a sequence of random variables \( X_n \) such that each \( X_n \) takes only a finite number of values and \( E(\|X_n - X\|^r) \to 0 \). Moreover, if \( r \geq 1 \), then we can choose the \( X_n \) in such a way that \( E(X_n) = E(X) \) for all \( n \).

3. The Gauss-Tchebyshev Inequality.

The Markov inequality states that

\begin{equation}
(3.1) \quad P(\|X - a\| \geq k) \leq E(\|X - a\|^r)/k^r,
\end{equation}

where \( X \) is a real random variable, \( a \in \mathbb{R} \), \( r > 0 \) and \( k > 0 \). If \( a = E(X) \) and \( r = 2 \), (3.1) gives the usual Tchebyshev inequality. If \( X \) has a distribution which is unimodal about \( M \), then the bound on the right side of (3.1) can be reduced by a factor which depends on \( r \). This is made precise by Theorem 3.1. below. For the special case \( r = 2 \), Theorem 3.1 goes back to Gauss (1821).
THEOREM 3.1. Let \( X \) have a distribution which is unimodal about \( M \). Then for every \( r > 0 \) and every \( k > 0 \),

\[
P(|X - M| \geq k) \leq \left( \frac{r}{r + 1} \right)^r \frac{(E(|X - M|)^r)}{k^r}.
\]

Moreover, this bound is sharp.

**Proof:** Without loss of generality, let \( M = 0 \). Since (3.2) is trivially true if \( E|X|^r = \infty \), we assume that \( E|X|^r < \infty \). Since \( X \) is unimodal about zero, by Theorem 2.1, \( X \) has the same distribution as \( UZ \), where \( U \) is uniform on \((0, 1)\) and \( U, Z \) are independent. Now \( E|X|^r = E(|Z|^r)/(r + 1) \). Therefore \( E|Z|^r < \infty \). Lemma 2.2 shows that it is sufficient to establish (3.2) in the case where \( Z \) takes only a finite number of values. Now the set of distributions of \( Z \), for which (3.2) is valid, is clearly convex. Therefore we need only consider the case where \( Z \) is degenerate. Finally, (3.2) is clearly unaffected by a change of scale. Therefore we may and do assume that \( Z \) is degenerate at \( 1 \), so that \( X \) has the uniform distribution on \((0, 1)\). In this case, \( E|X|^r = 1/(r + 1) \) and

\[
P(|X| \geq k) = \begin{cases} 
(1 - k), & \text{if } 0 < k \leq 1 \\
0, & \text{if } k \geq 1.
\end{cases}
\]

Therefore

\[
k^r P(|X| \geq k) = \begin{cases} 
k^r(1 - k), & \text{if } 0 < k \leq 1 \\
0, & \text{if } k \geq 1.
\end{cases}
\]

For fixed \( r \), the last quantity becomes maximum when \( k = r/(r + 1) \). The maximum value is \( r^r/(r + 1)^{r + 1} \). Therefore

\[
k^r P(|X| \geq k) \leq \left( \frac{r}{r+1} \right)^r \cdot \frac{1}{(r+1)} = \left( \frac{r}{r+1} \right)^r E|X|^r,
\]

which proves (3.2). Further the above calculation shows that the bound is sharp.
The special case $r = 2$ gives the Gauss inequality.

COROLLARY 3.1. (Gauss). If $X$ has a distribution which is unimodal about $M$, then, for all $k > 0$,

$$P(|X - M| \geq k) \leq 4 \frac{E(|X - M|^2)}{(9k^2)}$$

COROLLARY 3.2. Let $X$ have a symmetric and unimodal distribution. Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$. Then, for all $k > 0$,

$$(3.3) \quad P(|X - \mu| \geq k\sigma) \leq \frac{4}{(9k^2)}.$$

Proof. Immediate from corollary 3.1, because $M = \mu$.

Recently, Vysochanskii and Petunin (1979) showed that (3.3) holds for unimodal random variables without the assumption of symmetry provided that $k \geq \sqrt{(8/3)}$. We improve and generalize their results below (Theorem 3.2). Our proof is also considerably simpler because we use the convex structures introduced in Section 2.

THEOREM 3.2. Let $X$ have a unimodal distribution with mean $\mu$. Let $\tau_r = E(|X - \mu|^r)$. Then, for every $k > 0$,

$$P(|X - \mu| \geq k) \leq \max \left[ \frac{(r+1)\tau_{r+2}}{r+1+k}, \left(\frac{r}{r+1}\right)^{\frac{r}{r+1}} \frac{\tau_r}{k^r} \right].$$

Proof. Without loss of generality assume that $\mu = 0$. Suppose $X$ is unimodal about $M$. If $0$ is also a mode of $X$, then the theorem follows from Theorem 3.1. So, suppose that $X$ is not unimodal about $0$. Again, we may assume that $M > 0$. By Theorem 2.1, $X$ has the same distribution as $M + UZ$, where $U$ is uniform on $(0, 1)$ and $U, Z$ are independent. Now $0 = E(X) = M + \frac{1}{2}E(Z)$.
Fig. 1. Graph of the density $f$ in the proof of Theorem 3.2.
Therefore $E(Z) = -2M$. It is clear from Lemma 2.2 that it is sufficient to prove the theorem in the case where $Z$ takes only a finite number of values. Moreover, since the mean of $X$ is fixed at 0, the class of distributions of $X$ for which the theorem holds is convex. Therefore the second assertion of Lemma 2.2 and Lemma 2.1 show that it is sufficient to prove the theorem in the case where $Z$ takes exactly two values. We have thus reduced our problem to the case where $X$ has the density $f$ given by

$$
 f(x) = \begin{cases} 
 \alpha, & \text{if } -a < x < b, \\
 \beta, & \text{if } b < x < c, \\
 0, & \text{elsewhere}
\end{cases}
$$

Here $a$, $b$, $c$ are suitable positive constants. A graph of $f$ is given in Fig. 1. Since $f$ is not to be unimodal about 0, we must have $\alpha < \beta$. Further the condition $E(X) = 0$ requires that $b < c < a$. Three cases arise.

**Case 1.** Suppose $0 < k < b$. Here $P[|X| < k] = 2ak$ and so

$$
(3.4) \quad \int_{|t|<k} |t|^r f(t) dt = \frac{2ak^{r+1}}{(r+1)} = k^r P[|X| < k].
$$

**Case 2.** Suppose $b < k < c$. Here

$$
P[|X| < k] = \alpha(b + k) + \beta(k - b),
$$

and

$$
\int_{|t|<k} |t|^r f(t) dt = \frac{\alpha(b^{r+1} + k^{r+1}) + \beta(k^{r+1} - b^{r+1})}{(r+1)}.
$$
Simple algebraic manipulations yield

\[(3.5) \quad \int_{|t| < k} |t|^\tau f(t) dt - \frac{k^\tau P(|X| < k)}{(r+1)} = \frac{b(\beta - \alpha)}{(r+1)} (k^\tau - b^\tau);\]

Since \( \alpha < \beta \) and \( 0 < b < k \), the right side of (3.5) is positive.

Consider the two cases together. That is, let \( 0 < k < c \).

Then (3.4) and (3.5) show that

\[(3.6) \quad \int_{|t| < k} |t|^\tau f(t) dt \geq \frac{k^\tau P(|X| < k)}{(r+1)}.\]

Now

\[\tau_x = E|X|^\tau = \int_{|t| \geq k} |t|^\tau f(t) dt + \int_{|t| < k} |t|^\tau f(t) dt\]

\[\geq k^\tau P(|X| \geq k) + \frac{k^\tau P(|X| < k)}{(r+1)}, \quad [\text{using (3.6)}.]\]

Writing \( P(|X| < k) = 1 - P(|X| \geq k) \), we get

\[\tau_x \geq k^\tau \left[ \frac{r^\tau}{(r+1)} \right] P(|X| \geq k) + \frac{1}{(r+1)}.\]

Therefore

\[(3.7) \quad P(|X| \geq k) \leq \frac{(r+1) \tau_x - k^\tau}{r k^\tau}\]

**Case 3.** Suppose that \( c < k \). Define a new density \( g \) as follows.

\[g(x) = \left\{ \begin{array}{ll}
\gamma , & \text{if } 0 < x < c,
\text{f(x),} & \text{elsewhere}
\end{array} \right\} \]
Since \( g \) agrees with \( f \) outside the interval \((0, c)\), the constant \( \gamma \) must satisfy

\[
(3.8) \quad \gamma c = \int_0^c f(t) dt = \alpha b + \beta (c-b).
\]

Now let \( \delta = \int_{-\infty}^{\infty} |t|^r g(t) dt \). Then

\[
(r+1) (\tau - \delta) = (r+1) [\int_0^C t^r f(t) dt - \int_0^C t^r g(t) dt]
\]

\[
= \alpha b + \beta (c^r - b^r) - \gamma c^r + 1
\]

\[
= \alpha b + \beta (c^r - b^r) - c^r [\alpha + \beta (c-b)], \text{ using (3.8)}
\]

\[
= b(\beta - \alpha) (c^r - b^r).
\]

Since \( \alpha < \beta \) and \( 0 < b < c \), we see that \( \delta \leq \tau \). Let \( Y \) be a random variable with density \( g \). Since \( g \) is unimodal about 0, Theorem 3.1 shows that

\[
P(|Y| \geq k) \leq \left( \frac{\tau}{r+1} \right)^r \frac{\delta}{k^r} \leq \left( \frac{\tau}{r+1} \right)^r \frac{\tau}{k^r}.
\]

But since \( k > c \), the densities \( g \) and \( f \) agree on the set \((-\infty, -k] \cup [k, \infty)\). Therefore

\[
(3.9) \quad P(|X| \geq k) = P(|Y| \geq k) \leq \left( \frac{\tau}{r+1} \right)^r \frac{\tau}{k^r}.
\]

The theorem now follows from (3.7) and (3.9).

**Corollary 3.3.** Let \( X \) be a unimodal random variable with mean \( \mu \). Let

\[
\lambda_x = \{B(|X-\mu|^r) \}^{1/r}.
\]

Then, for every \( k > 0 \),

\[
P(|X-\mu| \geq k\lambda_x) \leq \max \left\{ \left( \frac{r+1}{rk^r} \right) \left( \frac{\tau}{(r+1)k} \right)^r \right\}.
\]
Proof. Immediate from Theorem 3.2, if we replace $k$ by $k\lambda_r$ and note $\lambda_r^R = \tau_r$. Observe that

$$\frac{(r+1) - k}{r} \leq \left(\frac{r}{r+1}\right)^R$$

whenever $k \geq k_r$, where

$$k_r = \left[\frac{(r+1)^R - r^R}{(r+1)^R}\right]^{1/r}.$$  \hspace{1cm} (3.10)

Therefore, the following corollary is immediate.

COROLLARY 3.4. With the same notation as in Corollary 3.3,

$$P(|X - \mu| \geq k\lambda_r) \leq \left(\frac{r}{r+1}\right)^R k^{-r},$$

for all $k \geq k_r$, where $k_r$ is given by (3.10).

For a comparison of our results with those given by Vyschanskii and Petunin, we write the special cases of the last two corollaries when $r = 2$.

COROLLARY 3.5. Let $X$ be a unimodal random variable with mean $\mu$ and variance $\sigma^2$. Then, for every $k > 0$,

$$P(|X - \mu| \geq k\sigma) \leq \max\left[\frac{3-k^2}{2k^2}, \frac{4}{9k^2}\right].$$  \hspace{1cm} (3.11)

Consequently, for every $k \geq \sqrt{19}/3$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{4}{9k^2}.\hspace{1cm} (3.12)$$

Proof. We only need to note that $k_2 = \sqrt{19}/3$.

REMARK. The inequality (3.11) is an improvement of the result of Vyschanskii and Petunin (1979). They have $(4-k^2)/3$ in place of our $3-k^2/2$. Consequently, they prove (3.12) for all $k \geq \sqrt{8}/3$.  

It is to be noted that (3.12) does not hold for all \( k > 0 \), if the distribution is not symmetric. The following detailed analysis of the example considered by Vysotskii and Petunin shows that (3.12) can fail if \( k = 1.385 \). We note that \( 1.385 < \sqrt{19}/3 \).

**EXAMPLE 3.1.** Let \( a \geq 1 \) and consider a random variable \( X \) such that

\[
P(X = 1) = \frac{(a - 1)}{(a + 1)}
\]

and

\[
P(X \leq x) = \frac{2(x + 1)}{(a + 1)^2}, \quad -a < x < 1.
\]

It is easy to check that \( \mu = E(X) = 0 \) and \( \sigma^2 = \text{Var}(X) = \frac{(2a - 1)}{3} \). Now

\[
P(|X| \geq 1) = \frac{a - 1}{a + 1} + \frac{2(a - 1)}{(a + 1)^2} = \frac{(a - 1)(a + 3)}{(a + 1)^2}.
\]

We now set \( k \sigma = 1 \). That is, \( k = (1/\sigma) \). Then

\[
k^2 P(|X - \mu| \geq k \sigma) = \sigma^{-2} P(|X| \geq 1)
\]

\[
= \frac{3(a - 1)(a + 3)}{(2a - 1)(a + 1)^2} = g(a), \text{ say}
\]

The condition \( g(a) > (4/9) \) reduces to

\[
(3.13) \quad 8a^3 - 15a^2 - 54a + 77 < 0.
\]

Numerical calculations show that (3.13) holds for \( 1.2816 \leq a \leq 3.05 \). Since \( k = \sigma^{-1} \), we see that (3.12) can fail if \( .767 \leq k \leq 1.385 \).
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