INTERRELATIONS AMONG GENERALIZED DISTRIBUTIONS

AND THEIR COMPONENTS

by

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I. INTRODUCTION

The following is a typical situation giving rise to a generalized distribution:

Let \( X_1 \) be a random variable with the probability generating function (p.g.f. for brevity) \( g_1(z) \), representing the number of egg masses laid in a sampling unit. Suppose that each egg mass gives rise to \( X_2 \) larvae with p.g.f. \( g_2(z) \). Then the distribution of \( X_2 \) averaged over \( X_1 \) is called "\( X_1 \) generalized \( X_2 \) distribution" and the random variable is denoted by \( X_1 \triangledown X_2 \). It can be shown that the p.g.f. of \( X_1 \triangledown X_2 \) is given by \( g_1 \left\{ g_2(z) \right\} \). Reference may be made to Gurland (1957) for additional details. Though the definition was in terms of egg masses and larvae, it is clear that it applies to any situation in which there is an original population and each member of the population gives rise to a new population. Thus it applies in accident statistics wherein we have a population of accidents per person and each accident gives rise to a population of losses.

There is another model called the model of compounding, which leads to the same distribution under certain conditions (c.f. Gurland (1957)).

These models are very important in areas such as accidents, entomology, bacteriology and demography. A number of generalized distributions have been

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developed by assuming specific forms for $g_1(z)$. Among them are the distributions developed by Katti and Gurland (1961), McGuire et al (1957), Neyman (1939) and Gurland (1958). Khatri and Patel (1961) have discussed properties of three classes of generalized distributions. The aim of this paper is firstly, to discuss interrelations between the generalized distributions with p.g.f. of the form $g_1 \left\{ g_2(z) \right\}$ and the component distributions $g_1(z)$ and $g_2(z)$ and secondly, to compare a number of distributions obtained through these models.

II. FACTORIAL MOMENTS

Here factorial moments of $g_1 \left\{ g_2(z) \right\}$ will be expressed in terms of those of $g_1(z)$ and $g_2(z)$.

Let $g(z) = g_1 \left\{ g_2(z) \right\}$. Denote by $\mu_i(1), 1^\mu(1), 2^\mu(1)$, the factorial moments of $g(z), g_1(z)$ and $g_2(z)$ respectively. Note that

$$\mu_i(1) = g^{(i)}(z) \bigg|_{z = 1} \quad i = 1, 2, \quad \text{(1)}$$

where $g^{(i)}(z)$ refers to the $i^{th}$ derivative of $g(z)$ with respect to $z$. Let $t = g_2(z)$ and write

$$g(z) = g_1(t) \quad \text{(2)}$$

and on differentiating (2) successively with respect to $z$, we have

$$g^{(1)}(z) = g^{(1)}(t) g_2^{(1)}(z) \quad \text{(3a)}$$

$$g^{(2)}(z) = g_1^{(2)}(t) \left\{ g_2^{(1)}(z) \right\}^2 + g_1^{(1)}(t) g_2^{(2)}(z) \quad \text{(3b)}$$

$$g^{(3)}(z) = g_1^{(3)}(t) \left\{ g_2^{(1)}(z) \right\}^3 + 3g_1^{(2)}(t) g_2^{(1)}(z) g_2^{(2)}(z) \quad \text{(3c)}$$

and so on. At $z = 1$, $t = g_2(z) = 1$. Hence, on substituting $z = 1$ in (1), (2), (3), we have
3.

\[ \mu_1 = 1^{\mu(1)} 2^{\mu(2)} \]  \hspace{1cm} (4a)

\[ \mu_2 = 1^{\mu(2)} 2^{\mu(1)} + 1^{\mu(1)} 2^{\mu(2)} \]  \hspace{1cm} (4b)

\[ \mu_3 = 1^{\mu(3)} 2^{\mu(1)} + 3 \left( 1^{\mu(2)} 2^{\mu(2)} 2^{\mu(1)} + 1^{\mu(1)} 2^{\mu(3)} \right) \]  \hspace{1cm} (4c)

and so on.

In connection with these formulae, it is of interest to note that the coefficients of various terms on the right hand side of (4) can be obtained from the relationship between moments and cumulants. This observation was communicated to the author by E. S. Pearson. If \( m(z) \) and \( C(z) \) denote the moment generating function (m.f.g.) and cumulant generating function (c.f.g.) respectively, then

\[ C(z) = \log m(z), \]

Hence, we have

\[ m^{(1)}(z) = C^{(1)}(z) m(z) \]

\[ = C^{(1)}(z) \exp \left\{ C(z) \right\}. \]  \hspace{1cm} (5)

This equation is the same as equation (3a) with the correspondence \( g^{(1)}(z) = m^{(1)}(z), t = C(z), \exp \left\{ C(z) \right\} = g_1^{(1)}(t) \) and \( C^{(1)}(z) = g_2^{(1)}(z) \). The only difference is that derivatives of \( g_1^{(1)}(t) \) at \( t = 0 \) give rise to \( \mu^{(i)}(i) \) while derivatives of \( \exp \left\{ C(z) \right\} \) with respect to \( C(z) \) at \( C(z) = 0 \) are all unity. The formulae connecting \( \mu^{(i)}(i) \) with \( 1^{\mu(1)}(i) \) and \( 2^{\mu(1)}(i) \) can therefore be obtained from the formulae connecting \( \mu^{(i)}(i) \) with \( \kappa^{(i)}_{(i)} \) by the following procedure:

(a) Replace \( \mu \) and \( \kappa \) by \( \mu^{(i)}(i) \) and \( 2^{\mu(i)}(i) \) respectively, and
(b) Multiply each term (which is product of \( K_i \)) by \( \mu^4(k) \) when \( k \) equals the sum of the powers of the various \( K_i \) that make up the product. Thus the term

\[
4200 K_4 K_3 K_1^3
\]

becomes

\[
4200 \mu(5) \mu(4) \mu(3) \mu(1)^3
\]

Formulae giving the first ten moments in terms of cumulants are given by M. G. Kendall (1945) in the chapter on moments and cumulants. Formulae giving the eleventh and twelfth moments along with the others are given by David and Kendall (1949). Formulae for values up to and including sixteen are available at Florida State University. These formulae have been punched on FORTRAN cards to facilitate computing on high speed computers. A computer program is available to compute additional relations. Brief descriptions are given in a research report by Katti and Singh (1963) The Public Health Service.

III. FACTORIAL CUMULANTS

A factorial cumulant generating function (f.c.g.f.) \( \kappa(u) \) is related to the p.g.f., \( g(z) \) by the formula

\[
\kappa(u) = \log g(1+u)
\]

and the factorial cumulants \( \kappa^{(i)} \) are obtained by the formula

\[
\kappa^{(i)} = \left. \frac{\partial^i}{\partial u^i} \kappa(u) \right|_{u=0}
\]
Since \( g(1+u) = \left[ g_2(1+u) \right] \), we have

\[
\log g(1+u) = \log g_1(1+t)
\]
or \( K(u) = 1^{K(t)} \) \hfill (8)

where \( (1+t) = g_2(1+u) \) and \( 1^{K(t)} \) is f.c.g.f. of \( g_1(z) \) with \( t \) as the dummy variable.

It is clear from (8) that expressions for \( K(i) \) in terms of \( 1^{K(i)} \) and \( 2^{\mu(i)} \) are the same as those for expressions \( \mu(i) \) in terms of \( 1^{\mu(i)} \) and \( 2^{\mu(i)} \), except that \( \mu(i) \) and \( 1^{\mu(i)} \) are now replaced by \( K(i) \) and \( 1^{K(i)} \) respectively.

The first three relations are given below for clarity:

\[
K(1) = 1^{K(1)} 2^{\mu(1)}
\]
\[
K(2) = 1^{K(2)} 2^{\mu(1)} + 1^{K(1)} 2^{\mu(2)}
\]
and
\[
K(3) = 1^{K(3)} 2^{\mu(1)} + 3 1^{K(2)} 2^{\mu(2)} + 1^{K(1)} 2^{\mu(3)} \cdot \cdot \cdot (9)
\]

These formulae simplify considerably for the following special cases:

If \( g_1(z) = \exp(\lambda(z-1)) \), then \( 1^{K(1)} = \lambda \) and \( 1^{K(i)} = 0 \), \( i \neq 1 \). Hence \( K(i) \) becomes \( \lambda 2^{\mu(1)} \). A generalized distribution with this \( g_1(z) \) is referred to as a Poisson generalized \( g_2(z) \) distribution or in brief, as a Poisson generalized distribution.

If \( g_1(z) \) has a general form but \( g_2(z) = \exp(\lambda(z-1)) \), then \( 2^{\mu(i)} = \lambda^i \).

From the nature of these relationships, as can be verified for the first three factorial moments from equations (4), \( \lambda^i \) factors out and the remainder is a linear function of \( 1^{K(i)} \).

Since factorial cumulants are not directly computable from data through simple explicit formulae, a recursion relation which expresses factorial
cumulants in terms of computable factorial moments and lower order factorial cumulants is given below for reference:

\[ K_{r+1} = \mu_{r+1} - \sum_{i=1}^{r} \binom{r}{i} \mu_{i+1} K_{r-i+1}. \]  

(10)

IV. USE OF ABOVE FORMULAE TO COMPUTE PROBABILITIES OF POSITIVE INTEGRAL VALUED GENERALIZED VARIABLES

If the random variable \( X \) corresponding to \( g_1(g_2(z)) \) takes on only positive integer values, then the following relation between its probabilities \( p_i \) and its factorial moments \( \mu^{(i)} \) is known:

\[ p_i = \sum_{r=i}^{\infty} (-1)^{r-i} \frac{\mu(r)}{r!}. \]  

(11)

Hence if the factorial moments \( \mu^{(i)} \) are known, then one may use the relationship in Section III between \( \mu^{(i)} \), \( \mu^{(1)} \) and \( \mu^{(2)} \) to compute \( \mu^{(i)} \) and then get \( p_i \) through (11). If the distribution is spread over a few points near the origin or if the tail of the distribution tapers off quickly, then one gets fairly accurate values for \( p_i \) by obtaining the first sixteen \( \mu^{(i)} \) and then using formula (11) after curtailing the infinite summation at 16.

In Table 1, an example is given using the Neyman Type A distribution for which \( g_1(z) = \exp\{\lambda_1(z-1)\} \) and \( g_2(z) = \exp\{\lambda_2(z-1)\} \) when \( \lambda_1 = 2 \) and \( \lambda_2 = 0.3 \). In column 3, exact probabilities obtained through a recursion relation are given and in column 2, factorial moments of Neyman Type A are given. These factorial moments were obtained through the use of the relations in Section III and then rechecked using recursion formulae. In columns 4, 5 and 6, \( p_i \) as computed from (11) by curtailing the infinite summations at various values are given. Here
the distribution essentially ends after the sixth count and the agreement between the exact $P_i$ and those obtained by curtailing after 16 is very good indeed.

When the distribution is spread over much more space, curtailing (11) after 16 could be misleading. To get better approximations one may follow the technique given by Hartley (1947). For the case of the positive integer valued variates, being discussed here, Hartley's technique involves solving the equations:

\[ 1P_0 + 1P_1 + 1P_2 + \ldots + 1P_n = 1, \]
\[ 0P_0 + 1P_1 + 2P_2 + \ldots + nP_n = \mu_1, \]
\[ 0^2P_0 + 1^2P_1 + 2^2P_2 + \ldots + n^2P_n = \mu_2, \]
\[ \vdots \]
\[ 0^nP_0 + 1^nP_1 + 2^nP_2 + \ldots + n^nP_n = \mu_n. \]

This assumes that $P_{n+1}, P_{n+2}$ etc. are so small as to be negligible. Hartley gives some discussion of the error in $P_i$ due to ignoring these probabilities. For large $n$, solving these equations numerically can be very cumbersome.
V. SPECIFIC GENERALIZED DISTRIBUTIONS

There are circumstances in which the choice of $g_1(z)$ and $g_2(z)$ is not uniquely specified. Rather, in a typical situation in which attention is directed to these distributions, a worker usually finds that the simple distributions fail to describe the data and that there is some reason to believe that the model underlying the distribution has some of the characteristics of the generalized model and he tries to see if one of the simpler generalized distributions will describe the data to any degree of satisfaction. The models actually underlying the data are invariably very complex and it is not clear at the outset which one of the simple generalized distributions will give an adequate description. In view of this, it was decided to derive a number of distributions obtained through the process of generalizing. All of the distributions considered here have at least two unknown parameters needing estimation. Hence comparison of these distributions was made along the lines of Anscombe (1950) who fixes the first two factorial cumulants as $kp$ and $kp^2$ and evaluates the ratios $\kappa_3 / kp^3$ and $\kappa_4 / kp^4$ where $\kappa_i$ stands for the $i$th factorial cumulant. The quantities $kp$, $kp^2$, $kp^3$ and $kp^4$ used here can be regarded as the first four factorial cumulants of the Neyman Type A distribution with probability generating function

$$\exp\{k[\exp(p(z-1))-1]\}.$$ 

The comparison is given in Table 2. When a distribution has more than two parameters, we give the range of values that the ratios take as the parameters vary over their admissible values. The following observations are made based on Table 2.
i. The three-parameter family of $\log c_0$ - Poisson has the range $(1-k, 2+k+k^2)$ for skewness and $((k-2)^2 - 3, 6+7k+3k^2+k^3)$ for kurtosis. These ranges cover the ranges of all generalized distributions given in Table 2. This distribution is being studied to see if it can act as an approximation to all of these distributions.

ii. The five relatively simple two-parameter distributions (a) Poisson with zeros (b) Poisson Binomial (c) Neyman Type A (d) Negative Binomial and (e) Logarithmic with zeros have their skewness scattered so widely that we can regard them as forming the basic set of generalized distributions. It appears that if the fit by all the five is extremely bad, the rest of the distributions are not likely to give satisfactory fit to the data. It is to be noted that two of the distributions i.e., Poisson with zeros and Logarithmic with zeros require very little work in addition to fitting the uniparameter Poisson and Logarithmic and tables are available to simplify the estimation further. While the Negative Binomial, the Neyman Type A and the Poisson Binomial distributions are more complex, the maximum likelihood equations and recursion formulae to compute probabilities have been worked out in the literature and are much simpler than those for the rest of the distributions in Table 2.

iii. If the value of $k$ happens to be around 0.5, fit by Poisson with zeros may be very much similar to the fit of the more difficult Poisson Binomial distribution.
iv. Distributions 3 and 4 in Table 2 have skewness in between those of Neyman Type A and Negative Binomial. If the data shows more skewness than that given by Neyman Type A but less than that given by Negative Binomial, these distributions may give a better description. One may, however, prefer to fit the three-parameter Poisson Pascal which covers the entire range between Neyman Type A and the Negative Binomial rather than try all of these two-parameter families.

v. The distributions 7, 8, 9, 12, 13 and 15 in Table 2 cover the same range of skewness and kurtosis and hence may be expected to give very similar description to the data if large skewness is associated with large kurtosis. In contrast with the two- and three-parameter families discussed above, these will have a higher flexibility in the sense of describing the fifth moment better and also in the sense of describing data with skewness in the right hand part of the interval (1, 2) and kurtosis in the left hand part of (1, 6) and data with skewness in the left hand part of (1, 2) and kurtosis in the right hand part of (1, 6). This flexibility has been gained at the expense of additional parameters.

vi. Discussion of the rest of the distributions is left to the reader.
REFERENCES


### TABLE 1

Comparison of the method of evaluating $P_i$ discussed here with the method using recursion formulae for the Neyman Type A distribution with $\lambda_1 = 2$, $\lambda_2 = 0.3$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\mu(i)$</th>
<th>$P_i$ from recursion formulae</th>
<th>$P_i$ from method in paper</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$k = 5$</td>
</tr>
<tr>
<td>0</td>
<td>1.0000000</td>
<td>0.59549425</td>
<td>0.5935315</td>
</tr>
<tr>
<td>1</td>
<td>0.6000000</td>
<td>0.26469180</td>
<td>0.27606751</td>
</tr>
<tr>
<td>2</td>
<td>0.5400000</td>
<td>0.09853032</td>
<td>0.07141498</td>
</tr>
<tr>
<td>3</td>
<td>0.5940000</td>
<td>0.03033430</td>
<td>0.06403502</td>
</tr>
<tr>
<td>4</td>
<td>0.7615400</td>
<td>0.00672689</td>
<td>$-0.01424260^*$</td>
</tr>
<tr>
<td>5</td>
<td>1.1032200</td>
<td>0.00206271</td>
<td>0.00919350</td>
</tr>
<tr>
<td>6</td>
<td>1.7714700</td>
<td>0.00047824</td>
<td>0.00096756</td>
</tr>
<tr>
<td>7</td>
<td>3.1006018</td>
<td>0.00010447</td>
<td>$-0.00022394$</td>
</tr>
<tr>
<td>8</td>
<td>5.8995021</td>
<td>0.00002170</td>
<td>0.00001716</td>
</tr>
<tr>
<td>9</td>
<td>1.2010212\times10</td>
<td>0.00000431</td>
<td>$-0.0003870^*$</td>
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<tr>
<td>10</td>
<td>2.6057131\times10</td>
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</tr>
<tr>
<td>11</td>
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<td>0.00000125</td>
</tr>
<tr>
<td>12</td>
<td>1.454269\times10^2</td>
<td>0.00000003</td>
<td>0.00000040*</td>
</tr>
<tr>
<td>13</td>
<td>3.709958\times10^2</td>
<td>0.00000013</td>
<td>0.00000123</td>
</tr>
<tr>
<td>$14^* $</td>
<td>9.916288\times10^2</td>
<td></td>
<td>$-0.0000002^*$</td>
</tr>
</tbody>
</table>

* The negative value here is due to the fact that we are ignoring probabilities beyond $P_k$.

** The probabilities beyond $P_{14}$ are not given since they are too small -- less than $10^{-9}$ in every case.
<table>
<thead>
<tr>
<th>Number</th>
<th>Distribution</th>
<th>Number of Parameters</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
</table>
| 1      | Poisson with zeros  
$g_1(z)$-Binomial with p.g.f. q+pz, $g_2(z)$-Poisson | 2         | $1 - k$        | $1 - 4k + k^2$ |
| 2      | Neyman Type A  
$g_1(z)$-Poisson, $g_2(z)$-Poisson | 2         | 1             | 1             |
| 3      | Neyman Type B  
$g(z) = \exp\left(\frac{\exp(\lambda_2(z-1)) - 1}{\lambda_2(z-1)} \right) - 1\right)$ | 2         | $9/8$        | $27/20$       |
| 4      | Neyman Type C  
$g(z) = \exp\left(\frac{\exp(\lambda_2(z-1)) - 1 - \lambda_2(z-1)}{\lambda_2^2(z-1)^2}\right) \right)$ | 2         | $6/5$        | $8/5$         |
| 5      | Negative Binomial (Pascal)  
$g_1(z)$-Poisson, $g_2(z)$-Logarithmic | 2         | 2             | 6             |
| 6      | Logarithmic with zeros  
$g_1(z)$-Binomial with p.g.f. q+pz, $g_2(z)$-Logarithmic | 2         | $2 + k + k^2$ | $6 + 7k + 8k^2 + k^3$ |
| 7      | Poisson Pascal  
$g_1(z)$-Poisson, $g_2(z)$-Negative Binomial. | 3         | ($1,2$)       | ($1,6$)       |
| 8      | Pascal Pascal  
$g_1(z)$ and $g_2(z)$-both Negative Binomials | 4         | ($1,2$)       | ($1,6$)       |
| 9      | Pascal Poisson  
$g_1(z)$-Pascal | 3         | ($1,2$)       | ($1,6$)       |
TABLE 2 (Continued)

<table>
<thead>
<tr>
<th>Number</th>
<th>Distribution</th>
<th>Number of Parameters</th>
<th>Skewness</th>
<th>Kurtosis</th>
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<tr>
<td>10</td>
<td>Type $H_1$ - Not generalized. Could be regarded as a Poisson, compounded with Beta. p.g.f. $F_1(k, \alpha \phi, \beta; \lambda)$. See Erdelyi (1953) for the definition of this function.</td>
<td>3</td>
<td>$(1-k, 2)$</td>
<td>$(1-4k^2, 6)$</td>
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<td>11</td>
<td>Type $H_2$ - Not generalized. p.g.f. $F(k_1, \alpha \phi, \beta; p_1(z-1))$. See Erdelyi (1953) for the definition of this function.</td>
<td>4</td>
<td>$(2, 2, \frac{2(k+1)}{\sqrt{k} + \sqrt{1+k}})$</td>
<td>see Katti (1958)</td>
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<td>5</td>
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<td>$(1, 6)$</td>
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<tr>
<td>$g_1(z)$</td>
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<td></td>
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<td>Type $H_1$</td>
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<td>Poisson Type $H_1$</td>
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<td>$(1, 6)$</td>
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<td>Poisson Type $H_2$</td>
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<td>$(1, 36)$</td>
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<tr>
<td>15</td>
<td>Pascal Logarithmic</td>
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<td>$(4.37, 6)$</td>
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<td>(0,6)</td>
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<td>Binomial Pascal</td>
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<td>(-3,6)</td>
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<td>$g_2(z) - $Pascal</td>
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<tr>
<td>19</td>
<td>Binomial Binomial</td>
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<tr>
<td></td>
<td>$g_1(z) - $Binomial</td>
<td>4</td>
<td>(-∞, 1)</td>
<td>covers</td>
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<tr>
<td></td>
<td>$g_2(z) - $Binomial</td>
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<td></td>
<td>(0,∞)</td>
</tr>
<tr>
<td>20</td>
<td>Binomial Poisson</td>
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<tr>
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<td>$g_1(z) - $Binomial</td>
<td>3</td>
<td>(-∞, 1)</td>
<td>(1,∞)</td>
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<td></td>
<td>$g_2(z) - $Poisson</td>
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<td>21</td>
<td>Log $c_o$ Poisson</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$g_1(z) - $Binomial</td>
<td>3</td>
<td>(rk,2+k+k²)</td>
<td>((k-2)²-3, 6+7k+8k²+k³)</td>
</tr>
<tr>
<td>22</td>
<td>Log $c_o$ Log $c_o$</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>4</td>
<td>($\frac{3k^2+2k+7}{4}, 2+k+k^2$)</td>
<td>see Katti (1958)</td>
</tr>
</tbody>
</table>