LUMPABILITY FOR NON-IRREDUCIBLE FINITE MARKOV CHAINS

by

Atef M. Abdel-Moneim*

and

Frederick W. Leysieffer

FSU Statistics Report M660

May, 1983

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

*Now at the Institute of Statistical Studies and Research, Cairo University.
Lumpability for Non-irreducible
Finite Markov Chains

by

Atef M. Abdel-Moneim

and

Frederick W. Leysieffer

Abstract

Conditions under which a function of a finite, discrete time Markov
chain, $X(t)$, is again Markov are given, when $X(t)$ is not irreducible. These
conditions are given in terms of an interrelationship between two partitions
of the state space of $X(t)$, the partition induced by the minimal essential
classes of $X(t)$ and the partition with respect to which lumping is to be
considered.

Key Words: Lumpability, Functions of Markov chains, Markov chains.
1. **Introduction.**

A function of a Markov chain is a random process of the form

\[ Y(t) = \sum_{i=1}^{m} i I_{A(i)}(X(t)) \]

where

\[ I_{A}(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A, 
\end{cases} \]

\( X(t), t = 0, 1, \ldots \) is a finite Markov chain with state space \( N = \{1, 2, \ldots, n\} \), and \( A = \{A(1), A(2), \ldots, A(m)\}, m \leq n \), is a partition on \( N \). The process \( Y(t) \) need not be Markov, but when it is, whatever the initial probability vector of \( X(t) \), \( X(t) \) is said to be lumped to \( Y(t) \) with respect to \( A \). The occurrence of lumping will be denoted by \( L(P, A) = P^* \) where \( P, (P^*) \) is the transition probability matrix (t.p.m.) for \( X(t), (Y(t)) \). In this paper the lumping of non-irreducible Markov chains is considered. This is a study of the interrelatedness of two partitions of the state space \( N \), the partition \( C \) induced by the maximal essential classes of the states of \( X(t) \) and the partition \( A \) with respect to which lumping is to be considered.

Previously lumping has been considered for irreducible Markov chains. See Burke and Rosenblatt [1], Hachigian and Rosenblatt [3] or Kemeney and Snell [4] for discussions of discrete time chains and Leysieffer [5] for continuous time chains. For discrete time irreducible Markov chains, a key result (see Burke and Rosenblatt [1] or Kemeney and Snell [4]) is the following theorem.
Theorem (1.1). A necessary and sufficient condition for \( L(P, A) = P^* \) is that for each pair of sets \( A(i), A(j) \) in \( A \), the quantities \( p^*_{ij} = \sum_{r \in A(j)} p_{kr} \) are constant for each \( k \in A(i) \). Here the \( p_{kr} \) are elements of the t.p.m. \( P \) and \( p^*_{ij} \) become the elements of \( P^* \), the t.p.m. for the lumped chain.

The most interesting interrelationship between partitions \( C \) and \( A \) occurs when they form an irreducible pair of partitions (Definition 2.1). The main results for this case are given in Section 3 after preliminaries are presented in Section 2. Section 4 contains the proofs for the theorems of Section 3. A general discussion covering all possible interrelationships between \( A \) and \( C \) is given in Section 5. Examples to illustrate results in this paper are in Section 6.

2. Preliminaries.

Consider two partitions on a set \( N = \{1, 2, \ldots, n\} \), \( A = \{A(i); 1 \leq i \leq m\} \) and \( C = \{C(j); 1 \leq j \leq r\} \), \( m \leq n \), \( r \leq n \).

Definition 2.1. The partitions \( A \) and \( C \) are said to form an irreducible pair of partitions of \( N \) if

\[
.u A(i) = u C(j) \quad \text{for sets } I \text{ and } J \text{ where} \\
i \in I \quad j \in J
\]

\( I \subseteq M = \{1, 2, \ldots, m\} \) and \( J \subseteq R = \{1, 2, \ldots, r\} \), implies \( I = M \) and \( J = R \).

Example 2.2. Let \( n = 6 \), \( m = 3 \) and \( r = 2 \) where \( A(1) = \{1,2\}, A(2) = \{3,4\}, A(3) = \{5,6\}, C(1) = \{1,2,3\}, C(2) = \{4,5,6\} \). It is clear that \( A \) and \( C \) form an irreducible pair of partitions. Now let \( C' = \{C'(1), C'(2)\} \) where
C'(1) = \{1, 2, 3, 4\} and C(2) = \{5, 6\}. Partitions \( A \) and \( C' \) do not form an irreducible pair of partitions, since \( A(1) \cup A(2) = C'(1) \).

**Definition 2.3.** Two partitions \( A \) and \( C \) of \( N \) are said to be commixing if for each \( A(i) \in A \) and \( C(j) \in C \), \( Z(i,j) \equiv A(i) \cap C(j) \neq \emptyset \).

Note that if \( A \) and \( C \) are commixing partitions of \( N \), then they form an irreducible pair.

3. **Main Results.**

Throughout the remainder of this paper, \( X(t) \), \( t = 0, 1, \ldots \) denotes a finite Markov chain with no transient states. The state space, \( N \), of \( X(t) \) has induced upon it a partition \( C = (C(i), 1 \leq i \leq r) \) of minimal essential classes. (The terminology is that of Chung [2].) The t.p.m., \( P \), of \( X(t) \) is thus of the form \( P = \text{diag}(P^1, P^2, \ldots, P^r) \) where each \( P^i \) is the \( n_i \times n_i \) t.p.m. corresponding to the \( i \)th minimal essential class, and where \( \sum_{i=1}^{r} n_i = n \).

Two main results of this paper can now be stated. Theorem 3.1 describes conditions on the partitions \( A \) and \( C \) under which lumping cannot occur. Theorem 3.2 characterizes the relationship between \( A \) and \( C \) when they form an irreducible pair of partitions on \( N \) and when lumpability holds.

**Theorem 3.1.** Suppose there exists a class \( C(u) \in C \) and a set \( A(s) \in A \) so that \( A(s) \subset C(u) \) and \( C(u) \setminus A(s) \) is not a union of sets of \( A \). Then lumpability of \( X(t) \) with respect to \( A \) cannot hold.
Theorem 3.2. Suppose there exists a t.p.m. P* such that L(P,A) = P*.

a. If A and C form an irreducible pair of partitions on N, then A and C are commuting.

b. If A and C are commuting, then L(P_u, Z(u)) = P*, u ∈ R, where Z(u) is a partition of C(u) given by Z(u) = \{Z(s,u); s ∈ M\} and where Z(s,u) = A(s) ∩ C(u).

4. Proofs and Further Theorems.

In the sequel A<q>, (C<q>) will denote that set of partition A, (C), to which the state q ∈ N belongs. Similarly, Z(<s>, u) = A<s> ∩ C(u), Z(s, <u>) = A(s) ∩ C<u> and Z(<s>, <u>) = A<s> ∩ C<u>.

Proof of Theorem 3.1. Let A(s) be a proper subset of C(u) and assume C(u)\A(s) is not a union of members of A. Then there exists an A(t) ∈ A such that A(t) ∩ C(u) ≠ ∅ and a C(v) ∈ C, v ≠ u such that A(t) ∩ C(v) ≠ ∅. Let t ∈ A(t) ∩ C(u) and let k ∈ A(t) ∩ C(v).

Suppose P is lumpable with respect to A. Then for any j ∈ A(t), by Theorem 1.1, we can define

\[ p_{ts}^* = \sum_{i \in A(s)} p_{ji}. \]

In particular, since k ∈ A(t), p_{ts} = \sum_{i \in A(s)} p_{ki} = 0 since k ∈ C(v), i ∈ C(u) and u ≠ v. Thus for any j ∈ A(t), \sum_{i \in A(s)} p_{ij} = 0. In particular, this is true for state t. Thus any i ∈ A(s) = A(s) ∩ C(u) cannot be reached from t ∈ A(t) ∩ C(u) in one transition of X(t).
Let q be any state in C(u), q ∉ A(t) so that q can be reached from \( \ell \) in one transition. There must be at least one such state or else \( \ell \) would be absorbing state contradicting the assumption that A(t) ∉ C(u). Note that if A<q> ∈ C(u), q could not be reached from \( \ell \) in one transition by the same reasoning that showed that i cannot be reached from \( \ell \) in one transition. If A<q> ∉ C(u), the reasoning given before shows that i cannot be reached from q in one transition.

Repeating the argument any integer, \( n \), number of times, one sees that \( \ell \) does not lead in \( n \) transitions to any state of C(u) which leads to state i in one transition. Thus \( \ell \) does not lead to i, contradicting the assumption that C(u) is a minimal essential class. Thus P is not lumpable with respect to A.

Another interrelationship between A and C which prohibits lumpability is given in the following theorem.

**Theorem 4.1.** Suppose there exists a set A(s) ∈ A and a non-absorbing state j ∈ A(s) with A(s) \{j\} = C(v) ∈ C for some v ∈ R. Then X(t) is not lumpable with respect to A.

**Proof.** Consider the quantities \( \gamma(k) = \sum_{i \in A(s)} p_{ki} \) for each \( k \in A(s) \).

Since A(s) = C(v) \cup \{j\}, \( \gamma(k) = \sum_{i \in C(v)} p_{ki} + p_{kj} \). Note that for \( k \neq j \), \( \gamma(k) = \sum_{i \in C(v)} p_{ki} = 1 \) whereas if \( k = j \), \( \gamma(k) = p_{jj} < 1 \) since C(v) is a minimal closed class, j ∈ C(v) and j is non-absorbing. Hence \( \gamma(k) \) is not constant for \( k \in A(s) \) and by Theorem 1.1, P is not lumpable with respect to A.
Proof of Theorem 3.2.a. Suppose A and C are not commixing. Then there exists an $s \in M$ and $v \in R$ such that $A(s) \cap C(v) = \emptyset$. Without loss of generality, assume $Z(1,1) = A(1) \cap C(1) = \emptyset$. Since $A$ partitions $N$, there exists an $s \in M$, $s \neq 1$ such that $Z(s,1) \neq \emptyset$. Let $k \in Z(s,1)$.

Since $L(P,A) = P^*$, $p_{s1}^*$ is defined by

$$
p_{s1}^* = \sum_{j \in A(1)} p_{ij} = \sum_{j \in Z(1,1)} p_{ij}.
$$

Using state $k$, $p_{s1}^*$ can be evaluated by

$$
p_{s1}^* = \sum_{j \in Z(1,k)} p_{kj} = \sum_{j \in Z(1,1)} p_{kj} = 0
$$

where the last equality follows since $Z(1,1) = \emptyset$. Thus $p_{ij} = 0$ for each $i \in A(s)$ and $j \in A(1)$.

The sets of $A$ can be partitioned into two classes:

$$
D(1) = \{ A(t) \in A: Z(t,1) = \emptyset \}
$$

$$
D(2) = \{ A(t) \in A: Z(t,1) \neq \emptyset \}.
$$

Note that neither $D(1)$ nor $D(2)$ are empty since $A(1) \in D(1)$ and $A(s) \in D(2)$. Let

$$
D(1) = \bigcup_{A(t) \in D(1)} A(t) \quad \text{and} \quad D(2) = \bigcup_{A(t) \in D(2)} A(t).
$$

Note $D(1) \cup D(2) = N$.

The previous argument showed that for any state $i \in D(2)$ and any state $j \in A(1)$, $p_{ij} = 0$. Thus no state in $A(1)$ can be reached from any state in $D(2)$ in one transition. By the same argument, no state of $D(1)$ can be reached
from any state in D(2) in one transition. Thus no state of D(1) can ever be reached from any state of D(2).

Furthermore, for each \( v \in R \), no state in \( D(1) \cap C(v) \) can be reached from any state in \( D(2) \cap C(v) \). Two cases are possible.

**Case 1.** There exists \( u \in R \) such that \( D(1) \cap C(u) \neq \emptyset \) and \( D(2) \cap C(u) \neq \emptyset \). This case cannot occur since the assumption that \( C(u) \) is an essential class of states is contradicted.

**Case 2.** For each \( u, 1 \leq u \leq r \), either \( D(1) \cap C(u) = \emptyset \) or \( D(2) \cap C(u) = \emptyset \).

Let

\[
E(1) = \{C(u): C(u) \cap D(1) = \emptyset\}
\]

\[
E(2) = \{C(u): C(u) \cap D(2) = \emptyset\}.
\]

Note that since \( D(1) \cup D(2) = N \), either \( C(u) \in E(1) \) or \( C(u) \in E(2) \) for each \( u, 1 \leq u \leq r \). Let

\[
E(1) = \bigcup_{C(u) \in E(1)} C(u) \quad \text{and} \quad E(2) = \bigcup_{C(u) \in E(2)} C(u)
\]

and note that \( E(1) \cup E(2) = N \).

By definition \( E(1) \cap D(1) = \emptyset \). Thus \( E(1) \subset E^c(1) = D(2) \). Similarly \( D(2) \subset E(2) \) and thus \( E(1) = D(2) \). Recalling that \( D(1) \neq \emptyset \) and \( D(2) \neq \emptyset \), one has a contradiction to the assumption that \( A \) and \( C \) form an irreducible pair of partitions on \( N \).

Thus \( A(1) \cap C(1) \neq \emptyset \). Since this argument applies to each pair \((s,u) \in M \times R\), \( Z(s,u) \neq \emptyset \) for each pair \((s, u)\) and thus \( A \) and \( C \) are commixing.
**Proof of Theorem 3.2.b.** Let \( s, t \in M \). Let \( i \in A(s) \) and \( u \in R \) be such that \( i \in C(u) \). Since \( L(P, A) = P^* \), \( P^* = \sum_{j \in A(t)} p_{ij} = \sum_{j \in Z(t, u)} p_{ij} \) where the second equality holds since \( p_{ij} = 0 \) for \( i \in C(u), j \notin C(u) \). Note that \( \sum_{j \in Z(t, u)} p_{ij} \) is constant in \( \ell \in Z(s, u) \), for each \( t \in M \). Note further that \( Z(u) = \{Z(t, u); t \in R\} \) is a partition for \( C(u) \). Thus there is a matrix \( P^u = (p^u_{st}), 1 \leq s, t \leq m \) such that \( L(P^u, Z(u)) = P^u \). Finally note that the elements of \( P^u \) were chosen independently of \( u \). Thus \( P^u = P^* \) for each \( u \in R \).

5. **General Lumping.**

Two additional fundamental relationships between \( A \) and \( C \) allow for lumping. These differ from the irreducible pair relationship discussed in sections 3 and 4, and are considered in this section. The most general relationship possible between the partitions \( A \) and \( C \) is a composition of the three fundamental relationships.

For two partitions \( G \) and \( H \), \( G \) is said to be a refinement of \( H \) (written as \( G \preceq H \)) if \( G \in G \) implies that there exists an \( H \in H \) such that \( G \subseteq H \).

**Lemma 5.1.** If \( C \preceq A \), then \( L(P, A) = I_m \) where \( I_m \) is the \( m \times m \) identity matrix.

**Proof.** For \( i \in A(s) \), consider

\[
\gamma(s, t) = \sum_{j \in A(t)} \gamma_{ij} = \sum_{j \in Z(t, <i>)} p_{ij}.
\]

If \( s \neq t \), \( Z(t, <i>) = \emptyset \) since \( C<i> \subseteq A(s) \) and \( \gamma(s, t) = 0 \). If \( s = t \), \( Z(s, <i>) = C<i> \) and \( \gamma(s, t) = 1 \), since \( C<i> \) is an essential class for \( X(t) \). Thus the condition of Theorem 1.1 is satisfied and lumping to \( I_m \) occurs.
If \( A = C \), each \( C(u), u \in R \) can be thought of as the state space of an irreducible Markov chain. If lumping occurs for each \( C(u) \in C \) with respect to the corresponding subset of \( A \), it occurs by composition with respect to \( A \). This is summarized without proof in the following lemma.

**Lemma 5.2.** Suppose \( A = C \) and suppose for each \( u \in R \), there exists an \( n_u \times n_u \) t.p.m., \( P^u \), such that \( L(P^u, Z^u) = P^u \) where \( Z^u = \{A(s): A(s) \subseteq C(u)\} \), \( n_u \) equals the number of states in \( C(u) \) and \( n = \sum_{u=1}^{r} n_u \). Then \( L(P, A) = P^* \) where \( P^* = \text{diag}(P^{*1}, P^{*2}, \ldots, P^{*r}) \).

Note that if \( A = C \), either lemma implies that \( L(P, A) = I_m \).

Any general interrelationship between two partitions can be expressed as a decomposition of the three fundamental relationships. This is summarized in Theorem 5.3 (where the partitions \( A \) and \( C \) could be arbitrary partitions on \( N \)).

**Theorem 5.3.** Let \( A \) and \( C \) be partitions on \( N \). Then for some integer \( s \), each of \( A \) and \( C \) can be partitioned into \( s+1 \) (possibly empty) classes of sets \( \{A(k), 0 \leq k \leq s\} \) and \( \{C(k), 0 \leq k \leq s\} \) where

(i) \( C(0) = A(0) \)

(ii) \( A(1) = C(1) \)

(iii) \( C(2) \subseteq A(2) \) and

(iv) for \( k = 3, 4, \ldots, s \), each pair \( (A(k), C(k)) \) forms an irreducible pair of partitions on \( N_k \) where \( N_k = U A(i) = U C(j). \)

Proof. Let \( C(0) = \{C \in C: \exists A \in A: A = C\} \) and \( A(0) = \{A \in A: \exists C \in C: A = C\} \).

Clearly \( C(0) = A(0) \).
Now let $C(1) = \{C \subset C(0) : C = \cup_{i \in I} A(i) \text{ for some subset } I \subset N\}$, and let

\[
A(1) = \{A(k) \subset A \setminus A(0) : A(k) \text{ is a set in a union of the form } \cup A(k) = C \text{ for some } C \subset C(0)\}.
\]

Clearly $A(1) = C(1)$.

Similarly define $C(2)$ in $C \setminus (C(0) \cup C(1))$ and $A(2)$ in $A \setminus (A(0) \cup A(1))$

where $C(2) = A(2)$.

If then $A^* = A \setminus (A(0) \cup A(1) \cup A(2))$ and $C^* = C \setminus (C(0) \cup C(1) \cup C(2))$ form an irreducible pair of partitions, in which case $s = 3$, $A(3) = A^*$ and $C(3) = C^*$, the partitioning is complete. If not there exist proper subsets $A^{**} = \{A(i) : i \in I\} \subset A^*$ and $C^{**} = \{C(j) : j \in J\} \subset C^*$ with $\cup_{i \in I^*} A(i) = \cup_{j \in J^*} C(j)$.

It then follows that $\cup_{i \in I^*} A(i) = \cup_{j \in J^*} C(j)$ where the union $i \in I^*$, $(j \in J^*)$ is taken over sets of $A^* \setminus A^{**}$, $(C^* \setminus C^{**})$. Either the pairs $(A^{**}, C^{**})$ and $(A^* \setminus A^{**}, C^* \setminus C^{**})$ form irreducible pairs of partitions or not. If so, the partitioning is complete. If not, the partitioning can continue in the same fashion and will terminate in a finite number of subpartition pairs, $(A(j), C(j))$, $j = 3, 4, \ldots, s$, since $N$ is finite.


Some examples will illustrate how lumping can occur for non-irreducible Markov chains. In this section consider $N = \{1, 2, \ldots, 8\}$ and a Markov chain with t.p.m. given by
Note $C = \{C(1), C(2), C(3)\}$ where $C(1) = \{1,2,3\}$, $C(2) = \{4,5,6\}$, and $C(3) = \{7,8\}$.

Let $P^*$ be the matrix

$$P^* = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$ 

**Example 6.1.** Let $A = \{A(1), A(2)\}$ where $A(1) = \{1,4,5,7\}$ and $A(2) = \{2,3,6,8\}$. Note $A$ and $C$ are an irreducible and commixing pair of partitions. Here it can easily be verified that $L(P,A) = P^*$.

**Example 6.2.** Let $A = \{A(1), A(2)\}$ where $A(1) = \{1,2,3,7,8\}$ and $A(2) = \{4,5,6\}$. It is clear that $C \preceq A$. Hence, by Lemma 5.1, $L(P,A) = I_2$. 

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{3}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\
\end{bmatrix}.
\]
Example 6.3. Let \( A = \{A(i), 1 \leq i \leq 6\} \) where \( A(1) = \{1\}, A(2) = \{2,3\}, A(3) = \{4,5\}, A(4) = \{6\}, A(5) = \{7\}, \) and \( A(6) = \{8\}. \) Here \( A = C \) and it can be verified that \( L(P,A) = \text{diag}(P^*, P^*, P^*). \)

Example 6.4. Let \( A = \{A(i), 1 \leq i \leq 4\} \) where \( A(1) = \{1,4,5\}, A(2) = \{2,3,6\}, A(3) = \{7\}, \) and \( A(4) = \{8\}. \) In this case \( \{C(1), C(2)\} \) and \( \{A(1), A(2)\} \) form an irreducible commixing pair of partitions on \( \{1, 2, \ldots, 6\} \) as do \( \{C(3)\} \) and \( \{A(3), A(4)\} \) on \( \{7,8\}. \) Here \( L(P,A) = \text{diag}(P^*, P^*) \) (c.f. (iv) in Theorem 5.3).

Example 6.5. Let \( A = \{A(i), 1 \leq i \leq 3\} \) where \( A(1) = \{1,2\}, A(2) = \{3,4,7\} \) and \( A(3) = \{5,6,8\}. \) Here \( A \) and \( C \) form an irreducible pair of partitions, but since \( A(1) \cap C(2) = \emptyset, \) this is not a commixing pair. Hence by Theorem 3.2a, no lumping is possible with respect to \( A. \)

REFERENCES


