TESTING WHETHER MEAN RESIDUAL LIFE CHANGES TREND

by

Frank Guess\(^1,2\), Myles Hollander\(^1\), and Frank Proschan\(^1\)
North Carolina State University, Florida State University,
and Florida State University

FSU Statistics Report No. M665

August, 1983

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

\(^1\)Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant Number AFOSR 82-K-0007. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

\(^2\)Research supported by the National Institute of Environmental Health Sciences under Grant 5 T32 ES 07011-05.


Key words and phrases. Nonparametric classes of distributions, mean residual life, exponentiality, increasing then decreasing mean residual life distributions, hypothesis testing, differentiable statistical functions, L-statistics.
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ABSTRACT

Given that an item is of age $t$, the expected value of the random remaining life is called the mean residual life (MRL) at age $t$. We propose two new nonparametric classes of life distributions for modeling aging based on MRL. The first class of life distributions consists of those with "increasing initially, then decreasing mean residual life" (IDMRL). The IDMRL class models aging that is initially beneficial, then adverse. The second class, "decreasing, then increasing mean residual life" (DIMRL), models aging that is initially adverse, then beneficial. We present situations where IDMRL (DIMRL) distributions are useful models. We propose two testing procedures for $H_0$: constant MRL (i.e., exponentiality) versus $H_1$: IDMRL, but not constant MRL (or $H_1^*$: DIMRL, but not constant MRL). The first testing procedure assumes the turning point, $\tau$, from IMRL to DMRL is specified by the user or is known. Our IDMRL($\tau$) test statistic, $T_n^*$, is a differentiable statistical function of order 1; thus, $T_n^*$, suitably normalized, is asymptotically normal. The second procedure assumes knowledge of the proportion, $\rho$, of the population that "dies" at or before the turning point (knowledge of $\tau$ itself is not assumed). We use L-statistic theory to show our IDMRL($\rho$) test statistic, $V_n^*$, appropriately normalized, is asymptotically normal. The exact null distribution of $V_n^*$ is established. Finally, for each procedure an application is given.
1. **Introduction and summary.** The mean residual life (MRL) has been used as far back as the third century A.D. (cf. Deevey (1947) and Chiang (1968)). In the last two decades, however, reliabilists, statisticians, and others have shown intensified interest in the MRL and derived many useful results concerning the MRL. Given that a unit is of age \( t \), the remaining life after time \( t \) is random. The expected value of this random residual life is called the mean residual life at time \( t \). Since the MRL is defined for each time \( t \), we also speak of the MRL function. See (2.1) for a more formal definition.

For a distribution with a finite mean, the MRL completely determines the distribution via an inversion formula (e.g., see Cox (1970, p. 128), Kotz and Shanbhag (1980), and Hall and Wellner (1981)). Hall and Wellner (1981) and Bhattacharjee (1982) derive necessary and sufficient conditions for an arbitrary function to be a MRL function. These authors recommend the use of the MRL as a helpful tool in model building.

Bryson and Siddiqui (1969) plot the empirical MRL for the survival times of patients with chronic granulocytic leukemia. They study the relationship between various nonparametric classes defined in terms of the MRL and in terms of other standard reliability functions (such as the failure rate). One of these classes is the decreasing mean residual life (DMRL) class. A life distribution is in the DMRL class if its MRL is decreasing.

Barlow, Marshall, and Proschan (1963) note that the DMRL class is a natural one in reliability theory and study some properties of this class. Morrison (1978) mentions that the increasing mean residual life (IMRL) distributions have been found useful as models in the social sciences.

Brown (1983) studies the problem of approximating IMRL distributions by exponential distributions. He mentions that certain IMRL distributions,
"... arise naturally in a class of first passage time distributions for Markov processes, as first illuminated by Keilson."

The exponential distribution is characterized as the life distribution with constant (≠ 0) MRL. The exponential distribution is both IMRL and DMRL. The exponential plays a role in reliability theory similar to the role the normal plays in classical statistical theory. Empirically, the exponential has been found to be a useful model (e.g., see Davis (1952)). For a theoretical development of this class see Barlow and Proschan (1981).


In this paper we propose two new nonparametric classes of distributions. The first class of distributions, called "increasing initially then decreasing mean residual life" (IDMRL) distributions, models aging that is initially beneficial, then adverse. (See Section 2 for more details.) The second class of distributions, called "decreasing initially, then increasing mean residual life" (DIMRL) distributions, is dual to the first class. It is used to model aging that is initially adverse, then beneficial. We develop tests of exponentiality versus the alternatives of IDMRL and DIMRL.

Hall and Wellner (1981) examine a parametric model in which MRL is decreasing (increasing), then constant. Such a MRL distribution is in the DIMRL (IDMRL) class. They apply their parametric model to a set of data from Bjerkedal (1960) on life lengths of guinea pigs injected with tubercle bacilli. Hall and Wellner (1981) estimate the parameters of their model by maximum likelihood. The empirical MRL plot (Hall and Wellner (1979)) is
suggestive of a DIMRL distribution. According to Hall and Wellner (1981),
the first stage of DMRL might be thought of as an "incubation period". The
next stage reflects, "an abrupt change in the mechanism of mortality".

Situations where it is reasonable to postulate an IDMRL model include:

(i) **Length of time employees stay with certain companies**: An
    employee with a company for four years has more time and career
    invested in the company than an employee of only two months.
    The MRL of the four-year employee is likely to be longer than
    the MRL of the two-month employee. After this initial IMRL
    (this is called "inertia" by social scientists), the processes
    of aging and retirement yield a DMRL period.

(ii) **Length of wars**: In the initial stages as negotiations deteriorate
    and conflict escalates, we expect the war to be longer as time
    goes by. Eventually, a DMRL will be applicable as resources
    and lives are expended.

(iii) **Life of certain television shows**: Many shows will initially
    be cancelled. The longer a show lasts the longer we expect it
    to continue. After this IMRL period, a DMRL models a waning
    period.

(iv) **Life lengths of humans**: High infant mortality explains the
    initial IMRL. Deterioration and aging explain the later DMRL
    stage.

With the procedures developed in this paper, we can test for the IDMRL
model (or the dual model of DIMRL). Our test statistics are motivated by
the HP(1975) test statistic. See Section 2 for more details.

In Section 2 we assume that the turning point $\tau$ (say) from IMRL to
DMRL is known. We then derive an IDMRL test statistic for testing constant
MRL (equivalent to exponentiality) versus IDMRL alternatives. By using the differentiable statistical function approach, we show asymptotic normality of the IDMRL test statistic. Knowledge of $\tau$ would be reasonable if we were working with a biological organism in a physical model of a disease process (e.g., the first two months form an incubation period). In a training program for future doctors or a recruiting program for a military service, the value of $\tau$ could be known by the length of the intensive stage designed to eliminate the weaker students or recruits.

Section 3 treats the case where we assume knowledge not of $\tau$, but of $\rho = F(\tau)$, the proportion of the population that dies (or leaves the program, etc.) at or before the turning point $\tau$. $L$-statistic theory is used to show asymptotic normality of the test statistic in this case. We also establish the exact sampling distribution of the test statistic for $\rho$ known and provide tables for ready application of the test procedure.

Section 4 contains two examples which illustrate the two tests.

2. The IDMRL test when the turning point $\tau$ is known. Let $F$ be a life distribution (i.e., $F(t) = 0$ for $t < 0$) with a finite first moment. Let $\bar{F}(t) \equiv 1 - F(t)$. The mean residual life function is defined as

$$m(t) = E[X - t | X > t] \quad \text{for } \bar{F}(t) > 0,$$

$$= 0 \quad \text{for } \bar{F}(t) = 0,$$

for $t \geq 0$. Note that we can express $m(t) = \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)} \, dx = \int_0^\infty \frac{\bar{F}(u)}{u} \, du$ when $\bar{F}(t) > 0$. Throughout, $F$ is assumed continuous.

**Definition 2.1.** A life distribution with a finite first moment is called an increasing then decreasing mean residual life (IDMRL) distribution if there exists a turning point $\tau \geq 0$ such that
(2.2) \[ m(s) \leq m(t) \quad \text{for } 0 \leq s \leq t < \tau \]
\[ m(s) \geq m(t) \quad \text{for } \tau \leq s \leq t. \]

The dual class of DIMRL distributions is obtained by reversing the inequalities on the MRL function in (2.2).

Notice that the exponential distributions are both IDMRL and DIMRL. If F is an IDMRL and a DIMRL distribution with turning point \( \tau \), then F is an exponential distribution. (Recall we assume F is continuous. For this result to be true we actually only need F to be continuous at \( \tau \) and not degenerate at 0.) But if F is IDMRL with turning point \( \tau_1 \) and DIMRL with turning point \( \tau_2 \neq \tau_1 \), F need not be exponential.

If F is IDMRL with turning points \( \tau_1 \) and \( \tau_2 > \tau_1 \), then F has constant MRL on \([\tau_1, \tau_2]\). The same conclusion is valid if IDMRL is replaced by DIMRL. (For F IDMRL we need not assume continuity. For F DIMRL we need to assume F is continuous at \( \tau_2 \).) In such cases, F will have an interval of turning points.

It is of interest to test

(2.3) \[ H_0: \text{F is constant MRL (equivalently, } F(x) = 1 - \exp(-x/\mu), x \geq 0, \mu > 0, \mu \text{ unspecified}) \]

versus

(2.4) \[ H_1: \text{F is IDMRL (and not constant MRL)} \]

based on a random sample \( X_1, X_2, \ldots, X_n \) from F. We assume in this section that the turning point \( \tau \) is known or has been specified by the user. If more than one turning point is specified (e.g., F is IDMRL(\( \tau_1 \)) and IDMRL(\( \tau_2 \))), for definiteness the user could simply take the smallest one.
Since the HP(1975) DMRL test statistic has desirable efficiency properties (cf. Hollander and Proschan (1975), Klefsjö (1983), and Hollander and Proschan (1982)), we modify it to get an IDMRL test statistic. Motivated by HP(1975), we consider the natural parameter of

\[
T(F) = \int_0^\tau \int_0^t F(s) \bar{F}(t) [m(t) - m(s)] dF(s) dF(t) \\
\quad + \int_0^\tau \int_t^\infty F(s) \bar{F}(t) [m(s) - m(t)] dF(s) dF(t).
\]

Using integration by parts, we can rewrite \(T(F)\) as

\[
T(F) = \int_0^\tau \left\{ \left[ \frac{2}{3} - F(\tau) + \frac{1}{2} F^2(\tau) \right] \bar{F}(x) + \left[ -1 + F(\tau) - \frac{1}{2} F^2(\tau) \right] \bar{F}^2(x) + \frac{1}{3} F^4(x) \right\} dx \\
\quad + \int_\tau^\infty \left\{ \left[ -\frac{1}{6} + \frac{1}{2} F(\tau) - \frac{1}{2} F^2(\tau) + \frac{1}{3} F(\tau) \right] \bar{F}(x) \\
\quad \quad + \left[ \frac{1}{2} - F(\tau) + \frac{1}{2} F^2(\tau) \right] \bar{F}^2(x) - \frac{1}{3} F^4(x) \right\} dx,
\]

a form which we find convenient. \(T(F)\) is a measure of the degree to which \(F\) satisfies the IDMRL property. If the MRL is constant (equivalently, if \(F\) is exponential), then \(T(F) = 0\).

Let \(F_n\) be the empirical distribution formed by a random sample \(X_1, \ldots, X_n\) from \(F\). \(T_n = T(F_n)\) (where \(T(\cdot)\) is defined by form (2.6)) is a natural statistic for testing \(H_0\) versus \(H_1\).

Integrating, we find the computationally simpler expression

\[
T_n = \sum_{i=1}^{i^*} B_1\left(\frac{n-i+1}{n}\right)(X_{(i)} - X_{(i-1)n}) + B_1\left(\frac{n-i^*}{n}\right)(\tau - X_{i^*n}) \\
\quad + B_2\left(\frac{n-i^*}{n}\right)(X_{(i^*+1)n} - \tau) + \sum_{i=i^*+2}^n B_2\left(\frac{n-i+1}{n}\right)(X_{(i-1)n} - X_{in}),
\]

where
\[ (2.8) \quad 0 = X_{0n} < X_{1n} < \ldots < X_{i^*n} \leq \tau < X_{(i^*+1)n} < \ldots < X_{nn}, \]

and

\[
\begin{align*}
B_1(u) & \overset{\text{def.}}{=} \left[ \frac{2}{3} - F_n(\tau) + \frac{1}{2} F_n^2(\tau) \right] u + \left[ -1 + F_n(\tau) - \frac{1}{2} F_n^2(\tau) \right] u^2 + \frac{1}{3} u^4, \\
B_2(u) & \overset{\text{def.}}{=} \left[ -\frac{1}{6} + \frac{1}{2} F_n(\tau) - \frac{1}{2} F_n^2(\tau) + \frac{1}{3} F_n^3(\tau) \right] u + \left[ \frac{1}{2} - F_n(\tau) + \frac{1}{2} F_n^2(\tau) \right] u^2 - \frac{1}{3} u^4.
\end{align*}
\]

(The dependence of \( B_i(u), i = 1, 2, \) on \( F_n(\tau) \) is suppressed for convenience.)

Under our continuity assumption on \( F \), \( w^1 \) ties will not occur and (2.8) is appropriate. In industrial, medical, and other settings ties may occur due to grouping of the data. For the case of ties, we find the computational expression

\[
(2.9) \quad T_n = \sum_{i=1}^{i^*} B_1 \left( \frac{S_{i-1}}{n} \right) (\tilde{X}_{ik} - \tilde{X}_{(i-1)k}) + B_1 \left( \frac{S_{i^*}}{n} \right) (\tau - \tilde{X}_{i^*k})
\]

\[
+ B_2 \left( \frac{S_{i^*}}{n} \right) (\tilde{X}_{(i^*+1)k} - \tau) + \sum_{i=i^*+2}^k B_2 \left( \frac{S_{i-1}}{n} \right) (\tilde{X}_{ik} - \tilde{X}_{(i-1)k}),
\]

where

\[
(2.10a) \quad 0 = \tilde{X}_{0k} < \tilde{X}_{1k} < \ldots < \tilde{X}_{i^*k} \leq \tau < \tilde{X}_{(i^*+1)k} < \ldots < \tilde{X}_{kk}
\]

are the distinct ordered times of deaths from the random sample,

\[
(2.10b) \quad n_i = \text{number of observed deaths at time } \tilde{X}_{ik}, \quad s_i = n - \sum_{k=0}^{i} n_k
\]

for \( i = 0, 1, \ldots, k < n \).

(Note that \( n_i \neq 0, i = 1, \ldots, k, \) while \( n_0 = 0 \) is allowed.)

To establish asymptotic normality of \( T_n \), we use the differentiable statistical function (DSF) approach of von Mises (1947) (cf. Boos (1977) and Serfling (1980)). The approach uses a Taylor expansion for functionals.
The first order Gâteaux differential of a functional $T$ at the point $F$ in the direction $G$ is defined as

\[
(2.11) \quad d_1 T(F, G-F) = \lim_{\lambda \to 0^+} \frac{T(F_{\lambda}) - T(F)}{\lambda},
\]

where $F_{\lambda} = F + \lambda(G-F)$, $F$ and $G$ are distributions in the domain of $T(\cdot)$, and $0 < \lambda \leq 1$. Notice that the differential is a function of two arguments: the distribution $F$, and the increment $D \overset{\text{def.}}{=} G-F$.

For the IDMRL functional $T$, the Gâteaux differential is

\[
(2.12) \quad d_1 T(F, D) = \left[ \int_0^\tau \left( -\bar{F}(\tau) \tilde{F}(x) + \bar{F}(\tau) \tilde{F}^2(x) \right) dx \right] D(\tau) \]

\[- \int_0^\tau \left( \frac{2}{3} - F(\tau) + \frac{1}{2}F^2(\tau) \right) + 2 \left[ 1 - F(\tau) - \frac{1}{2}F^2(\tau) \right] \bar{F}(x) \]

\[+ \frac{4}{3} \bar{F}^3(x) D(x) dx \]

\[+ \int_0^\infty \left( \frac{1}{2} - F(\tau) + F^2(\tau) \right) \bar{F}(x) - \bar{F}(\tau) \tilde{F}^2(x) dx \right] D(\tau) \]

\[- \int_0^\tau \left( \frac{1}{6} + \frac{1}{2}F(\tau) - \frac{1}{2}F^2(\tau) + \frac{1}{3}F^3(\tau) \right) + 2 \left[ \frac{1}{2} - F(\tau) + \frac{1}{2}F^2(\tau) \right] \bar{F}(x) \]

\[- \frac{4}{3} \bar{F}^3(x) D(x) dx.
\]

Set $D_n = F_n - F$. The differential is linear in the increment argument, and thus,

\[
(2.13) \quad d_1 T(F, D_n) = \frac{1}{n} \sum_{i=1}^n d_1 T(F, \delta_{X_i} - F),
\]

where $\delta_{X_i}(x) = 0$ if $x < X_i$ and = 1 if $x \geq X_i$. $d_1 T(F, D_n)$ is a random variable approximating $[T(F_n) - T(F)]$. The former is easier to use than the latter to show asymptotic normality.
Let
\[\mu(T,F) \overset{\text{def.}}{=} E_F[d_1^T(F, \delta_{X_1}, -F)]\]
and
\[\sigma^2(T,F) \overset{\text{def.}}{=} \text{Var}_F[d_1^T(F, \delta_{X_1}, -F)].\]

Note that \(\mu(T,F) = 0\). Assume \(F\) is a distribution such that \(0 < \sigma^2(T,F) < \infty\).

Applying the classical Lindeberg-Lévy Central Limit Theorem to (2.13), we have
\[(2.14) \quad \sqrt{n}[d_1^T(F,D_n)] \xrightarrow{D} N(0,\sigma^2(T,F)).\]

We sketch the proof that the remainder term
\[(2.15) \quad R_n = \sqrt{n}[T(F_n) - T(F) - d_1^T(F,D_n)] \xrightarrow{D} 0.\]

The key step of the proof involves that \(T(F)\) can be represented as
\[
T(F) = T_{11}(F)T_{12}(F) + T_{21}(F)T_{22}(F) + T_{3*}(F)
+ T_{41}(F)T_{42}(F) + T_{51}(F)T_{52}(F) + T_{6*}(F).
\]
(The functionals \(T_{ij}(\cdot), \ i = 1, 2, 3, \ j = 1, 2, *\) are defined in the obvious
way from (2.6), e.g., \(T_{11}(F) = \left[\frac{2}{3} - F(\tau) + \frac{1}{2}F^2(\tau)\right], T_{12}(F) = \int_0^\tau F(x)dx, T_{3*}(F) = \int_0^\tau \frac{1}{3}F^4(x)dx, \) etc.) Assuming we have a distribution \(F\) such that
\(0 < \sigma^2(T_{ij}, F) < \infty\), each of the statistics \(T_{ij}(F_n), i = 1, 2, 3, \ j = 1, 2, *\)
is a DSF. The sum (product) of DSF's is a DSF provided the sum (product)
variance term is strictly positive and finite. (I.e., if \(T_{11}(F_n)\) and \(T_{22}(F_n)\)
are DSF's, then \(T_{S}(F_n) = T_{11}(F_n) + T_{22}(F_n)\) \((T_{p}(F_n) = T_{11}(F_n)T_{22}(F_n))\) is a DSF
provided \(0 < \sigma^2(T_S, F)\) \((\sigma^2(T_p, F)) < \infty\).) Repeated application of this yields
\(T(F_n)\) is a DSF. To be a DSF implicitly means (2.15). (We work throughout
with DSF's of order \( m = 1 \), i.e., the first order Gâteaux differentials of the functionals exist and are not degenerate; hence, (2.15).

Consider that

\[
(2.16) \quad \sqrt{n}[T(F_n) - T(F)] = \sqrt{n} [d_1 T(F, D_n)] + R_n.
\]

Using (2.14)-(2.16) and Slutsky's Theorem, we have

\[
(2.17) \quad \sqrt{n}[T(F_n) - T(F)] \xrightarrow{d} N(0, \sigma^2(T,F)).
\]

(This also follows directly from \( T(F_n) \) being a DSF of order 1.)

Under \( H_0 \),

\[
\sqrt{n}[T(F_n) - T(F)] = \sqrt{n}[T(F_n)],
\]

and the appropriate variance conditions hold. (Assume \( \tau \neq 0 \) to avoid \( F(\tau) = 0 \). If \( \tau = 0 \), the test reduces to testing for DMRL (IMRL); see HP(1975).) From (2.17), we have

\[
\sqrt{n}[T(F_n)] \xrightarrow{d} N(0, \sigma^2(T,F)).
\]

Straightforward calculations show

\[
\sigma^2(T,F) = \mu^2 \left[ - \frac{1}{15} F^5(\tau) + \frac{1}{6} F^4(\tau) - \frac{1}{60} F^3(\tau) + \frac{1}{10} F^2(\tau) - \frac{1}{30} F(\tau) + \frac{1}{210} \right],
\]

where \( \mu = \int_0^\infty \bar{F}(x) dx \). \( \hat{\sigma}_n^2 \) def. \( \sigma^2(T,F_n) \) is a consistent estimator of \( \sigma^2(T,F) \); thus,

\[
(2.18) \quad \left\{ \sqrt{n}[T(F_n) - T(F)] / \hat{\sigma}_n \right\} \xrightarrow{d} N(0,1).
\]

The IDMRL test procedure rejects \( H_0 \) in favor of \( H_1 \) at the approximate level \( \alpha \) if \( T_n \) def. \( \sqrt{n}[T_n] / \hat{\sigma}_n \geq z_{\alpha} \) where \( z_{\alpha} \) is the upper \( \alpha \)-quantile of the standard normal distribution. If \( T_n < z_{\alpha} \), \( H_0 \) is accepted.
The DIMRL test rejects $H_0$ in favor of

$$H_1^*: F \text{ is DIMRL (and not constant MRL)}$$

at the approximate $\alpha$ level if $\tilde{T}_n \leq -z_\alpha$. If $\tilde{T}_n > -z_\alpha$, $H_0$ is accepted.

3. The IDMRL test when the proportion $\rho$ is known. In this section we do not assume knowledge of $\tau$, the turning point. Instead, we assume knowledge of the proportion $\rho$ of the population that "dies" at or before the turning point (e.g., in a training or recruiting program of students or military personnel, knowledge of $\rho$ would be reasonable). Let $F^{-1}(\rho) = \inf\{x | F(x) \geq \rho\}$ for $0 < \rho < 1$. Note that $\tau = F^{-1}(\rho)$.

Recall that we want to test $H_0$ (2.3) versus $H_1$ (2.4). In this section we find form (2.5) of $T(F)$ (a measure of the IDMRL property of $F$) convenient. A natural statistic to consider first is $U_n = T(F_n)$.

Let $X_{1n} \leq X_{2n} \leq \ldots \leq X_{nn}$ denote the order statistics from a random sample from $F$. Let $[x]$ = greatest integer less than or equal to $x$. Let $m_n(t)$ denote the empirical mean residual life function (i.e.,

$$m_n(t) = \left( \frac{1}{n} \sum_{k=i+1}^{n} (X_{kn} - t) \right)/(n-i) \text{ for } t \in [X_{in}, X_{(i+1)n}], \quad i = 0, 1, \ldots, n-1,$$

where $X_{0n} = 0$, $m_n(t) = 0$, for $t \geq X_{nn}$. Simplifying $U_n$ by integrating, we get

$$U_n = \sum_{i=1}^{j-1} \sum_{j=i+1}^{j-1} \frac{1}{n} \int_{X_{in}}^{X_{jn}} \int_{X_{in}}^{X_{jn}} \{m_n(X_{in}) - m_n(X_{jn})\}$$

$$+ \sum_{i=j}^{n-1} \sum_{j=i+1}^{n} \frac{1}{n} \int_{X_{in}}^{X_{jn}} \int_{X_{in}}^{X_{jn}} \{m_n(X_{in}) - m_n(X_{jn})\},$$

where
(3.2) \[ j^* = \begin{cases} n_0 & \text{if } n_0 \text{ is an integer} \\ [n_0]+1 & \text{if } n_0 \text{ is not an integer.} \end{cases} \]

Note that \( \hat{\tau} = F_n^{-1}(\rho) = X_{j^*} \).

The statistic \( U_n \) can be viewed as follows. Consider the first double sum in (3.1). For \( 1 \leq i < j \leq j^* \), we expect the difference \( [m_n(X_{jn}) - m_n(X_{in})] \) to be positive if \( F \) is strictly IDMRL. We weight the difference by \( \bar{F}_n(X_{in}) \bar{F}_n(X_{jn}) \). An "average" of these weighted differences is finally taken. Similar comments hold for the second double sum in (3.1).

Now we modify \( U_n \) to get another statistic \( V_n \). Notice that we have an estimator of the MRL at time \( X_{0n} = 0 \), namely, \( m_n(X_{0n}) = \frac{1}{n} \sum_{k=1}^{n} X_{kn} \), which is simply the sample mean. This uses information on the lifelengths of new items. We incorporate this additional information into \( V_n \) as follows:

(3.3) \[ V_n = U_n + \sum_{j=1}^{j^*} \frac{1}{n^2} \bar{F}_n(X_{jn}) [m_n(X_{jn}) - m_n(X_{0n})]. \]

The statistic \( V_n \) as expressed in (3.3) ((3.1)) implicitly uses the fact that \( F \) is continuous; and hence, no ties occur in the data. Recall our discussion in Section 2 that in actual practice ties may occur due to grouping of the data, even though the underlying distribution is continuous. For the case of ties, we find the computational expression:

(3.4) \[ V_n = \frac{1}{n^2} \sum_{i=0}^{j^*-1} \sum_{j=i+1}^{j^*} \bar{F}_n(X_{in}) \bar{F}_n(X_{jn}) [m_n(X_{jn}) - m_n(X_{in})] n_i n_j \]
\[ + \frac{1}{n^2} \sum_{i=j}^{k-1} \sum_{j=i+1}^{k} \bar{F}_n(X_{in}) \bar{F}_n(X_{jn}) [m_n(X_{jn}) - m_n(X_{in})] n_i n_j, \]

where \( k, n_i, \) and \( \bar{X}_{ik}, \ i = 1, \ldots, k \) are defined by (2.10 a,b) and
\( n_0 \overset{\text{def.}}{=} 1 + \text{number of observed deaths at time } \tilde{X}_{0k} = 0. \) (Note that \( j^* \neq i^* \) can happen.)

In establishing asymptotic normality, we find it useful to express \( V_n \) as a linear combination of order statistics (i.e., as an L-statistic). To represent \( V_n \) as an L-statistic, we reverse the triple summation implicit in (3.3); thus,

\[
(3.5) \quad V_n = \frac{1}{n^4} \sum_{k=1}^{n} c_{kn} X_{kn},
\]

where

\textbf{Case 1: } \( k < j^* \).

\[
c_{kn} = -\frac{4}{3} k^3 + \frac{1}{2} k^2 - \frac{1}{6} k + 4nk^2 - 2n^2 k - 2nkj^* + kj^* 2 + kj^* - nk + n^2 j^* - \frac{1}{2} nj^* - \frac{1}{2} nj^*.
\]

\textbf{Case 2: } \( k = j^* \).

\[
c_{jn} = -\frac{1}{2} n^3 + \frac{1}{2} n^2 + \frac{1}{6} j^* 3 + 2j^* 2 - \frac{1}{6} j^* + \frac{1}{2} n^2 j^* - \frac{5}{2} nj^*.
\]

\textbf{Case 3: } \( k > j^* \).

\[
c_{kn} = -\frac{1}{2} n^3 + \frac{1}{2} n^2 + \frac{1}{3} j^* 3 + \frac{1}{2} j^* 2 + \frac{1}{6} j^* + \frac{4}{3} k^3 - \frac{1}{2} k^2 + \frac{1}{6} k - kj^* + 3n^2 k + 2nkj^* - 4nk^2 + \frac{1}{2} nj^* 2 - nj^* - \frac{3}{2} n^2 j^* + kj^*.
\]

Note that \( c_{kn} \) depends on \( \rho \), as well as on \( k \) and \( n \).

From cases 1 and 3, we are led to the following weight function:

\[
(3.5) \quad J(u) = \begin{cases} 
J_1(u) = -\frac{4}{3} u^3 + 4u^2 + (\rho^2 - 2\rho - 2)u + (\rho - \frac{1}{2}\rho^2), & \text{if } 0 \leq u < \rho \\
J_2(u) = \frac{4}{3} u^3 - 4u^2 + (3 + 2\rho - \rho^2)u + (\frac{1}{3}\rho^3 + \frac{1}{2}\rho^2 - \frac{3}{2}\rho - \frac{1}{2}), & \text{if } \rho \leq u \leq 1.
\end{cases}
\]
(This comes from dividing by \(n^3\) in cases 1 and 3, equating \(\frac{j^*}{n}\) with \(\rho\) and \(\frac{k}{n}\) with \(u\), and ignoring terms that are not cubic.) Let \(S_n = \frac{1}{n} \sum_{k=1}^{n} J(\frac{k}{n})X_{kn}\). Note that

\[
(3.6) \quad \sqrt{n}[V_n - S_n] \xrightarrow{P} 0.
\]

To form our IDMRL test statistic, we modify \(V_n\) to get the scale invariant statistic

\[
(3.7) \quad V^*_n = \frac{V_n}{\bar{X}_n},
\]

where \(\bar{X}_n = m_n(0) = \frac{1}{n} \sum_{i=1}^{n} X_{in}\).

Now, we establish asymptotic normality of \(V^*_n\), appropriately normalized, using results of Stigler (1974) and Mason (1981). (See also Stigler (1979).)

Recall \(u = \int_0^\infty \bar{F}(x)dx\). Define

\[
(3.8) \quad \mu(J,F) = \int_0^\infty xJ(F(x)dF(x),
\]

and

\[
(3.9) \quad \sigma^2(J,F) = \int_0^\infty \int_0^\infty J(F(x))J(F(y))[F(\min(x,y)) - F(x)F(y)]dx dy.
\]

From (3.6), we have

\[
(3.10) \quad \sqrt{n}[V^*_n - S^*_n] \xrightarrow{P} 0,
\]

where \(S^*_n = S_n/\bar{X}_n\). Define

\[
(3.11) \quad J^*(u) = J(u) - \mu(J,F)/u,
\]

where \(0 \leq u \leq 1\). If we show
(3.12) \[ S_n^- \overset{\text{def.}}{=} \sqrt{n}(S_n^* - \mu(J, F)/\mu) \overset{d}{=} N(0, \sigma^2(J^*, F)/\mu^2), \]

then from (3.10) and Slutsky's Theorem, we will have

(3.13) \[ V_n^- \overset{\text{def.}}{=} \sqrt{n}(V_n^* - \mu(J, F)/\mu) \overset{d}{=} N(0, \sigma^2(J^*, F)/\mu^2). \]

Note that \( S_n^- \) can be rewritten as

(3.14) \[ S_n^- = \sqrt{n}\left[ \frac{1}{n} \sum_{i=1}^{n} \frac{J^*(\frac{i}{n})X_i}{\bar{X}_n} \right]/\bar{X}_n. \]

Assume \( F \) satisfies

\( A_1: \int_0^{\infty} x^2 dF(x) < \infty, \quad A_1^*: \int_0^{\infty} (F(x)\bar{F}(x))^{\frac{3}{2}} dx < \infty, \)

\( A_2: \sigma^2(J^*, F) > 0, \)

and

\( A_3: \) \( F \) has a unique \( \rho \)-quantile.

(Note that \( A_1^* \) is equivalent to \( A_1 \) when \( F \) has regularly varying tails with a finite exponent (see Feller (1966), p. 268, and Stigler (1974), p. 536). Also, note that \( A_3 \) is equivalent to \( F^{-1} \) is continuous at \( \rho \).) \( J^* \) is bounded on \([0,1]\). By \( A_3 \), \( J^* \) is continuous a.e. \( F^{-1} \). \( J^* \), also, satisfies a Hölder condition for \( \alpha > \frac{1}{2} \) (e.g., \( \alpha = 1 \), which is simply a Lipschitz condition) except at the one point \( \rho \). Under \( A_3 \), however, \( \rho \) is a continuity point of \( F^{-1} \).

Using Theorem 2 of Stigler (1974), then Theorem 1 of Stigler (1974) and Theorem 2 of Mason (1981), we have under \( A_1, A_1^*, A_2, \) and \( A_3 \) that
\begin{equation}
\sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^{n} J^*(\frac{i}{n}) X_{i\text{in}} - \mu(J^*, F) \right] \xrightarrow{d} N(0, \sigma^2(J^*, F)).
\end{equation}

(See Remark 2 of Stigler (1974) and consider $J^*(\frac{i}{n}) \overset{\text{def}}{=} J^*(\frac{i}{n+1})$ as it applies to Theorem 1 and Theorem 2. The proof of Theorem 2 of Mason (1981) can be modified in a straightforward fashion to handle our weight function, $J^*$, using $\frac{i}{n}$ instead of $\frac{i}{n+1}$.) Since $\mu(J^*, F) = 0$, we have from (3.15) that

\begin{equation}
\sqrt{n} \left[ \frac{1}{n} \sum_{i=1}^{n} J^*(\frac{i}{n}) X_{i\text{in}} \right] \xrightarrow{d} N(0, \sigma^2(J^*, F)).
\end{equation}

Applying Slutsky's Theorem, we get

\begin{equation}
S^* \overset{d}{=} \frac{N(0, \sigma^2(J^*, F)/\mu^2)}{n},
\end{equation}

hence, (3.13) is established.

With (3.13) we now formally develop the $\text{IDMRL}(\rho)$ test procedure. Note that $\mu(J, F) = T(F)$. From this, we have $\mu(J, F) = 0$ when $H_0$ holds. Also, $\sigma^2(J^*, F) = \sigma^2(J, F)$ under $H_0$. (Recall $J^*(u) = J(u) - \mu(J, F)/\mu$.) Since $V_n^*$ is a scale invariant statistic, the calculation of the asymptotic variance under $H_0$ can be made with scale parameter $\lambda$ taken to be 1. Note that $\mu = 1$.

Set $F_0(x) = 1 - \exp(-x), x \geq 0$. Calculations show that

\begin{equation}
\sigma^2(\rho) \overset{\text{def}}{=} \sigma^2(J, F_0) = - \frac{1}{150} + \frac{1}{60} - \frac{1}{30} + \frac{1}{10^5} - \frac{1}{30^5} + \frac{1}{210}.
\end{equation}

(Notice for $\rho = 0$ and $\rho = 1$, that $\sigma^2(\rho) = \frac{1}{210}$, which is the asymptotic variance of the HP(1975) DMRL test statistic.) Under $H_0$ and by (3.13), we have

\begin{equation}
\tilde{V}_n \overset{\text{def}}{=} \sqrt{n} \left[ V^* \right]/\sigma(\rho) \overset{d}{=} N(0, 1).
\end{equation}

(Under $H_0$, $A_1$, $A_1^*$, $A_2$, and $A_3$ are satisfied.)

The $\text{IDMRL}(\rho)$ test procedure rejects $H_0$ in favor of $H_1$ at the approximate $\alpha$-level if $\tilde{V}_n \geq z_\alpha$, where $z_\alpha$ is the upper $\alpha$-quantile of the standard normal distribution. If $\tilde{V}_n < z_\alpha$, $H_0$ is accepted.
The DIMRL\((\rho)\) test rejects \(H_0\) in favor of \(H_1\) (see (2.19)) at the approximate \(\alpha\)-level if \(\tilde{V}_n \leq -z_\alpha\). If \(\tilde{V}_n > -z_\alpha\), \(H_0\) is accepted.

By rewriting \(V_n\) as a sum of weighted normalized spacings, we can use the approach of Langenberg and Srinivisan (1979), (LS(1979)) to find the exact distribution of \(V^*_n = V_n / \bar{X}_n\). Set \(D_{jn} = (n-j+1)(X_{jn} - X_{(j-1)n})\), \(j = 1, 2, \ldots, n\). Using these normalized spacings in computing \(m_n(X_{1n})\) of \(V_n\), we can write

\[
(3.19) \quad \tilde{V}_n = \frac{1}{n} \sum_{k=1}^{n} \alpha_{kn} D_{kn}^*,
\]

where

**Case 1:** \(k < j^*\).

\[\alpha_{kn} = (-\frac{1}{3}k^3 + \frac{1}{2}k^2 - \frac{1}{6}k - nk + nk^2 - nkj^* + \frac{1}{2}kj^*^2 + \frac{1}{2}kj^*)/n^3.\]

**Case 2:** \(k = j^*\).

\[\alpha_{jn} = (\frac{1}{6}j^*^3 + j^*^2 - \frac{1}{6}j^* - nj^*)/n^3.\]

**Case 3:** \(k > j^*\).

\[\alpha_{kn} = (\frac{1}{3}j^*^3 + \frac{1}{2}j^*^2 + \frac{1}{6}j^* + \frac{1}{3}k^3 - \frac{1}{2}k^2 + \frac{1}{6}k + \frac{1}{2}nk - nk^2 + \frac{1}{2}jk^* - \frac{1}{2}nj^* - \frac{1}{2}kj^*^2 + \frac{1}{2}kj^*)/n^3.\]

Since \(V^*_n\) is scale invariant, we can without loss of generality set \(F(x) = 1 - \exp(-\frac{1}{2}x)\) under \(H_0\). The normalized spacings, \(D_{1n}, D_{2n}, \ldots, D_{nn}\), are then i.i.d. \(\chi^2_2\) under \(H_0\). (cf. Barlow and Proschan (1981), Corollary 2.6, p. 60). Following LS(1979), we use a result due to Box (1954) to find

\[
(3.20) \quad P[V^*_n > x] = P[\sum_{k=1}^{n} (\alpha_{kn} - x)D_{kn} > 0],
\]
or

\[ P[V_n > x] = \sum_{k=1}^{n} \delta_k \prod_{j=1}^{n} \frac{(\alpha_{kn} - x)/(\alpha_{kn} - \alpha_{jn})}{(\alpha_{kn} - x)/(\alpha_{kn} - \alpha_{jn})}, \]

when \( \alpha_{jn} \neq \alpha_{kn} \) for all \( j \neq k = 1, 2, \ldots, n \), and,

\[ P[V_n > x] = \sum_{k=1}^{m} \delta_{2k} \left\{ \left[ \prod_{j=1}^{n} \frac{[(\bar{\alpha}_{(2k)n} - x)/(\bar{\alpha}_{(2k)n} - \bar{\alpha}_{jn})]}{[(\bar{\alpha}_{(2k)n} - x)/(\bar{\alpha}_{(2k)n} - \bar{\alpha}_{jn})]} \right] \right. \\
\left. \cdot \left[ 1 - \sum_{j=1}^{n} \frac{[(\bar{\alpha}_{jn} - x)/(\bar{\alpha}_{(2k)n} - \bar{\alpha}_{jn})]}{[(\bar{\alpha}_{jn} - x)/(\bar{\alpha}_{jn} - \bar{\alpha}_{jn})]} \right] \right\} \\
+ \sum_{k=2m+1}^{n} \delta_k \left\{ \prod_{j=1}^{n} \frac{[(\bar{\alpha}_{kn} - x)/(\bar{\alpha}_{kn} - \bar{\alpha}_{jn})]}{[(\bar{\alpha}_{kn} - x)/(\bar{\alpha}_{kn} - \bar{\alpha}_{jn})]} \right\}, \]

when \( \bar{\alpha}_{(2k-1)n} = \bar{\alpha}_{2k}, k = 1, 2, \ldots, m \) (1 \( \leq m \leq n/2 \)) and \( \bar{\alpha}_{jn} \neq \bar{\alpha}_{kn} \) for \( j \neq k \), \( j, k \in \{2, 4, \ldots, 2m, 2m+1, 2m+2, \ldots, n\} \). (The \( \alpha_{in}'s \) are recorded as above and labeled \( \bar{\alpha}_{in}'s \).) Also,

\[ \delta_k = \begin{cases} 
1 & \text{if } \alpha_{kn}(\bar{\alpha}_{kn}) > x \\
0 & \text{if } \alpha_{kn}(\bar{\alpha}_{kn}) \leq x. 
\end{cases} \]

Tables 3.1-3.3 contain critical values of \( \bar{V}_n \) for \( \rho = 0.25, 0.5, \) and \( 0.75 \) for the sample sizes \( n = 2, \ldots, 30 \) in the lower and upper \( \alpha = 0.01, 0.05, 0.10 \) regions. (For \( \rho = 0(0.1)1, 1/3 \) and \( 2/3 \), see Guess (1983). Tables are available upon request.)
TABLE 3.1.

Exact critical values of the \( \hat{V}_n \), \( \rho = .25 \).

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TABLE 3.3.

Exact critical values of the IDMRL(\(p\)) test statistic \(\tilde{V}_n\), \(p = .75\).

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4. An example. We illustrate the use of the two tests on a data set from Bjerkedal (1960). We give a brief description of the data and then the results of the two tests. Bjerkedal (1960) studies the lifelengths of guinea pigs injected with different amounts of tubercle bacilli. (Guinea pigs are known to have a high susceptibility to human tuberculosis, which is one reason for choosing this species.) We describe the only study (M) in which animals in a single cage are under the same regimen. The regimen number is the common log of the number of bacillary units in .5 ml of the challenge solution, e.g., regimen 4.3 corresponds to $2.2 \times 10^4$ bacillary units per .5 ml $(\log_{10}(2.2 \times 10^4) = 4.342)$.

Before conducting such an experiment, it is reasonable to conjecture that the injection of tubercle bacilli causes an adverse stage of aging (DMRL). After the guinea pigs have survived this adverse stage, the guinea pigs' natural systems recoup to yield a beneficial stage (IMRL).

Hall and Wellner (1981) examine regimen 4.3 and fit a parametric distribution that is in the DIMRL class. They estimate the point at which the MRL changes trend as $\hat{\tau}_{4.3} = 91.9$ ($\tau$ corresponds to "a" in the notation of their parametric model.). We use $\hat{\rho}_{4.3} = F_n(\hat{\tau}_{4.3}) = 1/9$ as a natural estimator of $\rho_{4.3}$.

Using the information gained from regimen 4.3, we apply the DIMRL tests to regimen 5.5. (The sample sizes for regimen 4.3 and 5.5 are both 72.) For regimen 5.5, we use $\tau_{5.5} = 91.9$ and $\rho_{5.5} = 1/9$. Note that this is a reasonable a priori assertion concerning these two population parameters under regimen 5.5 since it is based on data from a closely related population.

For the DIMRL test with $\tau$ known, we get $T_n = -.6419$, $\hat{\sigma}_n^2 = 7.1072$, and $\bar{T}_n = -2.04$, yielding a P-value of .0207 in the normal approximation. For the
DIMRL test with $\rho$ known, we get $\mathbf{v}_n^* = -0.1106$, $\sigma^2(\rho) = 0.00209$, and $\mathbf{v}_n = -2.05$, yielding a $P$-value of 0.0202 in the normal approximation. Both the $\text{DIMRL}(\tau)$ and the $\text{DIMRL}(\rho)$ test procedures suggest significant evidence to reject $H_0$ in favor of the alternative $H_1'$.

Acknowledgement. We thank Dr. Joseph Travis, Department of Biological Sciences, and Dr. Duane Meeter, Department of Statistics, Florida State University, for the Deevey (1947) reference. We are also grateful to Mr. Kermit Rose of the Florida State University Computer Center for his suggestions and help with the computer program that produced the exact critical values for the $\text{DIMRL}(\rho)$ test.

REFERENCES


Given that an item is of age \( t \), the expected value of the random remaining life is called the mean residual life (MRL) at age \( t \). We propose two new nonparametric classes of life distributions for modeling aging based on MRL. The first class of life distributions consists of those with "increasing initially, then decreasing mean residual life" (IDMRL). The IDMRL class models aging that is initially beneficial, then adverse. The second class, "decreasing, then increasing mean residual life" (DIMRL), models aging that is initially adverse, then beneficial. We present situations where IDMRL (DIMRL) distributions are useful models. We propose two testing procedures for \( H_0 \): constant MRL (i.e., exponentiality) versus \( H_1 \): IDMRL, but not constant MRL (or \( H_1 \): DIMRL, but not constant MRL). The first testing procedure assumes the turning point, \( \tau \), from IMRL to DMRL is specified by the user or is known. Our IDMRL(\( \tau \)) test statistic, \( T_n \), is a differentiable statistical function of order 1; thus, \( T_n \), suitably normalized, is asymptotically normal. The second procedure assumes knowledge of the proportion, \( \rho \), of the population that "dies" at \( \alpha \) before the turning point (knowledge of \( \tau \) itself is not assumed). We use L-statistic theory to show our IDML(\( \rho \)) test statistic, \( V_n \), appropriately normalized, is asymptotically normal. The exact null distribution of \( V_n \) is established. Finally, for each procedure an application is given.