SOME IMPERFECT MAINTENANCE MODELS

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ABSTRACT

We develop optimum policies for several maintenance models based on the imperfect repair model of Brown and Proschan (J. Appl. Prob., 1983). In each of these we assume that planned replacement is perfectly executed. We assume further that unplanned maintenance is perfect with probability $p$ and is imperfect (minimal repair) with probability $1 - p$. We also assume that the cost of planned maintenance is higher than the cost of unplanned maintenance. In all our models we assume that after each unplanned repair there is an inspection requiring negligible time and yielding perfect information as to whether the repair is perfect or just minimal.
1. Introduction.

During the past twenty-five years a large number of articles on maintenance policies have appeared. Most of these papers overlook two important factors in real-world maintenance operations, the possibility of errors on the part of the maintenance performer and limitations, physical or otherwise, which make complete overhaul of the unit needing repair impossible.

Recently, however, several authors have treated imperfect maintenance, that is maintenance in which one or both of the factors just mentioned plays a role. T. Nakagawa in [5] discusses several models in which the repaired unit never has effective age zero and several other models in which the maintenance performer accomplishes planned periodic maintenance perfectly, i.e., the repaired unit is as good as new, with probability $p$ and performs only minimal repair (The unit is repaired so that it functions again, but it has the same failure rate and the same effective age as at the time of failure.) with probability $1 - p$. For the latter models, Nakagawa also assumes that unplanned maintenance, the repair of intermittent failures, is always perfect. Two other authors, M. Brown and F. Proschan, discuss general features of imperfect maintenance and inspection in [4] and develop properties of an imperfect repair model in [3]. For their imperfect repair model, the authors assume that unplanned repair is perfect with probability $p$ and is minimal repair with probability $q = 1 - p$.

The main interest of the authors in [3] is in studying properties of the distribution $F_p$ of the time between perfect repairs. For example, they show that the "failure rate" function of $F_p$ is $pr$ where $r$ is the failure rate function of the life distribution $F$ of the unit. They also show that if $F$ belongs to certain life distribution classes based on various notions of
aging, such as the classes of IFR and DFR distributions, then so does $F_p$. Finally, they prove stochastic inequalities concerned with $F_p$ and related random variables.

Our concern in the present paper is different. We develop several maintenance optimization models based on the imperfect repair model of Brown and Proschan. In each of these we assume that planned replacement is always perfect. We assume further that unplanned maintenance is perfect with probability $p$ and is imperfect (minimal repair) with probability $q = 1 - p$. We also assume that the cost of planned maintenance is higher than the cost of unplanned maintenance. Finally, in all of our models, we assume that after each unplanned repair there is an inspection, which takes negligible time and whose cost is included in the repair cost, which enables the operator of the unit to determine whether the repair was perfect or just minimal.

One possible interpretation of our models is the following. For unplanned maintenance the unit is repaired by a minimally trained or equipped repairman, whose fee is low; for planned maintenance, the unit is repaired by an extensively trained and well-equipped repairman, whose fee is rather high.

For information about minimal repair and other maintenance policies and for comparisons with our results in the imperfect maintenance setting, we refer the reader to Chapter 4 of [1].

We end the introduction to this paper by giving notation that we shall use and assumptions that we shall make throughout this article. The life distribution of the unit under consideration will be denoted by $F$. The function $\bar{F}$ is $1 - F$. We assume throughout that $F$ is absolutely continuous with density $f$ and that $F(0) = 0$. The failure rate, $r(t) = f(t)(\bar{F}(t))^{-1}$ is assumed to be continuous and increasing except in Theorem 5.2. The cumulative hazard
function is

\[ R(t) = \int_0^t r(u) \, du. \]

Note the relationship

\[ \tilde{F}(x) = e^{-R(x)}. \]

The distribution function of the time between successive perfect repairs, called \( F_p \) in [3], will be denoted by \( G \). We shall use the fact, proved in [3], that \( \tilde{G} = F^p \). The density of \( G \) is herein denoted by \( g \). Other notations that we shall use are

\[ r(\infty) = \lim_{t \to \infty} r(t), \]

\[ \mu = \int_0^\infty F(x) \, dx, \quad \gamma = \int_0^\infty F^p(x) \, dx, \]

and

\[ A(k) = \frac{1}{k!} \int_0^\infty e^{-R(x)} x^k \, dx \]

for \( k = 0, 1, 2, \ldots \). Note that \( A(0) = \mu \). For convenience we shall assume that \( p > 0 \) throughout this paper. It will be seen that our results apply to the case \( p = 0 \) also; we merely wish to avoid having to write down separate formulas for the case \( p = 0 \). Finally, we assume that repairs and replacements or overhauls take negligible time.
2. Model I.

Our first model, Model I, is a modified age replacement model. The unit under consideration is replaced at a cost of $c_2$ when it has effective age $T$ (a constant). Intermittent failures are corrected at a cost of $c_1$ ($c_1 < c_2$) by a repairman who performs perfect repair with probability $p$ and imperfect (minimal) repair with probability $q = 1 - p$. As stated in the introduction, we assume that each such repair is followed by an inspection, taking negligible time and at no extra cost, to determine whether or not the repair was perfect. The problem is to determine $T$ so as to minimize $L(T)$, the long-run expected cost per unit time.

We now present a lemma, two theorems, and an example. In Theorem 2.2, we give a formula for $L(T)$, and in Theorem 2.3, we discuss the optimal value of $T$. Example 2.4 concerns the Weibull distribution. It is interesting to note that the formula for $L(T)$ has the form of the long-run expected cost per unit time in a standard age replacement problem with life distribution $G$ and costs $\frac{c_1}{p}$ and $c_2$. As a result we could deduce most or all of the results in Theorem 2.3 by translating known results into our notation.

2.1 LEMMA. For the repair process described above, let $N(t)$ denote the number of failures in $[0,t)$, for $0 < t \leq T$, and let $X$ denote the time of the first perfect repair. Let $0 < x \leq T$.

(a) $P(N(x) = k | X > x) = \frac{(qR(x))^k e^{-qR(x)}}{k!}$, for $k = 0, 1, 2, \ldots$.

(b) $P(N(x) = k | X = x) = \frac{(qR(x))^k e^{-qR(x)}}{k!}$, for $k = 0, 1, 2, \ldots$.

The conditional mean in both cases (a) and (b) is $qR(x)$. 
PROOF. We prove (a). The proof for (b) is similar. Using Bayes' Rule, we obtain

\[
P(N(x)=k|X>x) = \frac{P(N(x)=k)P(X>x|N(x)=k)}{\sum_j P(N(x)=j)P(X>x|N(x)=j)}
\]

Since \( x < T \), \( N(x) \) has a Poisson distribution with mean \( R(x) \). Thus

\[
P(N(x)=k|X>x) = \frac{(R(x))^ke^{-R(x)}}{k!q^k} \sum_j \frac{R(x)^j e^{-R(x)}}{j!q^j} = \frac{(qR(x))^ke^{-qR(x)}}{k!}.
\]

The mean of this Poisson random variable is \( qR(x) \).

2.2 THEOREM.

\[
L(T) = \frac{c_1}{P} G(T) + c_2 \overline{\tilde{c}}(T)
\]

\[
= \frac{c_1}{T} \int_0^T \tilde{c}(x)dx.
\]

PROOF. The times of perfect repair are renewal times in a renewal process with interarrival distribution \( G \). From renewal theory, we have that

\[
L(T) = \frac{C(T)}{D(T)},
\]

where \( C(T) \) is the expected cost per renewal cycle and \( D(T) \) is the expected duration of a renewal cycle. It is clear that

\[
D(T) = \int_0^T \tilde{c}(x)dx.
\]

From Lemma 2.1, it follows that
\[ C(T) = c_1 \int_0^T \left[(1+qR(x))g(x)dx + (c_1qR(T)+c_2)\tilde{c}(T)\right].\]

Noting that \( R(x) = -\frac{\ln \tilde{G}(x)}{p} \) and using the substitution \( u = \tilde{c}(x) \), we obtain that

\[ \int_0^T R(x)g(x)dx = -\tilde{c}(T)R(T) + \frac{G(T)}{p}, \]

from which it follows that

\[ C(T) = \frac{c_1}{p} G(T) + c_2 \tilde{c}(T). \]

2.3 THEOREM. Assume, in addition to the hypotheses given in Section 1, that \( r \) is differentiable.

Let

\[ H(T) = (c_1 - pc_2) \left[ r(T) \int_0^T \tilde{c}(x) dx + \frac{1}{p} \tilde{c}(T) \right] - \frac{c_1}{p}. \]

(a) \( L^*(T) = \tilde{c}(T) \left[ \int_0^T \tilde{c}(x) dx \right]^{-2} H(T). \)

(b) If \( c_1 < pc_2 \), the optimal replacement age is \( T = \infty \).

(c) Assume that \( c_1 > pc_2 \). Then an optimal replacement age \( T^* \) exists. \( (T^* \) may be infinite.)

(d) The equation \( H(T) = 0 \) has at most one solution. If a solution exists, it must be \( T^* \).

(e) Suppose that \( c_1 > pc_2 \) and that

\[ r(\infty) > \frac{c_1}{pY(c_1 - pc_2)}. \]

Then \( T^* \) is finite.
PROOF. (a) Differentiate and use the identity $g = \text{pr} \tilde{g}$.

(b) If $c_1 \leq pc_2$, $L'(T) < 0$ for $T > 0$.

(c) First note that $H$ is increasing since an easy computation shows that

$$H'(T) = (c_1 - pc_2)L'(T) \int_0^T \tilde{g}(x) dx.$$ 

Also note that $H(0) = -c_2 < 0$. Thus if the equation $H(T) = 0$ has a solution $T^*$, then $L'$ is negative on $(0, T^*)$ and positive on $(T^*, \infty)$. If $H(T) = 0$ has no solution, then $L$ is decreasing and $T^* = \infty$.

(d) Look at $H'(T)$. Also note that if $H'(T) = 0$ has a solution, then $c_1 > pc_2$.

(e) We must show that the equation $H(T) = 0$ has a solution. The inequality involving $r(\omega)$ gives us what we need since it implies that

$$\lim_{t \to \infty} H(t) > 0.$$ 

(Recall that $H(0) < 0$.)

2.4 EXAMPLE. Let $F$ be an IFR Weibull distribution. Thus, for some $\lambda > 0$ and $\alpha \geq 1$,

$$\bar{F}(x) = e^{-\lambda x^\alpha}, \quad x \geq 0.$$ 

Assume that $c_1 > pc_2$. By Theorem 2.3, if $\alpha > 1$, a finite optimal replacement age $T^*$ exists, since $r(\omega) = \infty$. $T^*$ is the unique solution of the equation $H(T) = 0$. This equation reduces to the equation

$$\lambda \alpha T^{\alpha-1} \int_0^T e^{-p\lambda x^\alpha} dx + \frac{1}{p} e^{-p\lambda T^\alpha} = \frac{c_1}{p(c_1 - pc_2)},$$

which is easily solved by numerical methods (for specific $\lambda$ and $\alpha$).
If $a = 1$, the equation above has no solution (except in the uninteresting case $c_2 = 0$). Thus, for the exponential distribution, the optimal replacement age is $T^* = \infty$.

2.5 REMARK. Let $Q(T)$ denote the number of repairs (not including replacement) in one renewal cycle. We can give another derivation of the formula for $L(T)$ by first calculating the distribution of $Q(T)$. We obtain

$$P(Q(T) = 0) = \bar{F}(T)$$

and

$$P(Q(T) = k) = q^{k-1} p \left[ 1 - \sum_{j=0}^{k-1} e^{-R(T)} \frac{R(T)^j}{j!} \right] + q^k \frac{R(T)^k e^{-R(T)}}{k!} \quad \text{for } k \geq 1.$$ 

We illustrate the computation for the case $k = 2$.

$$P(Q(T) = 2) = qp \int_0^T \int_0^{t_1} \frac{f(t_2)}{\bar{F}(t_1)} \, dt_2 \, dt_1$$

$$+ q^2 \int_0^T \int_0^{t_1} \frac{f(t_2)}{\bar{F}(t_1) \bar{F}(t_2)} \, dt_2 \, dt_1$$

$$= qp(1 - e^{-R(T)}) (1 + R(T)) + q^2 \frac{e^{-R(T)} R^2(T)}{2}.$$ 

We can also obtain the formula for $P(Q(T) = k)$ by thinking about a process in which failures are corrected by minimal repair and in which no replacement occurs and an independent coin-tossing process in which the probability of heads is $p$. Since the failure process is a nonhomogeneous Poisson process with mean value function $R(t)$, it is clear that the claimed formula may be obtained.
3. **Model II.**

In this section we discuss Model II, an imperfect repair version of a model developed in [6]. Model II is the same as Model I except in one respect: we replace (or perfectly overhaul) the unit on the next failure after k-1 successive imperfect repairs (instead of at age T, as in Model I). The objective is to choose k so as to minimize \( L(k) \), the long-run expected cost per unit time.

The next result contains a formula for \( L(k) \). Recall the assumptions stated in Section 1.

3.1 THEOREM.

\[
L(k) = \frac{c_1}{D} \left( 1 - q^{k-1} \right) + c_2 q^{k-1} \sum_{\ell=0}^{k-1} q^\ell A(\ell).
\]

PROOF. From renewal theory, we have that

\[
L(k) = \frac{C(k)}{D(k)},
\]

where \( C(k) \) and \( D(k) \) are the long-run expected cost per renewal cycle and the expected duration of a renewal cycle, respectively.

We compute \( C(k) \) first. A renewal cycle can end in one of two ways:

1. there are \( k-1 \) successive imperfect repairs and another failure;
2. there are \( j-1 \) successive imperfect repairs followed by a perfect repair, for some integer \( j \), \( 1 \leq j \leq k-1 \). Considering the probabilities associated with these events, we obtain

\[
C(k) = (c_1 (k-1) + c_2) q^{k-1} + \sum_{j=1}^{k-1} c_j p_j q^{j-1}.
\]
Simplifying, we get

\[ C(k) = \frac{c_1}{p} (1 - q^{k-1}) + c_2 q^{k-1}. \]

Next we compute \( D(k) \). Toward this end, for \( j = 1, 2, 3, \ldots \), let \( T_j \) denote the time of the \( j \)th failure of a unit with life distribution \( F \), assuming that failures are corrected by minimal repair and that no overhauling or replacement is done. Let \( E_j = E(T_j) \). Since the number \( N(t) \) of failures in \([0, t]\) is a nonhomogeneous Poisson process with intensity function \( r(t) \) and mean value function \( R(t) \),

\[ E_j = \int_0^\infty P(T_j > t) dt = \int_0^\infty P(N(t) \leq j - 1) dt = \int_0^j \sum_{\ell=0}^{j-1} A(\ell). \text{ (See Section 1.)} \]

Thus, thinking about how a renewal cycle can terminate, we get

\[
D(k) = q k^{-1} E_k + \sum_{j=1}^{k-1} q^{j-1} p E_j
\]

\[
= q k^{-1} \sum_{\ell=0}^{k-1} A(\ell) + \sum_{j=1}^{k-1} q^{j-1} p \left( \sum_{\ell=0}^{j-1} A(\ell) \right) = \sum_{\ell=0}^{k-1} q^{\ell} A(\ell).
\]

Thus \( L(k) \) has the claimed form.

To discuss the optimization problem associated with Model II, we need a lemma due to T. Nakagawa [6].

3.2 LEMMA. The sequence \((A(k))\) is decreasing and

\[
\lim_{k \to \infty} A(k) = \frac{1}{r(\infty)}.
\]
3.3 THEOREM. Let \( \beta = \frac{c_1 q}{c_1 - pc_2} \) and

\[
Q(k) = \frac{pD(k)}{A(k)} + q^k
\]

for \( k = 1, 2, 3, \ldots \).

(a) The sequence \( (Q(k)) \) is increasing.

(b) If \( c_1 \leq pc_2 \), the sequence \( (L(k)) \) is decreasing, and the optimal value of \( k \) is \( k = \infty \).

(c) Suppose that \( c_1 > pc_2 \). Then for any positive integer \( k \), \( L(k+1) \geq L(k) \) if and only if \( Q(k) \geq \beta \). Also, an optimal value of \( k \), possibly infinity, exists.

(d) Suppose that \( c_1 > pc_2 \) and that \( r(\infty) > \beta (p \gamma)^{-1} \). Then a finite optimal value \( k^* \) of \( k \) exists. Furthermore,

\[
k^* = \min\{k : Q(k) \geq \beta\}.
\]

PROOF. (a) This follows from Lemma 3.2 and the identity \( D(k+1) = D(k) + q^k A(k) \).

(b) It is easy to see that for any positive integer \( k \),

\[
L(k+1) \geq L(k) \text{ if and only if } (c_1 - pc_2)Q(k) \geq c_1 q.
\]

From (1) it follows that \( L(k+1) < L(k) \) for all \( k \) if \( c_1 - pc_2 \leq 0 \).

(c) The first statement is immediate from (1). If the inequality \( Q(k) \geq \beta \) holds for some value of \( k \), there is a finite optimal value of \( k \); otherwise, the sequence \( (L(k)) \) is strictly decreasing, and the optimal value of \( k \) is \( k = \infty \).

(d) Lemma 3.2 asserts that \( \lim_{k \to \infty} A(k) = (r(\infty))^{-1} \). It follows from the Monotone Convergence Theorem that \( \lim_{k \to \infty} D(k) = \gamma \). Using these two facts and
the second inequality, we conclude that \( \lim_{k \to \infty} Q(k) > \beta \). From part (b), it follows that a finite optimal value \( k^* \) of \( k \) exists and that

\[
k^* = \min\{k : Q(k) \geq \beta\}.
\]

This concludes the proof of Theorem 3.3.

3.4 REMARK. Recalling the definition of \( \beta \) in Theorem 3.3, suppose that \( r(\infty) > \beta(pu)^{-1} \). It follows immediately that \( r(\infty) > \beta(py)^{-1} \) and, hence, from Theorem 3.3 that a finite optimal value \( k^* \) of \( k \) exists. It is possible to compute an upper bound \( k^{**} \) of \( k^* \), in this special case, by a method we now discuss. The method is useful because \( k^{**} \) is sometimes easier to compute.

Let

\[
k^{**} = \min\{k : (A(k))^{-1} > (pu)^{-1} \beta\}.
\]  

(2)

Since \( \lim_{k \to \infty} (A(k))^{-1} = r(\infty) \), \( k^{**} \) is well-defined. The inequalities

\[
Q(k) > \frac{pD(1)}{A(k)} = \frac{pu}{A(k)} > \beta
\]

for \( k \geq k^{**} \) show that \( k^{**} \geq k^* \).

3.5 EXAMPLE. Let \( F(x) = 1 - e^{-(\lambda x)^\alpha} \) for some \( \alpha > 1 \) and some \( \lambda > 0 \) and all \( x \geq 0 \). We find that

\[
A(k) = \frac{\Gamma(k + 1/\alpha)}{\lambda^k k!} = \frac{\Gamma(1/\alpha)}{\lambda^{k-1}} \prod_{\ell=0}^{k-1} (\ell + 1/\alpha).
\]

Since the expression for \( D(k) \) is rather complex, it is impossible to give a simple expression for \( k^* \); in practice, \( k^* \) must be found by evaluating \( Q(k) \) and checking whether \( Q(k) \geq \beta \). We shall indicate how to find an upper bound for \( k^{**} \). To do this, we use the inequality
\[ A(k) \leq \int_0^T e^{-R(t)} \frac{R(t)^k}{k!} dt + \frac{1}{r(T)} \]  

(3)

for \( k = 1, 2, 3, \ldots \) and \( T > 0 \) which is used in the proof of Lemma 3.2 in [6]. In the present example, \( R(t) = e^{-(\lambda t)^a} \). Replacing this factor in the integral in (3) by the constant 1, we get

\[ A(k) \leq \frac{\lambda a_1 k a + 1}{(k+1)k!} + \frac{1}{a \lambda a_1 a - 1} \]  

(4)

Choose \( T \) so that

\[ a \lambda a_1 a - 1 > 2 \beta (pu)^{-1} \]

Then choose \( k \) so that, for this value of \( T \),

\[ \frac{\lambda a_1 k a + 1}{(ka+1)k!} < \frac{pu}{2 \beta} \]

Using (2) and (4), we find that

\[ k^{**} \leq k \]

4. Model III.

Our third model, based on a model in [6], deals with a unit subject to two types of failures. We assume that the unit has life distribution \( F \) and that when the unit fails, with probability \( \eta \), the failure is a Type 1 failure and that, with probability \( 1-\eta \), the failure is a Type 2 failure. When the unit has a Type 1 failure, it is repaired at a cost of \( c_1 \). With probability \( p \), this repair is perfect; otherwise, it is minimal. When the unit has a Type 2 failure or when it fails again after \( k-1 \) successive
imperfect repairs following Type 1 failures, the unit is (perfectly) overhauled or replaced at a cost $c_2$ ($c_2 > c_1$). The problem is to choose $k$ so as to minimize $L(k)$, the long-run expected cost per unit time. Since the proofs of the basic results for this model are much like those for Model II, we shall state our results without proof.

4.1 THEOREM.

$$L(k) = \frac{c_1(n-n(nq)^{k-1}) + c_2(1-n+n(nq)^k - (nq)^k)}{(1-nq) \sum_{\ell=0}^{k-1} (nq)^\ell A(\ell)}.$$ 

4.2 THEOREM. Let

$$Q(k) = \frac{(1-nq)D(k)}{A(k)} + (nq)^k$$

where

$$D(k) = \sum_{\ell=0}^{k-1} (nq)^\ell A(\ell).$$

Also let

$$\delta = \int_0^\infty (1-nq)(x)dx$$

and

$$\beta = \frac{q(c_1n + c_2) - (1-n)}{c_1 - c_2 (1-nq)}.$$ 

(a) The sequence $(Q(k))$ is increasing.

(b) If $c_1 \leq c_2 (1-nq)$, then the sequence $(L(k))$ is decreasing and the optimal value of $k$ is $k = \infty$. 
(c) Suppose that $c_1 > c_2(1-nq)$. Then, for any positive integer $k$, $L(k+1) \geq L(k)$ if and only if $Q(k) \geq \beta$. Also, an optimal value of $k$, possibly infinity, exists.

(d) Suppose that $c_1 > c_2(1-nq)$ and that $r(as) > \beta(\delta(1-nq))^{-1}$. Then a finite optimal value $k^*$ of $k$ exists. Furthermore,

$$k^* = \min\{k: Q(k) \geq \beta\}.$$

5. **Increasing Repair Costs.**

In this section we present two modifications, based on the main cost structure considered in [2], of our first and second models.

The first of these two modified models, Model IV, is exactly like Model I except that the cost of the $k^{th}$ repair, beginning with a new or perfectly repaired unit, is $a + cl$ for some constants $a > 0$ and $c > 0$. We denote the cost of replacement or perfect overhaul by $c_0$. The objective is to find the replacement age $T$ which minimizes $L(T)$, the long-run expected cost per unit time.

In Theorem 5.1 and Theorem 5.2 we give a formula for $L(T)$ and discuss minimization of $L(T)$. Then we apply these results to the exponential distribution in Example 5.3.

5.1 **Theorem.**

$$L(T) = \left(\frac{\frac{a}{p} + \frac{c}{p^2}}{p^2}\right)G(T) + \left(\frac{c_0 - cA}{p} R(T)\right)\tilde{G}(T)$$

$$\int_0^T \tilde{G}(x)dx.$$
PROOF. From renewal theory, we have that

\[ L(T) = \frac{C(T)}{D(T)}, \]

where \( C(T) \) and \( D(T) \) have the expected interpretations. \( D(T) \) is, of course,

\[ \int_{0}^{T} \bar{g}(x) \, dx, \]

just as in Model I. For \( k = 1, 2, \ldots \), let

\[ c_k = \sum_{i=1}^{k} (a + ic) = (a + \frac{c}{2})k + \frac{c}{2}k^2. \]

Then from Lemma 2.1, we obtain that

\[
C(T) = \int_{0}^{T} g(x) \left( \sum_{k=0}^{\infty} \frac{(qR(x))^k}{k!} e^{-qR(x)} c_{k+1} \right) dx
\]

\[ + \bar{g}(T) \left( c_0 + \sum_{k=1}^{\infty} \frac{(qR(T))^k}{k!} e^{-qR(T)} c_k \right). \]

Simplifying the infinite series in the two parts of the expression, we obtain that

\[
C(T) = \int_{0}^{T} g(x)((a+2c)qR(x) + \frac{c}{2}q^2R^2(x) + (a+c)) dx
\]

\[ + \bar{g}(T)(c_0 + (a+c)qR(T) + \frac{c}{2}q^2R^2(T)). \]

Using the identities

\[
\int_{0}^{T} R(x)g(x) \, dx = \frac{G(T)}{p} - \bar{g}(T)R(T)
\]
and
\[ \int_0^T R^2(x)g(x)dx = \frac{2G(T)}{p^2} - \frac{2}{p} \tilde{g}(T)R(T) - \tilde{g}(T)R^2(T), \]

we have, after simplification, that
\[ C(T) = \left( a + \frac{c}{p^2} \right) G(T) + \left( c_0 - \frac{ca}{p} R(T) \right) \tilde{g}(T), \]

from which the claimed expression for \( L(T) \) follows.

5.2 THEOREM. Assume that \( r(t) \) is differentiable and nondecreasing and that \( cq > 0 \). For \( t \geq 0 \), let

\[ H(t) = \int_0^T \tilde{g}(x)dx(a + c - c_0 p + cqR(t))r(t) - \left( a + \frac{c}{p^2} \right) G(t) + \left( \frac{ca}{p} R(t) - c_0 \right) \tilde{g}(t). \]

(a) \( L^*(t) = \tilde{g}(t) \left( \int_0^T \tilde{g}(x)dx \right)^{-2} H(t). \)

(b) For any sufficiently large number \( t \), \( L \) is increasing on \( [t, \infty) \).

Thus \( L \) achieves a global minimum value at some finite value of \( T \).

(c) If \( a + c > pc_0 \) and either \( r(0) > 0 \) or \( r \) is increasing, then \( H \) is increasing. In this case, the optimal value of \( T \) is the unique solution of the equation \( H(T) = 0 \).

PROOF. (a) This is straightforward.

(b) Note that \( R(t) \to \infty \) as \( t \to \infty \) since \( r \) is nondecreasing and not identically zero. Also note that \( \tilde{g}(t) \to 0 \) as \( t \to \infty \). Thus \( H(t) \to \infty \) as \( t \to \infty \). The conclusion then follows from part (a).
(c) Since any optimal value of $T$ is a solution of the equation $H(T) = 0$, by part (a), it suffices to verify that $H$ is increasing. We do this by computing $H'(t)$ to obtain

$$H'(t) = \int_0^T \tilde{c}(x) dx ((a + c - c_0 p + cq R(t)) r'(t) + cr^2(t)).$$

This concludes the proof of the theorem.

5.3 EXAMPLE. Let $\bar{F}(x) = e^{-\lambda x}$, for some $\lambda > 0$ and all $x \geq 0$. The optimal age $T$ at which to replace a unit with life distribution $F$, assuming that $a + c > c_0 p$ and $cq > 0$, is the unique solution of the equation $H(T) = 0$. This equation reduces to the equation

$$c q e^{-p\lambda T} + cq \lambda T = c_0 p^2 + cq.$$

The last model that we consider, Model V, is the same as Model II except that the cost of the $k^{th}$ repair, beginning with a new or perfectly repaired unit, is $a + c \xi$, for some constants $a > 0$ and $c > 0$. We denote the cost of replacement or overhaul by $c_0$. The object is to find the value of $k$ which minimizes $L(k)$, the long-run expected cost per unit time.

In the remainder of this section, we give a formula for $L(k)$ (Theorem 5.4), discuss optimization of $L(k)$ (Theorem 5.5), and present an example involving the exponential distribution (Example 5.6).

5.4 THEOREM.

$$L(k) = \frac{C(k)}{D(k)}$$

where
\[ C(k) = p^{-2}((c+ap) + (c_0 - ck - a)q^{k-1} + (ck - c + a - 2c_0)q^k + c_0q^{k+1}) \]

and

\[ D(k) = \sum_{\ell=0}^{k-1} q^\ell A(\ell). \]

**PROOF.** Use the idea used in the proof of Theorem 3.1.

5.5 THEOREM. Let

\[ Q(k) = \frac{(pD(k) + q^kA(k))(pc + ap - p^2c_0)}{A(k)} + cq^{k+1}. \]

(a) \( L(k+1) \geq L(k) \) if and only if \( Q(k) \geq (ap+c)q. \)

(b) There is a finite optimal value \( k^* \) of \( k. \)

(c) If \( a+c \geq pc_0, \) then the sequence \( (Q(k)) \) is increasing. In this case,

\[ k^* = \min\{k: Q(k) \geq (ap+c)q\}. \]

Also, the sequence \( (Q(k)) \) is eventually increasing even if \( a+c < pc_0. \)

**PROOF.** (a) This is straightforward.

(b) This is immediate from (a) since \( Q(k) \to \infty \) as \( k \to \infty. \)

(c) The inequality

\[ Q(k+1) > Q(k) \]

is equivalent to the inequality

\[ p^2cA(k)D(k+1) + (pc + ap - p^2c_0)(pD(k) + pq^kA(k))(A(k) - A(k+1)). \]
Recall (Lemma 3.2) that the sequence \( (A(k)) \) is decreasing. Thus if \( ap - p^2c_0 \geq 0 \) or if \( k \) is sufficiently large, then \( Q(k+1) > Q(k) \).

5.6 EXAMPLE. Let \( F(x) = e^{-\lambda x} \) for some \( \lambda > 0 \) and all \( x \geq 0 \). A calculation involving the gamma function shows that

\[
A(k) = \frac{1}{\lambda} \quad \text{and} \quad D(k) = \frac{1-q^k}{p\lambda}.
\]

The inequality \( Q(k) \geq (ap+c)q \) is equivalent to the simple inequality

\[
pck + cq^{k+1} \geq cq - p^2(a-c_0),
\]

whose smallest solution is the optimal value \( k^* \).

REFERENCES


Some Imperfect Maintenance Models

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Maintenance models, optimum maintenance policy, imperfect repair, minimal repair, hazard function, renewal theory.

We develop optimum policies for several maintenance models based on the imperfect repair models of Brown and Proschan (J. Appl. Prob., 1983). In each of these we assume that planned replacement is perfectly executed. We assume further that unplanned maintenance is perfect with probability \( p \) and is imperfect (minimal repair) with probability \( 1 - p \). We also assume that the cost of planned maintenance is higher than the cost of unplanned maintenance. In all our models we assume that after each unplanned repair there is an inspection requiring negligible time and yielding perfect information as to whether the repair is perfect or just minimal.