ON THE LIMIT BEHAVIOR OF CERTAIN QUANTITIES
IN A SUBCRITICAL STORAGE MODEL

by

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Abstract

The limit behavior of the content of a subcritical storage model defined on a semi-Markov process is examined. This is achieved by creating a renewal equation using a regeneration point \((i_0, 0)\) of the process. By showing that the expected return time to \((i_0, 0)\) is finite, the conditions needed for the basic renewal theorem are established. The joint asymptotic distribution of the content of the storage at time \(t\) and the accumulated amount of the unmet (lost) demands during \((0, t)\) is then established by showing the asymptotic independence of these two.
1. **INTRODUCTION.**

The present work deals with a storage model which allows both inputs as well as releases occurring in random amounts and at random times according to an underlying semi-Markov process. While the reader may find other types of storage models elsewhere in the literature (see Moran [12], Prabhu [15], Lloyd [10], Ali Khan and Gani [1], for references) the present model is along the lines of Puri and Woolford [17], which itself is a generalization of a model considered previously by Senturia and Puri ([19], [20]) and Balagopal [3]. A special case of these models can be found in an earlier work (see Puri and Senturia [16]) which relates such models to a live situation arising in biology. The purpose of the present work is to answer a question left open by these authors in the so-called 'subcritical' case of these models and is concerned with the limit behavior of the storage level for the continuous-time case.

All our random variables will be considered as defined on a given underlying basic probability space \((Ω, A, P)\). Let \(J\) be a subset of nonnegative integers, and \(\{(X_n, T_n), n=0,1,2,...\}\) a Markov renewal process with semi-Markov matrix \(A(t) = (A_{ij}(t))\), (see Cinar [5]), where for all \(i,j ∈ J, t ≥ 0\),

\[
P(X_n = j, T_n - T_{n-1} ≤ t|T_0, X_0, T_1, X_1, ..., T_{n-1}, X_n = i) = A_{ij}(t).
\]

Then \(\{X_n, n=0,1,2,...\}\) is a positive recurrent, aperiodic irreducible Markov chain with state space \(J\) and transition matrix \(P = (p_{ij}) = (A_{ij}(∞))\). Also the moments of the sojourn times in a state \(i ∈ J\) are given by

\[
m_i(k) = \int_0^∞ t^k \sum_{j ∈ J} A_{ij}(dt), \quad k=1,2,...,
\]

where for simplicity we shall write \(m_i^{(1)} = m_i\). With each \(i ∈ J\) we associate a sequence \(\{U_n(i), n=0,1,2,...\}\) of I.I.D. real valued random variables, which are assumed to be independent of \(\{(X_n, T_n), n=0,1,2,...\}\) and of \(\{U_n(j), n=0,1,2,...\}\) for \(j ≠ i\), with \(E|U_1(i)| < ∞, \forall i ∈ J\). With these we define for \(n ≥ 1\),
\[
\begin{align*}
Z_n &= \max(0, Z_{n-1} + u_n(x_n)) \\
L_n &= L_{n-1} - \min(0, Z_{n-1} + u_n(x_n)),
\end{align*}
\]

with \( L_0 \equiv 0 \). From (1.3) it easily follows that

\[
\begin{align*}
Z_n &= \max[Z_0 + \sum_{i=1}^{n} u_i(x_i), \max_{1 \leq j \leq n} (\sum_{i=j+1}^{n} u_i(x_i))], \\
L_n &= Z_n - Z_0 - \sum_{i=1}^{n} u_i(x_i).
\end{align*}
\]

Finally to define these quantities for an arbitrary time \( t \), we let

\[
(X(t), Z(t), L(t)) \equiv (X_M(t), Z_M(t), L_M(t)),
\]

where

\[
M(t) = \sup\{ n : T_n \leq t \}.
\]

Also, we let

\[
\beta = \sum_{i \in J} \pi_i m_i,
\]

and

\[
E_{\pi} u = \sum_{i \in J} \pi_i E(u_i(i)).
\]

We shall adopt the terminology of saying that we are in the subcritical, critical or the supercritical case according as \( E_{\pi} u \) is less than, equal to or greater than zero. In [3], [17], [19] and [20], various authors studied the limit behavior of quantities such as \( Z(t) \) and \( L(t) \) but only for the critical and supercritical cases.
The methods used by these authors did not lend themselves to study the joint limit behavior of \((Z(t), L(t))\) for the subcritical case. Consequently this question was left open and will now be studied using a different approach based on a renewal equation argument. Thus throughout the paper we assume that \(E_U < 0\).

Section 2 deals with some results on arbitrary state Markov renewal processes. In Section 3, we establish the ergodicity of the process \(\{(X_n, Z_n)\}\). Section 4 deals with the study of asymptotic behavior of \(\{X(t), Z(t)\}\) via a renewal equation.

Finally in Section 5, the asymptotic independence of \(Z(t)\) and \(L(t)\) appropriately normalized is established. The joint asymptotic behavior of \(Z(t)\) and the normalized \(L(t)\) follows then from those of their marginals.

2. SEMI-MARKOV PROCESSES ON ARBITRARY STATE SPACE.

The purpose of this section is to derive some results about the times of first visit for semi-Markov processes with arbitrary state spaces. These results will be used in the later sections.

Let \(\{X_n\}\) be a Markov chain which has values in some arbitrary state space \((S, \mathcal{F})\) with \(P(x, A)\) a regular version of the stationary transition probabilities. Let us then recursively define for \(x \in S, A \in \mathcal{F}\), \(n > 1,\)

\[
(2.1) \quad P^n(x, A) = P(X_n \in A | X_0 = x) = \int_S P(x, dy) P^{n-1}(y, A).
\]

Let \(\phi\) be a non-trivial \(\sigma\)-finite measure on \(\mathcal{F}\).

**DEFINITION.** \(\{X_n\}\) is called \(\phi\)-irreducible if, whenever \(\phi(A) > 0\), for \(A \in \mathcal{F}\), then

\[
\sum_{n=1}^{\infty} 2^{-n} P^n(x, A) > 0, \text{ for all } x \in S.
\]

**DEFINITION.** A \(\sigma\)-finite, non-trivial measure \(\mu\) on \(\mathcal{F}\) is called subinvariant for \(\{X_n\}\) if \(\mu(A) = \int \mu(dy) P(y, A)\), for all \(A \in \mathcal{F}\), and called invariant if strict equality holds.
DEFINITION. If there is a finite invariant measure $\pi$ on $F$, we call $\{X_n\}$ ergodic, and we call $\pi$ the stationary measure.

Let $\{(X_n, T_n)\}$ be a semi-Markov process defined on the arbitrary state space $(S, F)$, where for all $y \in S$, $A \in F$, $t \geq 0$, $H_{yA}(t)$ is a regular version with respect to $\phi$ of the transition function, that is

$$H_{yA}(t) = P(X_{n \in A}, T_{n-1} \leq t | X_{n-1} = y).$$

For details of the definition of the semi-Markov process, see (Ginlar [6]). For all $A \in F$, let

$$T_A = \inf \{t > T_{1}: X(t) \in A\}.$$

Let us define

$$
\begin{align*}
\text{i)} & \quad P(x, A) = P(X_n \in A | X_{n-1} = x) = H_{xA}(\infty) \\
\text{ii)} & \quad E_H^n(x, A) = P(X_n = x, X_{n-1} \notin B, \ldots, X_1 \notin B, T_n \leq t | X_0 = x) \\
\quad & = \int_{\mathbb{B}} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} H_{x y_1} (dt_1) \int_{\mathbb{C}} H_{y_1 y_2} (dt_2) \cdots \int_{\mathbb{C}} H_{y_{n-2} y_{n-1}} (dt_{n-1}) \int_{\mathbb{C}} H_{y_{n-1} A} (t - t_i) \\
\text{iii)} & \quad P^n(x, A) = P(X_n = x, X_{n-1} \notin B, \ldots, X_1 \notin B | X_0 = x) = H^n_{x A}(\infty).
\end{align*}
$$

Also, for all $x \in S$, and any random variable $Z$, let us define

$$
\begin{align*}
\text{i)} & \quad E_x(Z) = E(Z | X(0) = x) \\
\text{ii)} & \quad P_x(Z \in A) = P(Z \in A | X(0) = x).
\end{align*}
$$

The following lemma, although straightforward, appears to be new, since we were unable to find it in the literature.
LEMMA 2.1. For a semi-Markov process \\{(X_n, T_n)\}, if \(A \in \mathcal{F}\) with \(\phi(A) > 0\), then for all \(x \in S\),
\[
E_x(T^A) = E_x(T_1) + \sum_{n=1}^{\infty} \int_A \gamma^n(x, dy) E_y(T_1).
\]

PROOF. Clearly, we have
\[
P_x(T^A > t) = P_x(T_1 > t) + \sum_{n=1}^{\infty} \int_A \gamma^n_x(y, dy) P_y(T_1 > t - s).
\]
Thus,
\[
E_x(T^A) = \int_0^\infty P_x(T^A > t) \, dt
\]
\[
= E_x(T_1) + \sum_{n=1}^{\infty} \int_A \int_0^\infty \int_0^\infty \gamma^n_x(y, dy) \, P_y(T_1 > t - s) \, dt
\]
\[
= E_x(T_1) + \sum_{n=1}^{\infty} \int_A \gamma^n(x, dy) E_y(T_1).
\]

Using Lemma 2.1, the following theorem can be established.

THEOREM 2.2. If the Markov chain \\{X_n\} is ergodic, then for \(A \in \mathcal{F}\) where \(\phi(A) > 0\), we have
\[
\int_A E_x(T^A) \pi(dx) \leq \int_S E_y(T_1) \pi(dy),
\]
with equality if for \(C = \{x \in A^c : \lim_{n \to \infty} \gamma^n(x, A^c) = 0\}\),
we have \(\mu(A^c/C) = 0\).

The idea of the proof is as follows. For our measure \(\pi\), from (Orey [14], p. 33) we have, for \(N \geq 1\) and \(E \in \mathcal{F}\),
\[
\pi(E) = \sum_{n=1}^{N} \int_A \gamma^n(x, E) \pi(dx) + \int_A \gamma^{N+1}(x, E) \pi(dx).
\]
From this, we have \(\pi(E) \geq \int(\sum_{n=1}^{\infty} \gamma^n(x, E)) \pi(dx)\), and combining this and Lemma 2.1, the theorem follows straightforwardly. Note: This result generalizes a well-known result (see Orey [14]), which states if \(\{X_n\}\) is an ergodic Markov chain on an arbitrary state space, then \(\int_A E_x(T^A) \pi(dx) = \int_S \pi(dx) = 1\); and also that of Çinlar [5], which states for \(\{(X_n, T_n)\}\) a semi-Markov process on a denumerable state space \(J\), \(\pi_i E^{i}(T^i_j) = \sum_{j \in J} E^{i}(T^i_j)\), for \(i \in J\).
3. THE ERGODICITY OF \{ (X_n, Z_n) \}.

Let \{ (\hat{X}_n, \hat{T}_n) \} be the semi-Markov process with denumerable state space \( J \), with semi-Markov matrix \( A(t) \) as in (1.1). Let \{ (\hat{X}_n, \hat{T}_n) \} be a semi-Markov process, independent of any variables thus far defined, with semi-Markov matrix \( \hat{A}(t) = (\hat{A}_{ij}(t)) \) defined for each \( i, j \in J \) by

\[
\hat{A}_{ij}(t) = \frac{\pi_j}{\pi_i} A_{ji}(t).
\]

Let the initial distribution of \( \hat{X}_0 \) be the stationary measure \( \pi \).

**Definition.** \{ (\hat{X}_n, \hat{T}_n) \} as defined above is said to be the dual semi-Markov process for \{ (X_n, T_n) \}. Likewise \( \hat{X}_n \) is called the dual Markov chain for \( X_n \).

Let \( \overline{X}_n \) be another Markov chain, independent of all random variables thus far defined, with transition matrix \( \mathcal{P} \), and initial distribution \( \mathcal{N} \).

**Definition.** The Markov chain \( \overline{X}_n \) as defined above is called the auxiliary Markov chain for \{ \( X_n \) \}.

For a listing of some of the useful properties of the dual semi-Markov process and the auxiliary Markov chain see (Çinlar [5], Woolford [22], and Hoel, Port and Stone [7]).

Let us now define

\[
\begin{align*}
S_n &= \sum_{i=1}^{n} U_i(X_i) \\
\hat{S}_n &= \sum_{i=1}^{n} U_i(\hat{X}_i) \\
\overline{S}_n &= \sum_{i=1}^{n} U_i(\overline{X}_i)
\end{align*}
\]

(3.2)

Using this and (1.4) it follows that

\[
Z_n = \max(Z_0 + S_n, \max_{1 \leq j \leq n} (S_n - S_j)).
\]

The following lemmas are needed for establishing the ergodicity of the process \{ (X_n, Z_n) \).
LEMMA 3.1. If $E_n U < 0$, then there exists an $i_0$, $n$, $\epsilon > 0$ and $\delta > 0$ such that

$$P(S_n < -\epsilon, \max(0, S_1, \ldots, S_n) = 0, X_n = i_0 | X_0 = i_0) > \delta.$$ 

COROLLARY 3.1. If $E_n U < 0$, then for $i_0$, $n$, $\epsilon$, and $\delta$ as in Lemma 3.1., we have

$$P(S_{nN} < -\epsilon N, \max_{0 \leq j \leq nN} (S_j) = 0, X_{nN} = i_0 | X_0 = i_0) > \delta N.$$ 

LEMMA 3.2. If $E_n U < 0$ and $X_0$ has initial distribution $\pi$, then there exists an $n > 0$ such that for all $A > 0$, there is an $N_A$ such that for $n > N_A$, we have

$$P(\max_{0 \leq j \leq n} (S_j), S_n + A = 0) > n.$$ 

The idea to prove the above results is to notice for any initial distribution of $X_n$, $n^{-1} S_n \rightarrow E_n U$ a.s., where $E_n U < 0$. Thus, for any $i$, there is an $n$ and a sequence $j_1, j_2, \ldots, j_{n-1}$ such that

$$P(S_n < -\epsilon, X_n = i, X_{n-1} = j_{n-1}, \ldots, X_1 = j_1 | X_0 = i) > n$$

for some $n > 0$. We can then show that for some $m$ using a cyclic permutation of the cycle $i, j_1, \ldots, j_{n-1}$ starting from $i_0 = j_m$, we get Lemma 3.1. Then, since $sup(S_j) < \infty$ a.s., it can be shown that for a determined number of the above cycles, the sequence $\{S_n\}$ will be forced sufficiently negative that the sequence will remain as negative as desired with positive probability, which yields Lemma 3.2.

LEMMA 3.3. If $E_n U < 0$, then for any arbitrary distribution of $(X_0, Z_0)$, there exists a $\delta > 0$ and an $N$ such that for $n > N$, we have $P(Z_n = 0) > \delta$.

PROOF. Let $T = \inf \{n > 0: X_n = \bar{X}_n\}$. Then $T < \infty$, a.s. (see [5]). Now choose $K_1, K_2$, such that $P(T > K_1) < n/3$ and $P(Z_{K_1} > K_2) < n/3$, where $n$ is as in Lemma 3.2. Then for $n > K_1$, we have

$$(3.3) \quad P(Z_n = 0) \geq P(Z_n = 0, T \leq K_1, Z_{K_1} \leq K_2)$$

$$\geq P(\max(K_2 + S_n - S_i, \max_{K_1 < i \leq n} (S_n - S_i)) = 0, T \leq K_1) - P(Z_{K_1} > K_2).$$
Using the definitions of $\overline{S}_n$ and $\hat{S}_n$ in (3.2) we have

\[(3.4) \quad P(\max(K_2 + S_n - S_{K_1}, \max_{1 \leq i \leq n} (S_n - S_i)) = 0, T \leq K_1)\]

\[= P(\max(K_2 + \overline{S}_n - \overline{S}_{K_1}, \max_{1 \leq i \leq n} (\overline{S}_n - \overline{S}_i)) = 0, T \leq K_1)\]

\[\geq P(\max(K_2 + \hat{S}_{n-K_1}, \max_{0 \leq i \leq n-K_1} (\hat{S}_i)) = 0) - P(T \leq K_1),\]

where the equality follows from the "indistinguishability" of $X_n$ and $\overline{X}_n$ for $n > T$ (see [5]) and the inequality follows from the reversibility of $\overline{X}_n$ and $\hat{X}_n$ (see [22]). Thus, for all $n > N_{K_2} + K_1$, it follows from Lemma 3.2 that $P(Z_n = 0) > \eta/3$.

Treating $\{(X_n, Z_n)\}$ as a Markov chain on state space $S = J \times [0, \infty]$ where $F$ is the $\sigma$-field generated by the sets $B = \{(i) \times [0, x]\}, i \in J, x \geq 0$, we will try to establish $\{(X_n, Z_n)\}$ is ergodic. To do this, we need to create a $\phi$-measure. Let

\[(3.5) \quad P^n((j,x), A) = P((X_n, Z_n) = (j, x), X_0 = j, Z_0 = x), \text{ for } A \in F.\]

Also, for $i_0$ of Lemma 3.1, define

\[(3.6) \quad \phi(A) = \sum_{n=1}^{\infty} 2^{-n} P^n((i_0, 0), A)\]

We then have,

**LEMMA 3.4.** If $E_{\Pi} U < 0$, then the chain $(X_n, Z_n)$ is $\phi$-irreducible for $\phi$ as defined in (3.6).

**PROOF.** For every $A \in F$ with $\phi(A) > 0$, we need to establish that

\[\sum_{n=1}^{\infty} 2^{-n} P^n((j,x), A) > 0, \text{ for } j \in J, x \geq 0.\]

For every $j$ and $x \geq 0$, there is an $n_j$ and a $B$ such that

\[P(X_{n_j} = i_0, Z_{n_j} \leq B | X_0 = j, Z_0 = x) > 0.\]
We denote this positive probability by \( \delta_j(x) \).

Then for \( m = n_j + n \left\lfloor \frac{B}{c} + 1 \right\rfloor \) (where \( n, c, \) are as in Corollary 3.1), we have

\[
(3.7) \quad P(X_m = i_0, Z_m = 0 | X_0 = j, Z_0 = x) 
\]

\[
\geq \delta_j(x) \cdot P \left( X_{m-n_j} = i_0, \max(B+S_{m-n_j}, \max_{0 \leq i \leq m-n_j} (S_i)) = 0 | X_0 = i_0 \right) 
\]

\[
= \delta_j(x) P(\hat{X}_{m-n_j} = i_0, \max(B+\hat{S}_{m-n_j}, \max_{0 \leq i \leq m-n_j} (\hat{S}_i)) = 0 | \hat{X}_0 = i_0). 
\]

\[
> \delta_j(x) \delta_{\frac{B}{c} + 1}. 
\]

From Corollary 3.1 we therefore have \( P(X_m = i_0, Z_m = 0 | X_0 = j, Z_0 = x) > 0 \), which in turn gives us that \( \sum_{n=1}^{\infty} 2^{-n} P^\infty((j,x), A) > 0 \), for all \( j, x \). \( \square \)

Now that we have established that there is a \( \phi \)-measure such that \( \{(X_n, Z_n)\} \) is \( \phi \)-irreducible, we have from Jain and Jamison [8] that there exists a subinvariant \( \sigma \)-finite measure \( \mu \). Using some results by Tweedie [21], we will now establish the ergodicity of \( \{(X_n, Z_n)\} \).

**THEOREM 3.5.** If \( E_U < 0 \), the Markov chain \( \{(X_n, Z_n)\} \) is ergodic.

**PROOF.** In view of [21], we only need to show that for some \((j, x)\) and some \( A \in \mathcal{F}_U \), that \( \lim(n^{-1} \sum_{i=1}^{n} P^i((j, x), A)) > 0 \), where \( \mathcal{F}_U = \{A \in \mathcal{F}: 0 < \mu(A) < \infty\} \).

Applying Lemma 3.1 to the dual process, choose an \( i_0 \) such that

\[
(3.14) \quad P(\max_{0 \leq j \leq n} (S_n - S_j) = 0, X_n = i_0 | X_0 = i_0) > 0. 
\]

Let us take \((j, x) = (i_0, 0), B = \{i_0\} \cup \{1, 2, \ldots, M\}, \) and \( A = B \times \{0\} \), where \( M \) is such that for \( \delta \) of Lemma 3.3,

\[
\prod_{i=1}^{M} \prod_{i=1}^{M} \prod_{i=1}^{M} \prod_{i=1}^{M} > 1 - \delta/A. 
\]
Since $\phi((i_0, 0)) > 0$, and $\mu > \phi$ (see [8]), we have that $\mu(A) > 0$. Also clearly by construction $\mu(A) < \infty$, so that $\Lambda \epsilon F_\mu$.

Let $N$ be as in Lemma 3.3. Then, for some $N_1 > N$ such that $P(X_n \epsilon B) > 1 - \delta/2$ for $n > N_1$, we have that for $n > N_1$ that $P(X_n \epsilon B, Z_n = 0 \mid X_0 = i_0, Z_0 = 0) > \delta/2$. Thus $\lim \frac{n}{n-1} \sum_{i=1}^{n} p((i_0, 0), A)) > \delta/2$, and in view of Tweedie [21], this establishes that $(X_n, Z_n)$ is ergodic. □

4. LIMIT BEHAVIOR OF $\{(X(t), Z(t))\}$ VIA A RENEWAL EQUATION.

We define

$$T^A = \inf \{t > T_1: (X(t), Z(t)) \in A\}.$$  \hspace{1cm} (4.1)

Then it can be readily seen that for $i_0$ of section 3,

$$P_{i_0, 0}(X(t) = j, Z(t) \leq x) \hspace{1cm} (4.2)$$

$$= P_{i_0, 0}(X(t) = j, Z(t) \leq x, T^{(i_0, 0)} > t) \hspace{1cm}$$

$$+ \int P_{i_0, 0}(X(t-\tau) = j, Z(t-\tau) \leq x) dP_{i_0, 0}(T^{(i_0, 0)} \leq \tau),$$

where $P_{j, x}(Z)$ and $E_{j, x}(Z)$ are defined as in (2.5).

Equation (4.2) is therefore our basic renewal equation for the process $(X(t), Z(t))$.

Under some appropriate conditions, the basic renewal theorem (see Karlin and Taylor [9]), will now yield the desired asymptotic behavior of $\{(X(t), Z(t))\}$ using (4.2). As a first step, in satisfying the conditions needed for the basic renewal theorem, we prove below that $E_{i_0, 0}(T^{(i_0, 0)}) < \infty$.

**Theorem 4.1.** If $E_{i_0, U} > 0$, and $\beta < \infty$, then $E_{i_0, 0}(T^{(i_0, 0)}) < \alpha$, for $i_0$ of Lemma 3.1 and $\beta$ as in (1.9).

**Proof.** Note that $\beta = \bar{m} / \bar{m}$, for $\bar{m}$ as in (1.2).
The observation that \(\{(X_n, Z_n, T_n)\}\) is a semi-Markov process on a general state space allows us to apply the results of section 2.

Now Theorem 3.5 guarantees the existence of an invariant probability positive measure \(\pi\) for \(\{(X_n, Z_n)\}\), since \(\{(X_n, Z_n)\}\) is ergodic. Letting \(A = (i_0, 0)\) and applying Theorem 2.2, we have

\[
(4.3) \quad \pi(i_0, 0) E_{i_0, 0}(T_{i_0, 0}^+) \leq \int_{\mathbb{R} \times [0, \infty]} E_{X, Y}(T_1) \pi(dx, dy) = \sum_{i=1}^{\infty} m_i < \infty,
\]

where the equality follows from the fact that \(T_1\) is independent of \(Z_0\), so \(E_{X, Y}(T_1)\) is constant for all \(y\).

Since \(\phi(i_0, 0) > 0\) implies \(\pi(i_0, 0) > 0\), the theorem follows.

We now establish the main result of this chapter, which tells us that in the subcritical case that \(\{(X(t), Z(t))\}\) converges in distribution.

**THEOREM 4.2.** If \(E\pi U < 0\), and \(\beta < \infty\), then as \(t \to \infty\),

\[
P(X(t) = i, Z(t) \leq x \mid X(0) = i_0, Z(0) = 0) \to P(X = i, Z \leq x),
\]

for every continuity point of \(P(Z \leq x)\), and for some random variables \(X\) and \(Z\).

**PROOF.** From (4.2), it follows from the basic renewal theorem that if \(T\) is non-arithmetic and

\[
P_{i_0, 0}(X(t) = i, Z(t) \leq x, T_{i_0, 0} > t)
\]

is directly Riemann integrable, then as \(t \to \infty\), \(P_{i_0, 0}(X(t) = i, Z(t) \leq x)\) will converge.

From Theorem 4.1, it follows that \(E_{i_0, 0}(T_{i_0, 0}^+) = \int_0^\infty P_{i_0, 0}(T_{i_0, 0} > t) dt\) is finite. Also, since

\[
(4.4) \quad P_{i_0, 0}(X(t) = i, Z(t) \leq x, T_{i_0, 0} > t) \leq P_{i_0, 0}(T_{i_0, 0} > t) ,\]

the desired direct Riemann integrability can be shown to be equivalent to right continuity of \(P_{i_0, 0}(X(t) = i, Z(t) \leq x, T_{i_0, 0} > t)\) by elementary arguments (see Royden [18]).
As such, it suffices to establish that
\[
\mathbb{P}^{(i_0,0)}_{(i_0,0)}(X(t) = i, Z(t) \leq x, T > t) \text{ is right continuous for } t.
\]
To this end, we note that for \( t > s \),
\[
\mathbb{P}^{(i_0,0)}_{(i_0,0)}(X(s) = i, Z(s) \leq x, T > s) \leq \mathbb{P}^{(i_0,0)}_{(i_0,0)}(M(t) - M(s) > 0)
\]

Thus, it is enough to show that
\[
\lim_{\varepsilon \to 0} \mathbb{P}^{(i_0,0)}_{(i_0,0)}(M(t + \varepsilon) - M(t) = 0) = 1.
\]
This is easily established using a renewal argument on \( M(t + \varepsilon) - M(t) \). Thus, the desired right-continuity is established, which completes the proof of the theorem for the case when the distribution of \( T \) is non-arithmetic. On the other hand, if \( T \) is arithmetic, then by a simple argument, the behavior of \( \{X_n, T_n\} \) is equivalent to \( \{\tilde{X}_n\} \), an appropriately defined Markov chain, and the result again follows from previous arguments.

\[\blacksquare\]

**Theorem 4.3.** If \( \mathbb{E}_n U < 0 \) and \( B < \infty \), then as \( t \to \infty \)
\[
\mathbb{P}(X(t) = i, Z(t) \leq x) \to \mathbb{P}(X = i, Z \leq x),
\]
for every continuity point \( x \) of \( \mathbb{P}(Z \leq x) \), where the random variables \( X \) and \( Z \) are defined as in Theorem 4.2.

**Proof.** We first need to establish that \( \mathbb{P}^{(i_0,0)}_{X(0), Z(0); (i_0,0)}(T \ll \infty) = 1 \). For this, it follows from Tweedie [21], that there exists a \( \phi \)-null set \( N(i_0,0) \) such that
\[
\mathbb{P}_{i_0}(\bigcup_{n=1}^{\infty} (X_n, Z_n) = (i_0,0)) < 1
\]
for \((j,x)\in\mathbb{N}(i_0,0)\). In order to establish that \(N(i_0,0) = \emptyset\), we note that since \\
\(\phi(i_0,0) > 0\) (which implies that \((i_0,0)\notin N(i_0,0)\)), using a straightforward argument, we have that for all \((j,x)\notin N(i_0,0)\)

\[(4.8) \quad P_{j,x}(X_n,Z_n) = (i_0,0), \text{ i.o.} = 1.\]

For every \(j\), there must be a \(y_j\) with \((j,y_j)\notin N(i_0,0)\). Then, since

\[(4.9) \quad P_{j,z}(X_n,Z_n) = (i_0,0), \text{ i.o.} = 1. \]

\[= P(X_n, \max[z + S_n, \max_{1 \leq j \leq n} (S_n - S_j)]) = (i_0,0), \text{ i.o.} | X_0 = j), \]

and since \(S_n \to -\infty\), a.s., we have

\[(4.10) \quad P((X_n, \max[z + S_n, \max_{1 \leq j \leq n} (S_n - S_j)] \neq (X_n, \max[y_j + S_n, \max_{1 \leq j \leq n} (S_n - S_j)]), \text{ i.o.} | X_0 = j)\]

\[\leq P(S_n > \min(-z, -y_j), \text{ i.o.} | X_0 = j) = 0.\]

Consequently from (4.10), we have

\[(4.11) \quad P_{j,z}(X_n,Z_n) = (i_0,0), \text{ i.o.} = 1. \]

Thus \(N(i_0,0) = \emptyset\), and hence \(P_{X(0),Z(0)}(i_0;0)^{T < \infty} = 1\), for arbitrary initial distributions.

As such, for all \(\epsilon\), there exist \(K_1\) and \(K_2\) such that

\[P_{X(0),Z(0)}(T \leq K_1) > 1 - \epsilon, \text{ and for } t > K_2\]

\[(4.12) \quad |P_{i_0,0}(X(t) = j, Z(t) \leq x) - P(X = j, Z \leq x)| < \epsilon, \]

for every continuity point \(x\) of \(P(Z \leq x)\). From this it follows that for \(t > K_1 + K_2\), we have
\begin{equation}
P(X = j, Z \leq x) - 2\varepsilon \leq P(X(t) = j, Z(t) \leq x) \leq P(X = j, Z \leq x) + 2\varepsilon,
\end{equation}

and the theorem follows. \[ \square \]

This completes the analysis of the asymptotic behavior of \{(X(t), Z(t))\} in the subcritical case. For the same case the following section deals with the joint asymptotic behavior of the storage level at time \( t \) and the cumulative amount of the output demands from the storage which were unmet during \([0, t]\).

5. SOME FURTHER RESULTS.

In this section, along with the level in the storage at time \( t \) in the subcritical case, we examine the behavior of a random variable \( L(t) \) which represents the cumulative amount of demands from the system which were unmet during time \((0, t]\) as given in (1.3) and (1.4). From these we also note that

\begin{equation}
L(t) = L_{M(t)} = Z(t) - S_{M(t)} - Z(0).
\end{equation}

To establish the joint behavior of \( L(t) \) and \( Z(t) \), we need a theorem which provides conditions for the asymptotic independence of the two variables. To this end, we define a new variable \( Z_{\tau}(t) \) by

\begin{equation}
Z_{\tau}(t) = \max_{M(t-\tau) \leq j \leq M(t)} (S_{M(t)} - S_j).
\end{equation}

Note that since \( Z(t) = \max(Z(0) + S_{M(t)}, \max_{1 \leq j \leq M(t)} (S_{M(t)} - S_j)) \),

we have that for \( 0 \leq \tau \leq t \), \( Z_{\tau_1}(t) \leq Z_{\tau_2}(t) \leq Z(t) \).

We now prove the following lemma, which establishes a useful property of \( Z_{\tau}(t) \).

**Lemma 5.1.** If \( E \bar{U} < 0 \) and \( \beta < \infty \), then for every \( \varepsilon > 0 \), there exist \( T_1 \) and \( T_2 \) with \( T_1 > T_2 \), such that \( P(Z(t) \neq Z_{\tau}(t)) < \varepsilon \), for all \( t > T_1 \) and \( \tau > T_2 \).

**Proof.** Consider the state \( i_0 \) used in the previous section, and let

\begin{equation}
N(i_0, 0)(t) = \sup \{1 \leq n \leq M(t) : (X_n, Z_n) = (i_0, 0)\}
\end{equation}
and \( T_t = T_{N(i_0, 0)(t)} \). Since \( E_{i_0, 0}(T_{i_0, 0}^{(i_0, 0)}) < \infty \), it follows from renewal theory (see Karlin and Taylor [9]) that for all initial distributions of \((X(0), Z(0))\), as \( t \to \infty \),

\[
(5.4) \quad P(t - T_t > z) = E_{i_0, 0}(T_{i_0, 0}^{(i_0, 0)})^{-1} \int_{i_0, 0}^{\infty} E_{i_0, 0}(T_{i_0, 0}^{(i_0, 0)} > x)dx.
\]

Thus we can choose a \( T_2 \) and a \( T_1 \) such that for \( t > T_1 \), \( P(t - T_t > T_2) < \epsilon \).

Note that \( t - T_t \leq T_2 \) implies there exists an \( x \), satisfying \( t - T_2 \leq x \leq t \), such that \( Z(x) = 0 \). Thus

\[
(5.5) \quad Z(t) = (Z(x) + S_M(t) - S_M(x), \max_{M(x) \leq j \leq M(t)} (S_M(t) - S_j)) = Z_{t-x}(t),
\]

and we have that \( t - T_t \leq T_2 \) implies \( Z(t) = Z_{\tau}(t) \), for \( \tau \geq T_2 \). From this, the lemma follows.

Another useful property of \( Z_{\tau}(t) \) is that \( Z_{\tau}(t) \) "ignores" the inputs and outputs which occurred before \( t-\tau \), in the sense that they do not enter into the formula for \( Z_{\tau}(t) \). Because of this, \( Z_{\tau}(t) \) is conditionally independent of anything that has happened in time \([0, t-\tau]\), given \( X_{M(t-\tau)+1} \) and \( T_{M(t-\tau)+1}' \). Thus, we can establish the following theorem.

THEOREM 5.2. Let \( \pi < 0 \), and \( \beta < \infty \). Also let \( Y(t) \) and \( Y'(t) \) be two processes such that,

1. \( P(Y(t) \leq x) \to P(Y \leq x) \) for all continuity points of \( P(Y \leq x) \) as \( t \to \infty \),
2. \( Y'(t-\tau) - Y(t) \to 0 \), as \( t \to \infty \), and
3. \( Y'(t-\tau) \) and \( Z_{\tau}(t) \) are conditionally independent given 

\[
X_{M(t-\tau)+1} \text{ and } T_{M(t-\tau)+1}'.
\]
Then as $t \to \infty$, $P(Z(t) \leq x, Y(t) \leq y) = P(Z \leq x)P(Y \leq y)$, for all continuity points $x$ and $y$ of the distributions of $Z$ and $Y$, respectively, where $Z$ is as described in Theorem 4.2.

**Remark.** Since $Y'(t-\tau) - Y(t) \mathcal{D} 0$, we have $P(Y'(t) \leq y) \geq P(Y \leq y)$, if and only if $P(Y(t) \leq y) \geq P(Y \leq y)$ for all continuity points of $P(Y \leq y)$.

**Proof of Theorem 5.2.** In view of Lemma 5.1, for every $\epsilon > 0$, there exist $T_1$ and $T_2$ such that for $t > T_1$, and $\tau > T_2$,

$$P(Z(t) \neq Z_\tau(t)) < \frac{\epsilon}{2}. \tag{5.6}$$

Also, for $\tau > T_2$, there exists a $T_\tau > T_1$ such that for $t > T_\tau$,

$$P(|Y(t) - Y'(t-\tau)| > \epsilon) < \frac{\epsilon}{2}. \tag{5.7}$$

Thus it follows that for $t > T_\tau$,

$$P(Z_\tau(t) \leq x, Y'(t-\tau) \leq y - \epsilon) - \epsilon \leq P(Z(t) \leq x, Y(t) \leq y) \leq P(Z_\tau(t) \leq x, Y'(t-\tau) \leq y + \epsilon) + \epsilon. \tag{5.8}$$

Now defining $\tau_t^* = T_{M(t)} + 1$, $X_t^* = X_{t, M(t)} + 1$ and $P_t^*(i, x) = P(X_t^* = i, T_t^* \leq x)$, we have

$$P(Z_\tau(t) \leq x, Y'(t-\tau) \leq y) = \sum_{i \in J} \int_{t-\tau}^\infty P(Z_\tau(t) \leq x \mid X_{t-\tau}^* = i, T_{t-\tau}^* = z)P(Y'(t-\tau) \leq y \mid X_{t-\tau}^* = i, T_{t-\tau}^* = z)P_{t-\tau}^*(i, dz), \tag{5.9}$$

where the equality follows from the fact that $Y'(t-\tau)$ and $Z_\tau(t)$ are conditionally independent given $X_{t-\tau}^*$ and $T_{t-\tau}^*$.

By Cinlar [5]), we have for any $\tau$

$$P(X_{t-\tau}^* = k, T_{t-\tau}^* \leq (t-\tau) > y) \geq (\sum_{j=1}^m m_j)^{-1} \int_0^\infty \sum_j (p_{jk} - A_{jk}(u)) du, \tag{5.10}$$

as $t \to \infty$, where $p_{jk}$, $A_{jk}(u)$, and $m_j$ are as defined in (1.1) and (1.2). Since
\[ \beta = \sum_{j} \pi_j m_j < \infty, \] for every \( \epsilon > 0 \), we can pick a finite set \( A \subseteq J \) a \( z > 0 \), and a \( K_1 \) such that for \( t > K_1 \)

\[ (5.11) \quad P((X^+_t \notin A) U (X^+_t \in A, T^+_t - t > z)) < \epsilon. \]

Thus for any \( \tau > K_1 + \tau \), we have

\[ (5.12) \quad | P(Z^+_\tau(t) \leq x, Y^+(t-\tau) \leq y) \]
\[ - \sum_{i \in A} \int_{t-\tau}^{t-\tau+2} P(Z^+_{t-\tau}(t) \leq x \mid X^+_t = i, T^+_t = \mu) P(Y^+(t-\tau) \leq y) X^+_{t-\tau} = i, T^+_{t-\tau} = \mu) \]
\[ \cdot P^+_{t-\tau}(i, d\mu) | < \epsilon. \]

From Theorem 4.3 it follows that for every \( \epsilon > 0 \), there exists an \( L \) such that for \( k \in A \) and \( t > L \),

\[ (5.13) \quad | P_{k^+}(U(k)^+) + (Z(t) \leq x) - P(Z \leq x) | < \epsilon, \]

and by the definition of \( Z^+_\tau(t) \), we have

\[ (5.14) \quad P(Z^+_\tau(t) \leq x \mid X^+_t = i, T^+_t = \mu) = P_{i^+}(U(i)^+) + (Z(t-\mu) \leq x). \]

Choose \( \tau > L + z \), for \( L \) in (5.13) and \( Z \) in (5.11). Then for \( t > K_1 + \tau \), since \( t-\tau \leq \mu \leq t-\tau + z \) implies \( t-\mu \geq \tau - z > L \), we have from (5.12), (5.13) and (5.14) that

\[ (5.15) \quad | P(Z \leq x) \sum_{i \in A} \int_{t-\tau}^{t-\tau+2} P(Y^+(t-\tau) \leq y \mid X^+_t = i, T^+_t = \mu) P^+_{t-\tau}(i, d\mu) \]
\[ - P(Z^+_\tau(t) \leq y, Y^+(t-\tau) \leq x) | \leq 2\epsilon. \]

From (5.11) and (5.15) we get

\[ (5.16) \quad | P(Z \leq x) P(Y^+(t-\tau) \leq y) - P(Z(t) \leq x, Y^+(t-\tau) \leq y) | \leq 3\epsilon. \]

Thus, to combine (5.8) and (5.16), we need to choose

\[ \tau > \max(L+z, T_2), \] and \( T > \max(T_\tau, K_1 + \tau) \) for \( T_2, T_\tau \) as in (5.6) and (5.7). With these it follows, that for \( t > T \),
(5.19) \[ P(Z \leq x)P(Y(t) \leq y - 2\varepsilon) - 4\varepsilon \]
\[ \leq P(Z(t) \leq x, Y(t) \leq y) \]
\[ \leq P(Z \leq x)P(Y(t) \leq y + 2\varepsilon) + 4\varepsilon, \]
which completes the proof of the theorem.

Before establishing the main theorem of this section, we need to define

(5.20) \[ t_1 = \inf\{n > 0: X_n = i_o\}. \]

and

(5.21) \[ \phi(x) = \int_{-\infty}^{x}(2\pi)^{-1/2}\exp(-y^2/2)dy. \]

To establish the joint asymptotic behavior of \(Z(t)\) and \(L(t)\), we use Theorem 5.2 to establish the asymptotic independence of the two random variables, which yields the following theorem.

**Theorem 5.3.** If \(E_{i_0}U \leq 0\), \(\beta < \infty\), and

\[
\sigma_{i_0}^2 = E[(\sum U_i(X_i)) + T_{t_1}(E_{i_0}U)]^2|X_{i_0} = i_0] < \infty,
\]

then for arbitrary distributions of \(X(0)\) and \(Z(0)\), as \(t \to \infty\), we have for all continuity points \(x\) of \(P(Z \leq x)\),

\[ P(Z(t) \leq x, L(t) + t\beta^{-1}(E_{i_0}U) \leq y(t\pi_i\beta^{-1}\sigma_{i_0}^2)^{1/2}) \to P(Z < x)\phi(y). \]

**Proof.** To appeal to Theorem 5.2, we define

(5.4) \[ Y(t) = (t\pi_i\beta^{-1}\sigma_{i_0}^2)^{1/2}(L(t) + t(E_{i_0}U)\beta^{-1}), \]

(5.5) \[ Y'(t) = ((t+\tau)\pi_i\beta^{-1}\sigma_{i_0}^2)^{1/2}(-S_M(t) + (t+\tau)(E_{i_0}U)\beta^{-1}). \]

Puri and Woolford [17] have shown that as \(t \to \infty\), \(P(Y'(t) \leq y) \to \phi(y)\).

Thus in view of the remark following Theorem 5.2, and of the fact that \(Y'(t-\tau)\) and \(Z(t)\) are by construction conditionally independent given \(X_{t-\tau}, T_{t-\tau}\), we only need to show that \(Y(t) - Y'(t-\tau) \leq 0\) in order to complete the proof of our theorem.
Since \( L(t) = Z(t) - S_{M(t)} - Z(0) \), it is easy to see that \( Y(t) - Y'(t-\tau) \) is equivalent to

\[
(5.26) \quad t^{-\beta} \left( S_{M(t-\tau)} - S_{M(t)} \right) \Rightarrow 0.
\]

For \( N_i(t) = N(i_0, [0, \infty)) \) as in (5.4), from (Puri and Woolford [17]), it follows that if \( \beta < \infty \), then \( t^{-\beta} \left( S_{N_i(t)} - S_{M(t)} \right) \Rightarrow 0 \), as \( t \to \infty \).

Also, by Lemma 5.4, established below, we have

\[
(5.27) \quad t^{-\beta} \left( S_{N_i(t)} - S_{N_i(t-\tau)} \right) \Rightarrow 0, \quad \text{as } t \to \infty.
\]

From this it follows that \( t^{-\beta} \left( S_{M(t-\tau)} - S_{M(t)} \right) \Rightarrow 0 \), and hence the proof.

**Lemma 5.4.** If \( \beta < \infty \), then \( t^{-\beta} \left( S_{N_i(t)} - S_{N_i(t-\tau)} \right) \Rightarrow 0 \), as \( t \to \infty \).

The idea of the proof is to first show \( P(S_{N_i(t)} - S_{N_i(t-\tau)} \leq x | X(0) = i_0) \) can be expressed as a renewal equation based on the first visit of \( X(t) \) to state \( i_0 \). Since \( E(T_{t_1} | X(0) = i_0) = \pi_{i_0}^{-1} \beta < \infty \), an argument identical to that of Theorem 4.2 will show that \( P(S_{N_i(t)} - S_{N_i(t+\tau)} \leq x | X(0) = i_0) \) converges as \( t \to \infty \). Since it is well known \( P(T_{t_1} < \infty) = 1 \) for an arbitrary distribution of \( X(0) \), the lemma follows directly.

**Remark.** The above lemma is valid as long as \( S_n = \sum_{i=1}^{n} Y_i(X_i) \) and, for every \( j \in J, \{ Y_n(j), n=1, 2, \ldots \} \) is an i.i.d. sequence, such that \( \{ Y_n(j) \} \) is independent of \( \{ Y_n(i) \} \) for \( j \neq i \).

6. **Concluding Remarks.**

The approach adopted here of using conditions such as those of Tweedie [21] to establish ergodicity followed by the use of a renewal equation argument appear to be more generally applicable to models defined on semi-Markov processes. How-
ever, the creation of the measure $\phi$ and the selection of an appropriate 'recurrent point' appear to be the major problems, and must be tackled with due considerations of the model at hand. It should be pointed out that under certain conditions, if no recurrent point is available, a special (recurrent) set may suffice for the establishment of the needed renewal equation in order to follow through the present approach (see Athreya, McDonald and Ney [2] and Nummelin [13]).

There are some questions about the model that have remained unanswered in this paper. In particular, while the existence of a limit distribution of $\{(X_n, Z_n)\}$ is exhibited, it would be of interest to find what this distribution is (or perhaps its Laplace transform). Also, the same question can be raised for the limit distribution of $\{(X(t), Z(t))\}$. Unfortunately, the techniques in this paper do not seem applicable to answering these questions.
REFERENCES


