AN INTEGRAL INEQUALITY WITH APPLICATIONS TO ORDER STATISTICS

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ABSTRACT

We say the life distribution function G majorizes the life distribution function F (written \( G \gtrless F \)) if

\[
\int_x^\infty G(t) dt \geq \int_x^\infty F(t) dt \quad \text{for all } x \geq 0
\]

and

\[
\int_0^\infty G(t) dt = \int_0^\infty F(t) dt < +\infty.
\]

An integral inequality is proved giving sufficient conditions on functions \( \psi \) and \( \phi \) in order to ensure that whenever \( G_i \gtrless F_i \) for \( i=1,\ldots,n \), then

\[
\int_0^\infty \psi(t) \phi(G_1(t),\ldots,G_n(t)) dt \leq \int_0^\infty \psi(t) \phi(F_1(t),\ldots,F_n(t)) dt.
\]

Applications in reliability theory and order statistics are given.
1. Introduction.

For given life distribution functions $F$ and $G$, the respective survival functions are $F = 1 - F$ and $G = 1 - G$. We define the partial ordering $\preceq_m$ on the class of life distributions with finite means by $G \preceq_m F$ (m for majorization) if

$$
\int_0^\infty G(t) dt \geq \int_0^\infty F(t) dt \quad \text{for all } x \geq 0
$$

and

$$
\mu_G = \int_0^\infty G(t) dt = \int_0^\infty F(t) dt = \mu_F < + \infty.
$$

If $X$ and $Y$ are nonnegative random variables with respective distribution functions $F$ and $G$, then Ross [11] says "$Y$ is more variable than $X$" (written $Y \geq_v X$ or $G \geq_v F$) if (1.1) holds. Stoyan [14] equivalently defines $Y$ to be "larger in mean residual life" than $X$ (written $G \preceq_c F$ or in previous publications $G \preceq F(2)$) if (1.1) holds. Bessler and Veinott [3] use the terminology "$Y$ is stochastically larger in mean than $X$." The notation of Stoyan (c for convex) is suggested by the following characterization:

$$
G \preceq_c F \iff \int_0^\infty \Psi(t) dG(t) \geq \int_0^\infty \Psi(t) dF(t)
$$

holds for all increasing (that is nondecreasing) convex functions $\Psi$, provided the integrals exist.

For life distribution functions $F$ and $G$, $G \preceq_m F$ if and only if $G \preceq_c F$ (or $G \geq_v F$) and $G$ and $F$ have equal finite means ($\mu_F = \mu_G$). For distribution functions with finite means, the following useful characterization of $G \preceq_m F$ (see for example Ross [11] or Stoyan [14]) is an immediate corollary of Theorem 2.1:

$$
G \preceq_m F \iff \int_0^\infty \Psi(t) dG(t) \geq \int_0^\infty \Psi(t) dF(t)
$$

holds for all convex functions $\Psi$, provided the integrals exist.
We note in particular that if $G \succcurlyeq F$, then
\[
\sigma_G^2 = \int_0^\infty (t-\mu_G)^2 dG(t) \geq \int_0^\infty (t-\mu_F)^2 dF(t) = \sigma_F^2.
\]

Hence $G \succcurlyeq F$ implies that the life distribution represented by $G$ is 'more dispersed' than that represented by $F$ around their common mean.

For life distribution functions $F$ and $G$ with a common mean, $G \succcurlyeq F$ is a more general relationship than $G \succ F$ ($F$ is star shaped with respect to $G$). When $F$ and $G$ are continuous life distributions (where $F(0) = G(0) = 0$, $F$ and $G$ have interval support and $G$ is strictly increasing on its support), then $G \succ F$ if $G^{-1}F(x)$ is star-shaped (that is $\frac{G^{-1}\mathbb{F}(x)}{x}$ is increasing for $x > 0$).

If $G \succ F$ and $F$ and $G$ have a common mean, then $\mathbb{F}(x)$ crosses $\mathbb{G}(x)$ once and from above as $x:0 \rightarrow \infty$, so that in particular $G \succcurlyeq F$ (see Barlow and Proschan [2]).

For a continuous life distribution function $F$ with mean $\mu$, let us define $G(x) = 1 - e^{-x/\mu}$ to be the exponential distribution with the same mean. Then $F$ is IFRA (increasing failure rate average) $\iff G \succ F$, and $F$ is HNBU (harmonic new better than used in expectation) $\iff G \succcurlyeq F$. See Klefsjö [6] for further properties of HNBU distributions.

If $F$ and $G$ are two life distribution functions with common mean and $\mathbb{F}(x)$ crosses $\mathbb{G}(x)$ once and from above as $x:0 \rightarrow \infty$, then $G \succcurlyeq F$, however the converse is clearly not true. For example let $F$ and $G$ be defined as follows:

\[
F(x) = \begin{cases} 
0 & x < 2 \\
1/2 & 2 \leq x < 4 \\
1 & 4 \leq x
\end{cases} \quad G(x) = \begin{cases} 
0 & x < 1 \\
1/4 & 1 \leq x < 3 \\
3/4 & 3 \leq x < 5 \\
1 & 5 \leq x
\end{cases}
\]

Then $G \succcurlyeq F$ and $G$ 'crosses' $F$ three times.
A vector \( \underline{b} = (b_1, \ldots, b_n) \) majorizes the vector \( \underline{a} = (a_1, \ldots, a_n) \) if
\[
\sum_{i=k}^{n} b[i] \geq \sum_{i=k}^{n} a[i] \quad \text{for } k=2, \ldots, n
\]

and
\[
\sum_{i=1}^{n} b[i] = \sum_{i=1}^{n} a[i],
\]
where the \( b[i] \)'s and \( a[i] \)'s are the components of \( \underline{b} \) and \( \underline{a} \) respectively in ascending order. When \( \underline{b} \) majorizes \( \underline{a} \) we write \( \underline{b} \succ_m \underline{a} \).

Suppose now that \( \underline{b} \) and \( \underline{a} \) are \( n \) dimensional vectors with nonnegative components such that \( \underline{b} \succ_m \underline{a} \). If \( G \) and \( F \) are respectively the distribution functions for the uniform distributions on the components of \( \underline{b} \) and \( \underline{a} \), then \( G \succ_m F \). This is our motivation for using the letter \( m \) for our partial ordering on the family of life distribution functions with finite means.

2. An Integral Inequality.

The following theorem is a variant of an integral inequality obtained by Fan and Lorentz [4].

**Theorem 2.1.** Let \( \phi = [0,1]^n \rightarrow [0,\infty) \) be a continuous increasing function, and assume that for \( i=1, \ldots, n \), \( F_i \) and \( G_i \) are life distribution functions where \( G_i \succ_m F_i \).

a) If \( \psi \) is nonnegative decreasing, \( \phi \) is convex in each variable separately and \( \phi \) satisfies the following property:

\[
\phi(u_i + h, u_j + k) - \phi(u_i, h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \geq 0
\]

for all \( i \neq j \), \( 0 \leq u_i \leq u_i + h \leq 1 \), \( 0 \leq u_j \leq u_j + k \leq 1 \)

(where we have used the notational simplification of omitting those arguments of \( \phi \) which are the same in a given formula),

then providing the integrals exist,

\[
\int_0^\infty \psi(t)\phi(\overline{G}_1(t), \ldots, \overline{G}_n(t))dt \leq \int_0^\infty \psi(t)\phi(\overline{F}_1(t), \ldots, \overline{F}_n(t))dt.
\]
b) If \( \psi \) is nonnegative increasing, \( \phi \) is concave in each variable separately and \( \phi \) satisfies the following property:

\[
\phi(u_i + h, u_j + k) - \phi(u_i + h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \leq 0
\]

for all \( i \neq j, 0 \leq u_i \leq u_i + h \leq 1, 0 \leq u_j \leq u_j + k \leq 1 \),

then providing the integrals exist

\[
\int_0^\infty \psi(t) \phi(G_1(t), \ldots, G_n(t)) dt \geq \int_0^\infty \psi(t) \phi(F_1(t), \ldots, F_n(t)) dt.
\]

**Proof:** We prove only a), the proof of b) following in a similar fashion.

(i) Initially we show that it suffices to prove the result for the case when \( F_1, G_1, \ldots, F_n, G_n \) all have finite support. In turn to establish this we show that if the inequality is valid whenever \( F_1 \) and \( G_1 \) have finite support, then it is true in general.

Suppose now that \( F_1, G_1, \ldots, F_n, G_n \) are arbitrary life distributions where \( G_i \geq F_i \) for \( i = 1, \ldots, n \). Given \( \varepsilon > 0 \), we can find \( S \) so that

\[
\int_S^\infty \psi(t) \phi(G_1(t), \ldots, G_n(t)) dt < \varepsilon.
\]

Now define \( F_1' \) and \( G_1' \) by

\[
F_1'(t) = \begin{cases} 
F_1(t) & t < S \\
0 & t \geq S
\end{cases}
\]

\[
G_1'(t) = \begin{cases} 
G_1(t) & t < S \\
S \int_0^t F_1(\tau) d\tau - \int_0^S G_1(\tau) d\tau & S \leq t \leq S + \frac{G_1(S)}{G_1(S)} \\
0 & \text{otherwise,}
\end{cases}
\]
(if \( \overline{G}_1(S) = 0 \), then both \( G_1 \) and \( F_1 \) have finite support). Then \( G_1' > F_1' \), and

\[
\int_0^\infty \psi(t) \phi(F_1(t), F_2(t), \ldots, F_n(t)) \, dt \geq \int_0^\infty \psi(t) \phi(G_1'(t), G_2(t), \ldots, G_n(t)) \, dt \\
\geq \int_0^\infty \psi(t) \phi(G_1'(t), G_2(t), \ldots, G_n(t)) \, dt \\
\geq \int_0^\infty \psi(t) \phi(G_1(t), G_2(t), \ldots, G_n(t)) \, dt - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, the conclusion follows.

(ii) It now remains to show that

\[
\int_0^\infty \psi(t) \phi(G_1(t), G_2(t), \ldots, G_n(t)) \, dt \leq \int_0^\infty \psi(t) \phi(F_1(t), F_2(t), \ldots, F_n(t)) \, dt
\]

whenever \( G_1 > F_1 \) for all \( i = 1, \ldots, n \), and where the support of \( F_1 \) and \( G_1 \) is \([0, S]\) for all \( i = 1, \ldots, n \).

Let \( \varepsilon > 0 \) be given. As \( \phi \) is continuous, there exists a \( \delta > 0 \) such that whenever \( u, v \in [0,1]^n \) and \( \|u - v\| = \max_{i=1,\ldots,n} |u_i - v_i| < \delta \), then \( |\phi(u) - \phi(v)| < \varepsilon / 2S\psi(0) \).

There exist only a finite number of points \( r \) in \([0, S]\) where at least one of \( F_1, G_1, \ldots, F_n, G_n \) has a jump discontinuity with jump \( \geq \delta / 2 \). Hence we can find an integer \( N \) large enough so that

1. \( \psi(0)4rS \sup_{\varepsilon} |\phi| < N \)

and

2. on all but at most \( r \) of the \( N \) intervals \([0, S/N], \ldots, [(N-1)S/N, NS/N]\),

\[
\max_i \left[ F_i \left( \frac{jS}{N} \right) - F_i \left( \frac{(j+1)S}{N} \right) \right] < \delta \quad \text{and} \quad \max_i \left[ \overline{G}_i \left( \frac{jS}{N} \right) - \overline{G}_i \left( \frac{(j+1)S}{N} \right) \right] < \delta.
\]
Hence for each $i = 1, \ldots, n$, we define the following simple survival functions:

$$F_i''(t) = \left( \int_{jS/N}^{(j+1)S/N} F_i(t) \, dt \right) / S/N$$

and

$$G_i''(t) = \left( \int_{jS/N}^{(j+1)S/N} G_i(t) \, dt \right) / S/N$$

when $t \in \left[ \frac{jS}{N}, \frac{(j+1)S}{N} \right]$ for some $j = 0, \ldots, N-1$, and zero otherwise.

Note that $G_i'' \leq F_i''$ for all $i = 1, \ldots, n$.

Moreover,

$$\left| \int_0^S \psi(t) \phi(F_1(t), \ldots, F_n(t)) \, dt - \int_0^S \psi(t) \phi(F_1''(t), \ldots, F_n''(t)) \, dt \right|$$

$$= \left| \sum_{j=0}^{N-1} \frac{(j+1)S}{N} \psi(t) \left[ \phi(F_1(t), \ldots, F_n(t)) \, dt - \phi(F_1''(t), \ldots, F_n''(t)) \right] \right|$$

$$< \psi(0)2r \sup \left| \phi \right| \frac{S}{N} + \frac{\varepsilon}{2S} N \left( \frac{S}{N} \right)$$

$$< \varepsilon .$$

Similarly,

$$\left| \int_0^S \psi(t) \phi(G_1(t), \ldots, G_n(t)) \, dt - \int_0^S \psi(t) \phi(G_1''(t), \ldots, G_n''(t)) \, dt \right| < \varepsilon .$$

Therefore, it suffices to prove (2.2) for the case when all $F_i, G_i$ are step functions which are constant on $\left[ \frac{jS}{N}, \frac{(j+1)S}{N} \right]$, $j = 0, \ldots, N-1$. Furthermore, without loss of generality we may assume that $\psi$ is constant on each interval of the form $\left[ \frac{jS}{N}, \frac{(j+1)S}{N} \right]$ for $j = 0, \ldots, N-1$. 
(iii) Assume now that $G_i > F_i$ for $i = 1, \ldots, n$ and that all $2n$ functions have support in $[0, S]$ and are constant on each interval $\left[ \frac{jS}{N}, \frac{(j+1)S}{N} \right]$ for $j = 0, \ldots, N-1$. We also assume $\psi$ is constant on each of these intervals and use the notational simplification $\psi(j) = \psi\left(\frac{jS}{N}\right)$ for $j = 0, \ldots, N-1$.

Each $\bar{G}_i$ may be transformed into $\bar{F}_i$ by a finite succession of
transformations $\tau$ of the following type (see Hardy, Littlewood and Pólya [5]). $\tau$ changes the value $v_{ji}$ of $\bar{G}_i$ on the interval $\left[ \frac{jS}{N}, \frac{(j+1)S}{N} \right]$ into $v_{ji} + h$ and the value $v_{ki}$ of $\bar{G}_i$ on $\left[ \frac{kS}{N}, \frac{(k+1)S}{N} \right]$ into $v_{ki} - h$ where $j < k$ and

$$0 \leq v_{ki} - h \leq v_{ki} \leq v_{ji} \leq v_{ji} + h \leq 1.$$

Letting $\Delta_{\tau}$ denote the change in the integral $\int_0^\infty \psi(t) \phi(\bar{G}_1(t), \ldots, \bar{G}_n(t)) dt$ resulting from such a transformation $\tau$, we complete the proof by showing that $\Delta_{\tau} \geq 0$. Without loss of generality $i = 1$, and hence

$$\Delta_{\tau} = \sum_{j=1}^{N} \psi(j) \left[ \phi(v_{j1} + h, v_{j2}, \ldots, v_{jn}) - \phi(v_{j1}, v_{j2}, \ldots, v_{jn}) \right]$$

$$- \psi(k) \left[ \phi(v_{k1}, v_{k2}, \ldots, v_{kn}) - \phi(v_{k1} - h, v_{k2}, \ldots, v_{kn}) \right]$$

$$\geq \psi(k) \sum_{j=1}^{N} \left[ \phi(v_{j1} + h, v_{j2}, \ldots, v_{jn}) - \phi(v_{j1}, v_{j2}, \ldots, v_{jn}) \right]$$

$$- \phi(v_{j1} + h, v_{k2}, \ldots, v_{kn}) - \phi(v_{j1}, v_{k2}, \ldots, v_{kn}).$$

(since $\phi$ is convex in each variable separately)
= \psi(k)^{S_N} \{ \phi(v_{j1} + h, v_{k2} + h, \ldots, v_{kn} + h_n) - \phi(v_{j1}, v_{k2} + h, \ldots, v_{kn} + h_n) - \phi(v_{j1} + h, v_{k2} + h, \ldots, v_{kn}) + \phi(v_{j1}, v_{k2} + h, \ldots, v_{kn}) \} + \ldots + \phi(v_{j1}, v_{k2} + h, \ldots, v_{kn}) - \phi(v_{j1}, v_{k2} + h, v_{k3}, \ldots, v_{kn}) - \phi(v_{j1} + h, v_{k2}, \ldots, v_{kn}) + \phi(v_{j1}, v_{k2}, \ldots, v_{kn}) \}

\geq 0

(since \phi satisfies property (2.1) and \psi is nonnegative).

Here \( h_i = v_{ji} - v_{ki} \) for \( i = 2, \ldots, n \).

**Corollary 2.2.** Let \( G \) and \( F \) be life distribution functions with finite means. Then \( G \succcurlyeq F \) if and only if

a) For all nonnegative increasing continuous convex \( \phi \) and nonnegative decreasing \( \psi \),

\[
\int_0^\infty \psi(t) \phi(G(t)) dt \leq \int_0^\infty \psi(t) \phi(F(t)) dt
\]

and

b) For all nonnegative increasing continuous concave \( \phi \) and nonnegative increasing \( \psi \),

\[
\int_0^\infty \psi(t) \phi(G(t)) dt \geq \int_0^\infty \psi(t) \phi(F(t)) dt,
\]

provided the integrals exist.

**Proof.** The only if part follows immediately from Theorem 2.1. Assume now a) and b) hold. Letting \( \phi(u) = u \) and \( \psi_x(t) = \chi_{[x, +\infty)} \) (that is the characteristic function of the interval \([x, +\infty)\)) it follows from b) that

\[
\int_0^\infty G(t) dt \geq \int_0^\infty F(t) dt \text{ for all } x \geq 0.
\]

Taking \( \psi(t) = 1 \), it follows from a) that

\[
\mu_F = \mu_G.
\]
Corollary 2.3. If $G$ and $F$ are life distributions with finite means, then

$$G \geq M F \iff\int_0^\infty \psi(t)dG(t) \geq \int_0^\infty \psi(t)dF(t)$$

holds for all convex functions $\psi$, provided the integrals exist.

Proof. The if part of the result is immediate. Now suppose $G \geq M F$. It suffices to prove (2.5) for the case where $\psi$ has derivative $\psi$ and $\psi(0)=0$.

Then

$$\int_0^\infty \psi(t)dG(t) = \int_0^\infty \psi(t)G(t)dt$$

$$= \int_0^\infty [\psi(t)-\psi(0)]G(t)dt + \psi(0)\mu_G$$

$$\geq \int_0^\infty [\psi(t)-\psi(0)]F(t)dt + \psi(0)\mu_F \ (by \ Theorem \ 2.1)$$

$$= \int_0^\infty \psi(t)dF(t).$$

Remark 2.4. Another approach to (2.5) in the proof of Corollary 2.3 is as follows. Suppose $G \geq M F$. Let $Z_G$ and $Z_F$ be the random variables with respective densities $\frac{1}{\mu_G} \int_0^t G(s)ds$ and $\frac{1}{\mu_F} \int_0^t F(s)ds$. Then $Z_G \geq_{st} Z_F$ ($Z_G$ is stochastically larger than $Z_F$) and hence (see for example Ross [11]) $E(\psi(Z_G)) \geq E(\psi(Z_F))$ for all increasing $\psi$. But

$$\int_0^\infty \psi(t)G(t)dt = E(\psi(Z_G)) \geq E(\psi(Z_F)) = \int_0^\infty \psi(t)F(t)dt.$$

3. Applications.

Theorem 3.1. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be independent nonnegative random variables where $X_i \sim F_i$ and $Y_i \sim G_i$ for $i=1, \ldots, n$, and let $X_{[1]}', \ldots, X_{[n]}'$ and $Y_{[1]}', \ldots, Y_{[n]}'$ be respectively the $X$ ($Y$) observations in increasing order.
Assume that $G_i > F_i$ for $i = 1, \ldots, n$. Then

\begin{align*}
\text{a)} \quad \int_0^\infty \mathbb{P}[Y_n > \ldots > Y_k > t] \, dt & \geq \int_0^\infty \mathbb{P}[X_n > \ldots > X_k > t] \, dt \\
& \quad \text{for all } x \geq 0 \text{ and } k = 1, 2, \ldots, n.
\end{align*}

\begin{align*}
\text{b)} \quad (EY_1, \ldots, EY_n) & \geq (EX_1, \ldots, EX_n).
\end{align*}

**Proof.** b) follows immediately from a). In what follows $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ will denote any vector whose components are zeroes or ones. For $i = 1, \ldots, n$, we define $\phi_i : [0,1] \to [0, +\infty)$ by

$$
\phi_i(u_1, \ldots, u_n) = \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \geq u_1 + u_2 + \ldots + u_n \geq n-i+1} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n (1-u_1) (1-u_2) \ldots (1-u_n).
$$

We note that $EX_i = \int_0^\infty \phi_i(\overline{F}_1(t), \ldots, \overline{F}_n(t)) \, dt$ for $i = 1, \ldots, n$. Now for $k = 1, \ldots, n$ we define

$$
\phi_k(u_1, \ldots, u_n) = \sum_{i=k}^n \phi_i(u_1, \ldots, u_n)
$$

$$
= \sum_{i=k}^n \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \geq u_1 + u_2 + \ldots + u_n \geq n-i+1} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n (1-u_1) (1-u_2) \ldots (1-u_n)
$$

$$
= \sum_{j=1}^n \text{min}(j, n-k+1) \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \geq u_1 + u_2 + \ldots + u_n \geq j} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n (1-u_1) (1-u_2) \ldots (1-u_n).
$$

Since $\int_0^\infty \mathbb{P}[X_n > \ldots > X_k > t] \, dt = \int_0^\infty \phi_k(\overline{F}_1(t), \ldots, \overline{F}_n(t)) \, dt$, it suffices by Theorem 2.1 b to show that each $\phi_k$ satisfies (2.3) and is concave increasing in each variable separately.

Now $\frac{\partial}{\partial u_1} \phi_k(u_1, \ldots, u_n) = \sum_{j=0}^{n-k} \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \geq j} u_1 \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n (1-u_1) (1-u_2) \ldots (1-u_n)$

where $\varepsilon_1$ represents an $n-1$ component vector of zeroes and ones.
As \(\phi_k(u_1, \ldots, u_n)\) is symmetric in \(u_1, \ldots, u_n\), it follows that \(\phi_k\) is an increasing function linear (and hence concave) in each variable separately. For a continuously twice differentiable function \(\phi\) on \([0,1]^n\), it is easy to verify that the following conditions are equivalent (see Lorentz [7]):

\[
\begin{align*}
(3.1) \quad & \phi(u_i + h, u_j + k) - \phi(u_i + h, u_j) - \phi(u_i, u_j + k) + \phi(u_i, u_j) \geq 0 \\
& \text{for all } i \neq j, \quad 0 \leq u_i \leq u_i + h \leq 1, \quad 0 \leq u_j \leq u_j + k \leq 1.
\end{align*}
\]

\[
\begin{align*}
(3.2) \quad & \phi(u_i + h, u_j + h) - \phi(u_i + h, u_j) - \phi(u_i, u_j + h) + \phi(u_i, u_j) \geq 0 \\
& \text{for all } i \neq j, \quad 0 \leq u_i \leq u_i + h \leq 1, \quad 0 \leq u_j \leq u_j + h \leq 1.
\end{align*}
\]

\[
\begin{align*}
(3.3) \quad & \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \phi(u_1, \ldots, u_n) \geq 0 \\
& \text{for all } i \neq j.
\end{align*}
\]

Therefore, due to the symmetry of \(\phi_k\) and the above equivalence, it suffices to note that

\[
\frac{\partial}{\partial u_1} \frac{\partial}{\partial u_2} \phi_k(u_1, \ldots, u_n) = - \sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n = 0}^{n-k} \varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots \varepsilon_n (1-u_1)^{1-\varepsilon_1} (1-u_2)^{1-\varepsilon_2} \cdots (1-u_n)^{1-\varepsilon_n} \leq 0
\]

(where \(\varepsilon_{12}\) represents an \(n-2\) component vector of zeroes and ones).

**Remark 3.2.** Let \((X_1, \ldots, X_n)\) and \((Y_1, \ldots, Y_n)\) be random samples of size \(n\) from populations with life distribution functions \(F\) and \(G\) respectively. Barlow and Proschan [1] show that if \(G \succeq F\) where \(G\) and \(F\) have common mean, then

\[
(\text{EY}_1, \ldots, \text{EY}_n) \succ (\text{EX}_1, \ldots, \text{EX}_n).
\]
Shaked [13] proves the same result under the more general assumption that $G \geq F$. His proof uses the characterization of Corollary 2.3 together with the fact that

$$
\psi_k(t_1, \ldots, t_n) = t_{[n]}^+ \ldots + t_{[k]}
$$

is (separately) convex for each $k$. It follows that

$$
EY_{[n]}^+ \ldots + EY_{[k]} = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \psi_k(t_1, \ldots, t_n)dG(t_1) \ldots dG(t_n)
$$

$$
\geq \int_0^\infty \int_0^\infty \ldots \int_0^\infty \psi_k(t_1, \ldots, t_n)dF(t_1) \ldots dF(t_n)
$$

$$
= EX_{[n]}^+ \ldots + EX_{[k]}.
$$

Remark 3.3 Suppose that for each $a \in A$, $F^{(a)}$ is distribution function on $\mathbb{R}$, and that $\gamma$ is a probability measure defined on a $\sigma$-field of subsets of $A$. One may define the $n$-variate distribution function assuming appropriate measurability conditions on $F^{(a)}$

$$
F(x_1, \ldots, x_n) = \int_A F^{(a)}(x_1) \ldots F^{(a)}(x_n) d\gamma(a).
$$

If random variables $X_1, \ldots, X_n$ have such a joint distribution function, they are said to be 'positively dependent by mixture'. Given $X_1, \ldots, X_n$ positively dependent by mixture, let $Y_1, \ldots, Y_n$ be independent random variables where $Y_i$ is distributed as $X_i$ for $i = 1, \ldots, n$. Shaked [12] (See also Marshall and Olkin [9] and Proschan [10]) has shown that in this case

$$
(EY_{[1]}^+ \ldots , EY_{[n]}) \overset{\text{m}}{\succ} (EX_{[1]} \ldots , EX_{[n]}).
$$

Remark 3.4 Theorem 3.1 shows that if $G_i \overset{\text{m}}{\succeq} F_i$ for all $i=1, \ldots, n$, then for any $k \sum_{i=k}^n Y_{[i]}$ is "more variable" than $\sum_{i=k}^n X_{[i]}$ (in the terminology of Ross [11]) or that $\sum_{i=k}^n Y_{[i]}$ is "larger in mean residual life" than $\sum_{i=k}^n X_{[i]}$.
(in the terminology of Stoyan [14]). Since \( \psi_k(t_1, \ldots, t_n) = t_{[n]}^* + \cdots + t_{[k]} \)

is convex, this also follows by using the result that if \( X_1, \ldots, X_n, Y_1, \ldots, Y_n \) are independent and \( Y_i \) is "more variable" than \( X_i \) for \( i = 1, \ldots, n \), then \( \psi_k(Y_1, \ldots, Y_n) \) is "more variable" than \( \psi_k(X_1, \ldots, X_n) \) (see Ross [11]).

**Remark 3.5** If \( X_1, \ldots, X_n \) are independent HNBUE random variables, then Theorem 3.1 (b) could be useful in constructing bounds on the expected order statistics \( EX_1, \ldots, EX_n \).

**Example 3.6** Let us consider the following problem of general interest. \( n \) components are to be purchased in order to form a coherent system (for example a \( k \) out of \( n \) system), and all of the components are to be purchased from either company A or company B. Let us suppose that each company makes the claim that components of type \( i \) have mean life \( \mu_i \) (\( i = 1, \ldots, n \)), but that company B is known to be 'more variable' than company A in the production of any type of component. If we wish to maximize the mean life of the system, from which company should we buy?

Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be random variables representing the lifetimes of the components from A and B respectively. If we can assume that the components function independently within the system and that \( Y_i \) is more variable than \( X_i \) in the sense that \( G_i \overset{m}{\succ} F_i \) (where \( X_i \sim F_i \) and \( Y_i \sim G_i \)) for all \( i = 1, \ldots, n \), then we know that

\[
(EY_{[1]}, \ldots, EY_{[n]}) \overset{m}{\succ} (EX_{[1]}, \ldots, EX_{[n]}).
\]

In particular \( EY_{[1]} - EX_{[1]} \leq 0 \) and \( EY_{[n]} - EX_{[n]} \geq 0 \). Therefore if our system is a series system we would buy from A, while if it is parallel we would buy from B. This result was observed by Marshall and Proschan [8].
For a more general k out of n system, we would be interested in the expected order statistics \( EX[n-k+1] \) and \( EY[n-k+1] \) in order to compare companies A and B. Although

\[
(\text{EY}[1], \ldots, \text{EY}[n]) \cong (\text{EX}[1], \ldots, \text{EX}[n]),
\]

\( \text{EY}[i] - \text{EX}[i] \) may theoretically at least undergo many sign changes as \( i:1 \to n \) even in the case when \( F_1 = F \) and \( G_1 = G \) for all \( i=1, \ldots, n \). However under the assumption that \( G \cong F \) where \( G \) and \( F \) are continuous, \( G \) is strictly increasing on its interval support and \( G(0)=F(0)=0 \), one may show that the number of sign changes in \( \text{EY}[i] - \text{EX}[i] \) is no greater than the number of sign changes in \( \bar{G}(x) - \bar{F}(x) \) as \( x:0 \to \infty \). Since \( \binom{n-1}{i-1} F^{i-1}(t) \bar{F}^{n-i}(t) \) is totally positive of order \( \infty \) in \( i \) and \( t \), this follows using the variation diminishing property of totally positive functions and the identity

\[
\text{EY}[i] - \text{EX}[i] = \int_0^\infty n(G^{-1}F(t) - t) \binom{n-1}{i-1} F^{i-1}(t) \bar{F}^{n-i}(t) dt
\]

(see Barlow and Proschan [1]). In particular if \( \bar{F} \) crosses \( \bar{G} \) once then there exists a constant \( C \) (depending on \( n, F \) and \( G \)) such that

\[
\text{EY}[i] - \text{EX}[i] \leq 0 \quad \text{for} \quad i < C
\]

and

\[
\text{EY}[i] - \text{EX}[i] \geq 0 \quad \text{for} \quad i > C.
\]
References


