GENERATING POSITIVELY CORRELATED RANDOM VARIABLES FROM A
SEQUENCE OF INDEPENDENT RANDOM VARIABLES WITH SYMMETRIC
LOGARITHMICALLY CONCAVE DENSITIES

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GENERATING POSITIVELY CORRELATED RANDOM VARIABLES FROM A SEQUENCE OF INDEPENDENT RANDOM VARIABLES WITH SYMMETRIC LOGARITHMICALLY CONCAVE DENSITIES

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ABSTRACT

Let \( \mathbf{X} = (X_1, \ldots, X_n) \) be independent random variables with logarithmically concave symmetric densities. We show that for any logarithmically concave functions \( f \) and \( g \) on \( \mathbb{R}^n \) that are invariant under sign changes,

\[
\text{Cov}(f(X), g(X)) \geq 0.
\]

Bounds on the values of logarithmically concave densities on \( \mathbb{R}^n \) evaluated at the mean vector are also given.
I. INTRODUCTION AND SUMMARY.

The area of positive and negative dependence of multivariate distributions has attracted the attention of many authors over the past decade. (See Abdel-Hameed and Sampson (1978), Jogdeo (1977), Kanter (1975), Karlin and Rinott (1980), Dykstra (1980), and Prekopa (1973).) Pitt (1977) proves that if \( n(x) \) is the standard normal density on \( \mathbb{R}^2 \) and if \( A \) and \( B \) are balanced convex subsets of \( \mathbb{R}^2 \) (i.e., \( A = -A \) and \( B = -B \)) then

\[
\int_{A \cap B} n(x) \, dx \geq \left( \int_A n(x) \, dx \right) \left( \int_B n(x) \, dx \right).
\]

The question whether Pitt's result is true for any standard normal density on \( \mathbb{R}^n, n > 2 \), remains unanswered.

In this paper we investigate covariance inequalities for a class of logarithmically concave densities. We show that if \( X_1, \ldots, X_n \) are independent random variables each having a symmetric logarithmically concave symmetric density, then the random variables \( Y_1 = f(X_1, \ldots, X_n) \) and \( Y_2 = g(X_1, \ldots, X_n) \) are positively correlated whenever \( f \) and \( g \) belong to a certain class of logarithmically concave functions on \( \mathbb{R}^n \). In particular it follows that if \( A \) and \( B \) are subsets of \( \mathbb{R}^n \) that are symmetric along all the axes, then

\[
\int_{A \cap B} h(x) \, dx \geq \left( \int_A h(x) \, dx \right) \left( \int_B h(x) \, dx \right),
\]

where \( h \) is the joint density of the independent random variables \( X_1, \ldots, X_n \). We remark that a subset of \( \mathbb{R}^n \) that is symmetric along all the axes is necessarily a balanced set and it follows that if
(X_1, \ldots, X_n) is a standard normal vector, with density \( n \) then

\[
\int_{A \cap B} n(x) \, dx \geq \left( \int_A n(x) \, dx \right) \left( \int_B n(x) \, dx \right)
\]

for all subsets \( A \) and \( B \) of \( \mathbb{R}^n \) that are symmetric along all axes.

Throughout, the word "symmetric" will be used to mean "symmetric about the origin". For \( n=1, 2, \ldots \), let

\[
H_n = \{ f: \mathbb{R}^n \rightarrow \mathbb{R}_+, f \text{ is logarithmically concave and symmetric} \},
\]

\[
A_n = \{ A: A \text{ is a } n \times n \text{ diagonal matrix with diagonal elements } \pm 1 \},
\]

\[
G_n = \{ f \in H_n : f(xA) = f(x) \text{ for all } x \text{ in } \mathbb{R}_+ \text{ and all } A \in A_n \}, \text{ and}
\]

\[
L_n = \{ K: K \text{ is a convex symmetric subset of } \mathbb{R}^n \}.
\]

**Section 2. Positive Correlations of Functions of Multivariate Random Variables with Logarithmically Concave Densities.**

In this section we will show that if \( X_1, \ldots, X_n \) are independent random variables each having a logarithmically concave symmetric density, then the random variables \( Y_1 = f(X_1, \ldots, X_n) \) and \( Y_2 = g(X_1, \ldots, X_n) \) are positively correlated whenever \( f \) and \( g \) belong to \( G_n \).

**2.1 Theorem.** Let \( H \) be a convex subset of \( \mathbb{R}^n \). Then \( f: H \rightarrow \mathbb{R}_+ \) is in \( H_n \) if and only if the set \( H^* = \{(x,z): f(x,z) : f(x) \geq e^z \} \) is a symmetric subset of \( \mathbb{R}^n \) for each \( a \in \mathbb{R}_+ \).

**Proof.** (If) Suppose that \( f \) is not in \( H_n \). Then \( f \) is either not logarithmically concave or not symmetric. First assume that \( f \) is not logarithmically concave on \( H \). Then there exists \( x_1, x_2 \) in \( H \) and \( a \) in \( (0,1) \) such that

\[
f(ax_1 + (1-a)x_2) < f^a(x_1) f^{1-a}(x_2).
\]
Thus the point \((ax_1 + (1-a)x_2, \ln f(x_1) + (1-a)\ln f(x_2))\) belongs to the line segment joining \((x_1, \ln f(x_1)), (x_2, \ln f(x_2))\) but not in \(H^+\). Since \((x_1, \ln f(x_1)), (x_2, \ln f(x_2))\) are in \(H^+\), then \(H^+\) is not convex.

If \(f\) is not symmetric on \(H\), then there exists \(x_0\) in \(H\) such that \(f(x_0) \neq f(-x_0)\). Let \(a_0 = (f(x_0) \vee f(-x_0))\) and \(K_{a_0} = \{x: f(x) \geq a_0\}\). Then either \(x_0\) or \(-x_0\) is in \(K_{a_0}\) but not both, contradicting the assumptions that \(K_{a_0}\) is symmetric.

(Only if). Let \(f \in H^n, (x_1, z_1), (x_2, z_2)\) be any two points in \(H^+\) and \(\ell\) is the line joining them. Let \((x, z_2)\) be any two points in \(H^+\) and \(\ell\) is the line joining them. Let \((x, z)\) be any point on \(\ell\). Then there exists \(0 \leq a \leq 1\) such that

\[ x = ax_1 + (1-a)x_2, \]
\[ z = az_1 + (1-a)z_2. \]

Since \(f(x_1) \geq e^{z_1}\) and \(f(x_2) \geq e^{z_2}\), it follows that \(f(x) \geq e^{z}\). Therefore, \((x, z)\) is in \(H^+\). Thus \(H^+\) is convex.

The fact that \(\{x: f(x) > a\}\) is symmetric for each \(a \in R_+\) is immediate since \(f\) is symmetric. ||

2.2 Corollary. Let \(H\) be a convex subset of \(R^n\) and assume that \(f: H^n \rightarrow R_+\) is in \(H^n\). Then, \(\{x: f(x) \geq a\}\) is a convex and symmetric subset of \(R^n\) for each \(a \in R_+\).

Proof: The symmetry of the set \(\{x: f(x) \geq a\}\) is proved in Theorem 2.1. Now assume that \(x_1, x_2\) are in \(\{x: f(x) \geq a\}\). Then for \(a\) in \((0,1)\) we have

\[ f(ax_1 + (1-a)x_2) \geq f^a(x_1) f^{1-a}(x_2) \geq a. \]
Thus, $ax_1 + (1-a)x_2$ is in the set and hence it is convex; since $a$ is arbitrary, the result follows.||

The converse to Corollary 2.2 is not true:

Let $f$ be a concave symmetric function on $R$ which is not in $H_1$.

Then the sets \{x : f(x) \geq a\} are convex symmetric subsets of $R^1$. However $f$ is not in $H_1$.

The following lemma is due to Hoeffding and is a restatement of Lemma 2 of Lehmann (5).

2.3 Lemma. Let $X$ and $Y$ be extended-valued random variables. Then

$$\text{Cov}(X,Y) = \int_{R^2} \text{Cov}(I_{x_1 < \infty}^{X}, I_{y_1 < \infty}^{Y}) \, dx \, dy,$$

where the equality is valid even if one side is infinite.

2.4 Lemma. For any $K \in L_{n-1}$, $h \in H_{n-1}$, and $f \in H_n$, the function $g: \mathbb{R} \to \mathbb{R}_+$ defined by

$$g(x) = \int_{K} f(x_1, \ldots, x_{n-1}, x) h(x_1, \ldots, x_{n-1}) \, dx_1 \ldots dx_{n-1}$$

is in $H_1$.

Proof: The logconcavity of $g$ follows from Theorem 6 of Pukemia [1973]. The symmetry of $g$ follows from the symmetry of $f$, $h$, and $k$.

Thus $g$ is in $H_n$, as desired.||

2.5 Theorem. Let $f$ and $g$ be in $H_1$. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $X$ is an extended-value random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then for each $f$, $g$ in $H_1$ we have

$$\text{Cov}(f(X), g(X)) \geq 0.$$

Proof: By Lemma 2.3 we have

$$\text{Cov}(f(X), g(X)) = \int_{R^2} \text{Cov}(I_{[x, \infty]} \circ f(X), I_{[y, \infty]} \circ g(X)) \, dx \, dy$$

since

$$I_{[x, \infty]} \circ f(X) = I_{f^{-1}[x, \infty]} \circ X.$$
From Corollary 2.2 we know that there exists a constant $a > 0$ such that $f^{-1}[x, \infty] = [-a, a]$. Thus there exists an $a > 0$ such that

$$I_{[x, \infty]} \circ f(X) = I_{[-a, a]} \circ X$$

Similarly we conclude that there is a $b > 0$ such that

$$I_{[y, \infty]} \circ g(x) = I_{[-b, b]} \circ X.$$ 

Therefore,

$$\text{Cov}(I_{[x, \infty]} \circ f(X), I_{[y, \infty]} \circ g(X))$$

is clearly nonnegative. From Lemma 2.3 it follows that $	ext{Cov}(f(X), g(X))$ must be nonnegative.

2.6 Theorem. Let $X_1, \ldots, X_n$ be independent random variables each having a density that belongs to $H$. Then for all $f$ and $g$ in $G_n$ we have $\text{Cov}(f(X), g(X)) \geq 0$.

Proof: We proceed by induction on $n$. By Theorem 2.5 the result is true for $n=1$. Now assume the result is true for some $n_0$. For $f$ and $g$ in $H_{n_0+1}$ and $X = (X_1, \ldots, X_{n_0+1})$, we write

$$\text{Cov}(f(X), g(X)) = E[\text{Cov}(f(X), g(X)|X_{n_0+1})]$$

$$+ \text{Cov}[E(f(X)|X_{n_0+1}), E(g(X)|X_{n_0+1})].$$

For a fixed $x_{n_0+1}$, the function $f_{x_{n_0+1}} : R^n + R_+$ is defined for any $x = (x_1, \ldots, x_{n_0})$ by $f_{x_{n_0+1}}(x_1, \ldots, x_{n_0}) = f(x_1, \ldots, x_{n_0+1})$. For $x = (x_1, \ldots, x_{n_0})$, $x_1 = (x_{11}, \ldots, x_{1n_0})$, $x_2 = (x_{21}, \ldots, x_{2n_0})$, and $x^*_{n_0+1}$ and $x^*_{n_0+1} = (x_{21}, \ldots, x_{2n_0+1})$. Then for $\lambda$ in $(0, 1)$,

$$f_{x_{n_0+1}}(ax_{n_0+1} + (1-a)x_2) = f(ax_{n_0+1} + (1-a)x^*_{n_0+1}) \geq f^a(x^*_{n_0+1}) f^1-a(x^*_{n_0+1})$$

$$= f^a_{x_{n_0+1}}(x_1) f^1-a_{x_{n_0+1}}(x_2).$$
Moreover, for any matrix $A$ in $A^*_{n_0}$ and $x = (x_1, \ldots, x_{n_0})$ in $R^{n_0}$ and $x^* = (x_1, \ldots, x_{n_0}, x_{n_0+1})$, we have $f_{x^*_{n_0+1}}(xA) = f(x^*A^*)$,

where $A^*$ is the $(n_0+1) \times (n_0+1)$ diagonal matrix defined by $A^* = (A^0)_{01}$.

Since $f$ is in $G^*_{n_0+1}$, then we have that

$$f_{x^*_{n_0+1}}(xA) = f(x^*A^*)$$

$$= f(x^*)$$

$$= f_{x^*_{n_0+1}}(x).$$

Therefore $f_{x^*_{n_0+1}}(x_1, \ldots, x_{n_0})$ is in $G^*_{n_0}$. The induction hypothesis combined with the above argument gives

$$E[\text{Cov}\{f(X), g(X)\} \mid x^*_{n_0+1} = x^*_{n_0+1}] = 0$$

$$E[\text{Cov}\{f_{x^*_{n_0+1}}(X^*), g_{x^*_{n_0+1}}(X^*)\}] \geq 0$$

where $X^* = (X_1, \ldots, X_{n_0})$ when $x = (x_1, \ldots, x_{n_0+1})$

From Lemma 2.4 and the hypothesis, we deduce that

$Ef(X \mid X_{n_0+1} = x_{n_0+1})$ as well as $Eg(X \mid X_{n_0+1} = x_{n_0+1})$ are in $H_1$.

By Theorem 2.5 we have $\text{Cov}(Ef(X \mid X_{n_0+1})) \geq 0$. Thus we finally conclude that $\text{Cov}(f(X), g(X)) \geq 0$.

The proof of the following theorem can be obtained by imitating the proof of Theorem 2.6.

2.7 Theorem. Let $X = (X_1, \ldots, X_n)$ be a standard normal vector with density $n$. Then

$$\int A^n B^n (\int (x) dx) (\int (x) dx).$$

for all subsets $A, B$ that are symmetric along all the axes.


In this section we derive some inequalities for strongly unimodal densities. First we define for $n=1,2,...$,

$\mathcal{U}_n = \{f: \mathbb{R}^n \rightarrow \mathbb{R}_+; f$ is a logarithmically concave density\}.

$\mathcal{V}_n = \{f: \mathbb{R}^n \rightarrow \mathbb{R}_+; f$ is a logarithmically convex density\}.

3.1 Lemma. Let $f$ and $g$ be functions mapping $\mathbb{R}^n$ into $\mathbb{R}_+$. Then

$$\int f \ln (f/g) \geq (\int f) \ln (\int f/\int g).$$

In particular, if $f$ is a density function on $\mathbb{R}^n$, then

$$\int f \ln (f/g) \geq -\ln \int g,$$

for any measurable function $g: \mathbb{R}^n \rightarrow \mathbb{R}_+$.

**Proof:** First assume $\int f = \int g$. Then

$$\int f \ln(f/g) = -\int f \ln(g/f) \geq -\int f((g/f)-1)$$

[since $\ln x \leq (x-1)$, $x \geq 0$]

$$= \int f - \int g = 0.$$

Thus the inequality is satisfied in this case.

Now, if $\int f \neq \int g$, then define $g^* = (\int f/\int g)g$, and note that $\int f = \int g^*$. Therefore, using the above inequality we have

$$\int f \ln (f/g^*) \geq 0.$$ Using the definition of $g^*$ and simplifying we get

$$\int f \ln (f/g) \geq (\int f) \ln (\int f/\int g).$$

3.2 Theorem. Let $X$ be a random vector with density $f$ belonging to $\mathcal{U}_n$ and finite mean $\mu$. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $\int \exp(-g) = 1$, then

$$f(\mu) \geq \exp(-\int gf).$$

**Proof:** Take $f_1 = f$, $f_2 = e^{-g}$. By Lemma 3.1, we have $\int f_1 \log(f_1/f_2) \geq 0$. Take $f_1 = f$, $f_2 = e^{-g}$. Using Jensen's inequality and the fact that $f$
belongs to \( U_n \), we deduce that

\[ 0 \leq \int f(\log f + g) \leq \log f(\mu) + \int fg, \]

completing the proof. ||

3.3 Theorem. Let \( \mathbf{X} \) be a nonnegative random vector with density \( f \) belonging to \( U_n \) with finite mean \( \mu \). Then \( f(\mu) \geq \frac{1}{\mu_1 \cdots \mu_n} e^{-n} \).

Equality is attained for \( f(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\mu_i} e^{-x_i/\mu_i} \) and therefore the bound is sharp.

Proof: Choose \( g(x) = \sum \log \frac{\mu_i}{a_i} + \sum \frac{x_i}{a_i} \) for \( x \geq 0 \). Then

\[ \int e^{-g} = \prod_{i=1}^{n} \int_{0}^{\infty} \frac{a_i}{\mu_i} e^{-a_i x_i/\mu_i} \, dx_i = 1. \]

Also \( \int fg = \sum \log \frac{\mu_i}{a_i} + \sum a_i \). Thus by Theorem 3.2, \( f(\mu) \geq \prod_{i=1}^{n} \left( \frac{a_i}{\mu_i} e^{-a_i} \right) \). The right hand side is maximized by choosing \( a_i = 1, i=1, \ldots, n \). Equality is attained for \( f(\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\mu_i} e^{-x_i/\mu_i} \), as may be directly verified. ||

The following theorem gives a lower bound on the peak of density function belonging to \( U_n \) in terms of the determinant of its covariance matrix.

3.4 Theorem. Let \( \mathbf{X} \) be a multivariate vector with mean \( \mu \), covariance matrix \( \Sigma \), and density \( f \) belonging to \( U_n \). Then

\[ f(\mu) \geq (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp(-n/2). \]

Proof: Let \( g(x) = \frac{1}{2} \ln |\Sigma| + (n/2) \ln (2\pi) + \frac{1}{2} (x-\mu)^\top \Sigma^{-1} (x-\mu) \).

Then \( \int e^{-g} = 1 \), and \( \int g(x) f(x) \, dx = Ef(x) = \frac{1}{2} \ln |\Sigma| + (n/2) \ln (2\pi) + n/2. \)

The desired conclusion then follows from Theorem 3.2. ||
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Let $X = (X_1, \ldots, X_n)$ be independent random variables with logarithmically concave symmetric densities. We show that for any logarithmically concave functions $f$ and $g$ on $\mathbb{R}^n$ that are invariant under sign changes,

$$\text{Cov}(f(X), g(X)) \geq 0.$$  

Bounds on the values of logarithmically concave densities on $\mathbb{R}^n$ evaluated at the mean vector are also given.