SADDLEPOINT THEORY FOR NONDIFFERENTIABLE
MULTIOBJECTIVE FRACTIONAL PROGRAMMING

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FSU Statistics Report M691

January, 1985

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† This research was partially supported by the FOCUS Grant from the GTE Company.
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ABSTRACT

Saddlepoint theory is developed for the nondifferentiable multiobjective fractional programming problem. Necessary and sufficient conditions of Fritz John and Kuhn-Tucker type are established under convexity restrictions. All functions involved are assumed to be nondifferentiable.

KEY WORDS: Fractional, nondifferentiable, multiobjective, saddlepoint, Fritz John, Kuhn-Tucker.
1. INTRODUCTION

Multicriteria optimization (vector optimization) has its origin in economics. Economic equilibrium theory, game theory, production theory and welfare theory are some areas in which substantial applications have been made. For a survey paper of the results prior to 1960, see Stadler [7]. In recent years, the topic has been developed along the lines of classical linear and nonlinear programming. The reader may refer (for partial references) to, [3] for efficiency, [6] for scalarization, [4] and [8] for necessary and sufficient conditions and [2] for duality. The contribution of Datta [3] is with fractional objectives.

This paper is based on the ideas presented in [1] and [4]. We generalize these ideas in the setting of a fractional objective function. Saddle-point theory of the type presented in [5] is developed in the nondifferentiable framework. We consider the following vector minimization problem

\[(\text{VP}_0) \quad \text{Min} \left( \frac{f_1(x)}{h_1(x)}, \ldots, \frac{f_p(x)}{h_p(x)} \right) \]

subject to \(g_j(x) \leq 0, \quad x \in S, \quad j = 1, \ldots, m\)

where \(S\) is a convex, compact subset of \(\mathbb{R}^n\); \(f_i, h_i, g_j:\)

\(S \rightarrow \mathbb{R} (i = 1, \ldots, p; \quad j = 1, \ldots, m)\). All functions are assumed to be continuous. Further, \(f_i, h_i, g_j (i = 1, \ldots, p; \quad j = 1, \ldots, m)\) are considered to be convex and all \(h_i(x) > 0\) for all \(x \in S\). Without loss of generality, we also assume that \(f_i (i = 1, \ldots, p)\) is nonnegative. This is possible because if some \(f_i\) is not nonnegative, we can assume that it is bounded from below by some number \(d_i\) and \(f_i\) can be replaced by \(f_i - d_i\), which is nonnegative and convex. Consider the following associated problem:

\[(\text{VP}_1) \quad \text{Min} \left( F_1(x, y_1^0), \ldots, F_p(x, y_p^0) \right) \]

subject to \(g_i(x) \leq 0, \quad i = 1, \ldots, m\)
where $F_i(x, y_i^0) = f_i(x) - y_i^0 h_i(x)$ and the $y_i^0$ are some nonnegative fixed real numbers.

Let $X$ be the set of feasible solutions of $(VP_0)$ or $(VP_1)$.

2. NOTATIONS, PRELIMINARIES

For $x, y \in \mathbb{R}^n$,

$x \preceq y$ means $x_i \preceq y_i$ for all $i$

$x \preceq y$ means $x_i \preceq y_i$ for all $i$ and $x_i < y_i$

for at least one $i$.

The zero vector of any appropriate dimension is denoted by $\mathbf{0}$.

For $W \subseteq \mathbb{R}^q$, define $w^* \in W$ to be a minimal element of $W$ if there exists no $w^0 \in W$ such that $w^0 \leq w^*$. The set of minimal elements of $W$ is denoted by $\min(W)$. In this terminology, $x^0$ is an efficient solution of $(VP_0)$ if $x^0 \in X$ and $(f_1(x^0)/h_1(x^0), \ldots, f_p(x^0)/h_p(x^0)) \in \min_1 \{ (f_1(x)/h_1(x), \ldots, f_p(x)/h_p(x)) \}$. A similar statement can be made for an efficient solution of $(VP_1)$.

The following two results will be needed. The first is analogous to a result in [1].

**LEMMA 2.1.** If Problem $(VP_0)$ has an efficient solution $x^0$ with

$$(f_1(x^0)/h_1(x^0), \ldots, f_p(x^0)/h_p(x^0)) = (y_1^0, \ldots, y_p^0)$$

then

$$\bar{u} = (F_1(x^0, y_1^0), \ldots, F_p(x^0, y_p^0)) \in \min_1 \{ (F_1(x, y_1^0), \ldots, F_p(x, y_p^0)) \}.$$ Conversely

if for some $x^0 \in X$ with $y_i^0 = f_i(x^0)/h_i(x^0)$ so that

$$\bar{u} = (F_1(x^0, y_1^0), \ldots, F_p(x^0, y_p^0)) \in \min_1 \{ (F_1(x, y_1^0), \ldots, F_p(x, y_p^0)) \}$$

then

$$y_i^0 = \min_1 \{ (f_1(x^0)/h_1(x^0), \ldots, f_p(x^0)/h_p(x^0)) \}$$

$x \in X$

That is, $x_0$ is an efficient solution of $(VP_0)$.

**PROOF.** Suppose $x^0$ is an efficient solution of $(VP_0)$ with

$$(f_1(x^0)/h_1(x^0), \ldots, f_p(x^0)/h_p(x^0)) = (y_1^0, \ldots, y_p^0).$$

Then $f_i(x^0)/h_i(x^0) = y_i^0$
implies \( f_i(x^0, y^0_i) = f_i(x^0) - y^0_i h_i(x^0) = 0 \) for \( i = 1, \ldots, p \). Now suppose 
\[
\bar{\delta} = \min_{x \in X} \{ F_1(x, y^0_1), \ldots, F_p(x, y^0_p) \}.
\]
Then there exists \( x^* \in X \) such that \( F(x^*, y^0_1), \ldots, F(x^*, y^0_p) \leq \bar{\delta} \). Therefore 
\[
f_i(x^*)/h_i(x^*) \leq y^0_i \text{ for all } i \text{ and } f_i(x^*)/h_i(x^*) < y^0_i \text{ for at least one } i.
\]
So \( x^0 \) is not an efficient solution of \((VP_0)\). This contradiction implies that \( x^0 \) is an efficient solution of \((VP_1)\).

Conversely suppose there exists \( x^0 \in X \) with \( f_i(x^0)/h_i(x^0) = y^0_i \) for \( i = 1, \ldots, p \) so that \( \bar{\delta} = \min_{x \in X} \{ F_1(x, y^0_1), \ldots, F_p(x, y^0_p) \} \). Suppose \( x^0 \) is not an efficient solution of \((VP_0)\). That is there exists \( x^* \in X \) such that 
\[
(f_1(x^*)/h_1(x^*), \ldots, f_p(x^*)/h_p(x^*)) \leq (f_1(x^0)/h_1(x^0), \ldots, f_p(x^0)/h_p(x^0)) \leq (y^0_1, \ldots, y^0_p).
\]
From (2.1), we have \( F_1(x^*, y^0_1), \ldots, F_p(x^*, y^0_p) \leq 0 \) which means that \( \bar{\delta} \) is not \( \min_{x \in X} \{ F_1(x, y^0_1), \ldots, F_p(x, y^0_p) \} \) which gives a contradiction. Hence \( x^0 \) is an efficient solution of \((VP_0)\). The proof is now complete. \( \square \)

Next we state Kanniappan's Lemma 3.1 [4] in the framework of problem \((VP_1)\):

**LEMMA 2.2.** Problem \((VP_1)\) has an efficient solution \( x^0 \) with \( f_i(x^0)/h_i(x^0) = y^0_i \) (for \( i = 1, \ldots, p \)) if and only if \( x^0 \) minimizes each \( F_i(x, y^0_i) \) on the constraint set \( C_i = \{ x \in X : F_j(x, y^0_j) \leq F_j(x^0, y^0_j), j \neq i \} \) for \( i = 1, \ldots, p \).

3. NECESSARY AND SUFFICIENT CONDITIONS

We define the Fritz John Saddlepoint problem (FJSPP) and the Kuhn-Tucker Saddlepoint problem (KTTP) associated with Problem \((VP_1)\).

The Fritz John Saddlepoint Problem (FJSPP)

For some fixed \( y^0_1, \ldots, y^0_p \) find \( x^0 \in X, a^0_i = (a^0_{i1}, \ldots, a^0_{ip}), b^0_i = (b^0_{i1}, \ldots, b^0_{in}), (a^0_i, b^0_i) \geq 0 \) (for \( i = 1, \ldots, p \)) if they exist such that
\[
\begin{align*}
a_{ij}^0 F_i(x^0, y_i^0) + \sum_{j=1}^{p} a_{ij} F_j(x^0, y_j^0) + \sum_{k=1}^{m} b_{ik} g_k(x^0) \leq \sum_{j=1}^{p} a_{ij}^0 F_j(x^0, y_j^0) + \sum_{k=1}^{n} b_{ik}^0 g_k(x^0)
\end{align*}
\]

for \(i = 1, \ldots, p\) and for all \(x \in X\), \(a_{ij} \geq 0\), \(a_{ij} \in R\), \(b_{ik} \geq 0\), \(b_{ik} \in R\).

The Kuhn-Tucker Saddlepoint Problem (KTSPP)

For some fixed \(y_1^0, \ldots, y_p^0\), find \(x^0 \in X\), \(u_i^0 = (u_{i1}^0, \ldots, u_{i(i-1)}^0, u_{ii}^0, u_{i(i+1)}^0, \ldots, u_{ip}^0)\), \(v_i^0 = (v_i1^0, \ldots, v_i n^0)\), \((u_i^0, v_i^0) \geq 0\) (for \(i = 1, \ldots, p\)) if they exist such that

\[
\begin{align*}
F_i(x^0, y_i^0) + \sum_{j=1}^{p} u_{ij} F_j(x^0, y_j^0) + \sum_{k=1}^{m} v_{ik} g_k(x^0) \\
\leq F_i(x^0, y_i^0) + \sum_{j=1}^{p} u_{ij}^0 F_j(x^0, y_j^0) + \sum_{k=1}^{m} v_{ik}^0 g_k(x^0)
\end{align*}
\]

for \(i = 1, \ldots, p\), \(u_{ij} \geq 0\), \(u_{ij} \in R\) \((i \neq j)\), \(v_{ik} \geq 0\), \(v_{ik} \in R\) for all \(i, k\).

THEOREM 31. (Sufficiency). If \((x^0, u_i^0, v_i^0)\) with \(f_i(x^0)/h_i(x^0) = y_i^0\) (for \(i = 1, \ldots, p\)) is a solution of (KTSPP) then \(x^0\) is an efficient solution of (VP$_1$) and hence of (VP$_0$). If \((x^0, a_i^0, b_i^0)\) with \(f_i(x^0)/h_i(x^0) = y_i^0\) (for \(i = 1, \ldots, p\)) is a solution of (FJSPP) and \(a_{ii}^0 > 0\) for \(i = 1, \ldots, p\) then \(x^0\) is an efficient solution of (VP$_1$) and hence of (VP$_0$).

PROOF. As in [5], we establish the theorem under the first hypothesis because the second statement follows from the first. We note that \(F_j(x^0, y_j^0) = 0\) for \(j = 1, \ldots, p\). From the first inequality in (KTSPP) and taking \(v_{ik} = v_{ik}^0\) for \(k = 1, \ldots, m\); \(i \neq j\) and \(v_{ij} = v_{ij}^0 + 1\), it follows that \(g_j(x^0) \leq 0\). Repeating this process for all \(j\), we have \(g(x^0) \leq 0\). This means \(x^0\) is a feasible solution of (VP$_1$). Since \(v_{ik}^0 \geq 0\), \(g_k(x^0) \leq 0\) for \(k = 1, \ldots, m\), it follows that

\[
\sum_{k=1}^{m} v_{ik}^0 g_k(x^0) \leq 0
\]

for \(i = 1, \ldots, m\). Again using the first inequality in (KTSPP) and letting \(v_{ik} = 0\) for all \(k\), we obtain

\[
\sum_{k=1}^{m} v_{ik}^0 g_k(x^0) \geq 0.
\]

Hence
\begin{align}
\sum_{k=1}^{m} v_{ik}^0 g_k(x^0) = 0. \text{ Now from the second inequality in (KTSSPP) and for any } x \in C_i
\end{align}

\begin{equation}
F_i(x^0, y_i^0) - F_i(x, y_i^0) + \sum_{j=1}^{p} u_{ij}^0 (F_j(x^0, y_j^0) - F_j(x, y_j^0)) - \sum_{k=1}^{m} v_{ik}^0 g_k(x) \leq 0
\end{equation}

for \( i = 1, \ldots, p \). Since \( g_k(x) \leq 0 \) and \( v_{ik}^0 \geq 0 \), we have \( -\sum_{k=1}^{m} v_{ik}^0 g_k(x) \geq 0 \). So from (3.1) we have

\begin{equation}
F_i(x^0, y_i^0) - F_i(x, y_i^0) + \sum_{j=1}^{p} u_{ij}^0 (F_j(x^0, y_j^0) - F_j(x, y_j^0)) \leq 0.
\end{equation}

Since it is assumed that \( x \in C_i \) and \( u_{ij}^0 \geq 0 \) for all \( j, j \neq i \), we have

\begin{equation}
\sum_{j=1}^{p} u_{ij}^0 (F_j(x^0, y_j^0) - F_j(x, y_j^0)) \geq 0 \text{ for } j = 1, \ldots, p.
\end{equation}

From (3.2) and (3.3) we have \( F_i(x^0, y_i^0) \leq F_i(x, y_i^0) \) \( \forall x \in C_i \) for \( i = 1, \ldots, p \). Hence \( x^0 \) minimizes \( F_i(x, y_i^0) \) over \( C_i \) for each \( i \). Therefore by Lemma 2.2, \( x^0 \) is an efficient solution of (VP) and hence (by Lemma 2.1) of (VP_0).

This completes the proof. \( \square \)

The next result is along the lines of Theorem 5.4.1 in [5].

THEOREM 3.2. (Necessary Conditions). Suppose

(i) \( S \) is a convex set

(ii) \( f_i, -h_i(i = 1, \ldots, p) \), \( g_j(j = 1, \ldots, m) \) are convex functions and \( h_i(x) > 0 \) for all \( x \in X \) for \( i = 1, \ldots, p \)

(iii) \( f_i(x) \geq 0 \) for all \( x \in X \)

(iv) \( x^0 \) is an efficient solution of (VP) with \( f_i(x^0)/h_i(x^0) = y_i^0 \) for \( i = 1, \ldots, p \).

Then there exist \( a_i^0 \in \mathbb{R}^p \), \( b_i^0 \in \mathbb{R}^m \), \( (a_i^0, b_i^0) \geq 0 \) for \( i = 1, \ldots, p \) such that \( (x^0, a_i^0, b_i^0) \) satisfy (FJSPP) and \( b_i^0 g(x^0) = 0 \) for \( i = 1, \ldots, p \).

PROOF. By Lemma 2.1, \( x^0 \) is an efficient solution of (VP). Now it follows
by Lemma 2.2, that the system

\[ F_i(x, y_i^0) - F_i(x^0, y_i^0) \leq 0 \]

\[ F_j(x, y_j^0) - F_j(x^0, y_j^0) \leq 0 \quad \forall j \neq i \]

\[ g(x) \leq 0 \]

has no solution \( x \). By Cor. 4.2.2 [5], there exist \( a_i^0 \in \mathbb{R}^p, b_i^0 \in \mathbb{R}^m, (a_i^0, b_i^0) \geq 0 \) for \( i = 1, \ldots, p \) such that

\[ \sum_{i=1}^{p} a_{ij}^0 (F_i(x, y_i^0) - F_i(x^0, y_i^0)) + \sum_{j=1}^{p} a_{ij}^0 (F_j(x, y_j^0) - F_j(x^0, y_j^0)) + \sum_{k=1}^{m} b_{ik}^0 g_k(x) \geq 0 \quad (3.4) \]

But since \( b_{ik}^0 \geq 0 \) and \( g_j(x^0) \leq 0 \), we have

\[ \sum_{k=1}^{m} b_{ik}^0 g_k(x^0) \leq 0 \quad (3.5) \]

Letting \( x = x^0 \) in (3.4) we have

\[ \sum_{k=1}^{m} b_{ik}^0 g_k(x^0) \geq 0 \quad (3.6) \]

Therefore

\[ b_{ik}^0 g(x^0) = \sum_{k=1}^{m} b_{ik}^0 g_k(x^0) = 0 \quad (3.7) \]

Now from (3.4), in view of (3.7),

\[ \sum_{i=1}^{p} a_{ij}^0 F_i(x, y_i^0) + \sum_{k=1}^{m} b_{ik}^0 g_k(x^0) \leq \sum_{j=1}^{p} a_{ij}^0 F_j(x, y_j^0) + \sum_{k=1}^{m} b_{ik}^0 g_k(x) \quad (3.8) \]

Since \( g_k(x^0) \leq 0, F_j(x^0, y_j^0) = 0 \), for all \( a_{ii}^0 \geq 0, a_{ij}^0 \geq 0, a_{ij}^0 \geq 0, b_{ik} \geq 0, b_{ik}^0 \geq 0 \),

\[ a_{ii}^0 F_i(x^0, y_i^0) + \sum_{j=1}^{n} a_{ij}^0 F_j(x^0, y_j^0) + \sum_{k=1}^{m} b_{ik}^0 g_k(x^0) \]

\[ \leq \sum_{j=1}^{p} a_{ij}^0 F_j(x^0, y_j^0) + \sum_{k=1}^{n} b_{ik}^0 g_k(x^0) \quad (3.9) \]
Combining (3.8) and (3.9), the proof is completed for \( i = 1, \ldots, p \). □

The final result follows Theorem 5.4.7 in [5]. We state it without proof.

**THEOREM 3.3.** Under the hypotheses of Theorem 3.2 and the assumption that \( g \) satisfies one of the constraint qualifications, if \( x^0 \) is an efficient solution of \( (VP_0) \) with \( f_i(x^0)/h_i(x^0) = y^0_i \) for \( i = 1, \ldots, p \), then there exist

\[
\begin{align*}
  u^0_i & = (u^0_{i1}, \ldots, u^0_{i,i-1}, u^0_{i,i+1}, \ldots, u^0_{ip}), \\
  v^0_i & = (v^0_{i1}, \ldots, v^0_{in}), \quad (u^0_i, v^0_i) \geq 0
\end{align*}
\]

such that \((x^0, u^0_i, v^0_i)(for \ i = 1, \ldots, p)\) solve \( (KTSP) \).


