OPTIMALITY CRITERIA IN MATHEMATICAL PROGRAMMING
IN VOLVING GENERALIZED INVEXITY

BY

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1. **INTRODUCTION**

The concept of invexity was introduced by Hanson [2] as a generalization of convexity for constrained optimization problems of the form

\[ \min f(x) \text{ for } x \in X \subseteq \mathbb{R}^n \text{ subject to } g(x) \leq 0 \quad (1.1) \]

where \( f: X \rightarrow \mathbb{R} \) and \( g: X \rightarrow \mathbb{R}^m \) are differentiable functions on a set \( X \subseteq \mathbb{R}^n \).

Hanson [2] showed that weak duality and sufficiency of the Kuhn-Tucker conditions hold when invexity is required instead of the usual requirement of convexity.

Subsequently, Hanson and Mond [3] introduced two new classes of functions which are not only sufficient but are also necessary for optimality in primal and dual problems, respectively.

Let

\[ P = \{ x \mid x \in X, g(x) \leq 0 \}, \text{ and} \]

\[ D = \{ x \mid (x, y) \in Y \}, \]

where \( Y = \{ (x, y) \mid x \in X, y \in \mathbb{R}^m, \nabla_x f(x) + y^T [\nabla_x g(x)] = 0, y \geq 0 \} \).

In [3], Hanson and Mond defined \( f(x) \) and \( g(x) \) as Type I objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in P \) such that

\[ f(x) - f(x_0) \succeq [\nabla_x f(x_0)]^T \eta(x) \quad (1.2) \]

and

\[ -g(x_0) \succeq [\nabla_x g(x_0)] \eta(x), \quad (1.3) \]

and \( f(x) \) and \( g(x) \) as Type II objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in D \) such that

\[ f(x_0) - f(x) \succeq [\nabla_x f(x)]^T \eta(x) \quad (1.4) \]

and

\[ -g(x) \succeq [\nabla_x g(x)] \eta(x). \quad (1.5) \]
2.

In the definitions of Type I and Type II functions we shall consider $x_0$ to be fixed.

If $x_0$ is not fixed the definition of Type I would be written as

$$f(x) - f(x_0) \geq \sum x f(x_0) \eta(x, x_0)$$

and

$$- g(x_0) \geq \sum x g(x_0) \eta(x, x_0)$$

and the definition of Type II would be equivalent to this.

**Example 2.1.** The functions $f(x) = -\frac{1}{x}$ and $g(x) = -x + 1$ are Type I with respect to $\eta(x) = 1 - \frac{1}{x}$ at $x_0 = 1$ but $f(x)$ and $g(x)$ are not Type II at $x_0$.

**Example 2.2.** The functions $f: [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \sin^2 x$ and $g: [0, \pi/2] \to \mathbb{R}$ defined by $g(x) = -\cos x$ are Type II with respect to $\eta(x) = 0$ at $x_0 = \pi/2$ but $f(x)$ and $g(x)$ are not Type I at $x_0$.

In [1], Ben-Israel and Mond gave a sufficient condition for invexity. The next two theorems use that idea to find sufficient conditions for Type I and Type II functions.

**Theorem 2.1.** If $f(x)$ and $g(x)$ are differentiable at $x_0$ and there exists an $n$-dimensional vector function $\eta(x)$ such that

$$f(x_0 + \lambda \eta(x)) \leq \lambda f(x) + (1 - \lambda)f(x_0) \quad 0 \leq \lambda \leq 1$$

and

$$g(x_0 + \beta \eta(x)) \leq (1 - \beta)g(x_0) \quad 0 \leq \beta \leq 1$$

for all $x \in \mathbb{R}$, then $f(x)$ and $g(x)$ are Type I.
PROOF. Since (1.2) follows from (2.1) by Theorem 2 of [1], it suffices to show that (1.3) follows from (2.2).

The last inequality can be rewritten as
\[ g(x_0 + \beta \eta(x)) - g(x_0) \leq -\beta g(x_0) \]

Assume \( \beta > 0 \) and divide by \( \beta \) to obtain
\[ \frac{1}{\beta} [g(x_0 + \beta \eta(x)) - g(x_0)] \leq -g(x_0) \]
The limit as \( \beta \to 0^+ \) gives (1.3).

THEOREM 2.2. If \( f(x) \) and \( g(x) \) are differentiable on \( D \) and there exists an \( n \)-dimensional vector \( \eta(x) \) such that
\[ f(x + \lambda \eta(x)) \leq \lambda f(x_0) + (1 - \lambda) f(x) \quad 0 \leq \lambda \leq 1 \tag{2.3} \]
and
\[ g(x + \beta \eta(x)) \leq (1 - \beta) g(x) \quad 0 \leq \beta \leq 1 \tag{2.4} \]
for all \( x \in D \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are Type II.

PROOF. Since (1.4) follows from (2.3) by Theorem 2 of [1], it suffices to show that (1.5) follows from (2.4).

The last inequality can be rewritten as
\[ g(x + \beta \eta(x)) - g(x) \leq -\beta g(x) \]
Assume \( \beta > 0 \) and divide by \( \beta \) to obtain
\[ \frac{1}{\beta} [g(x + \beta \eta(x)) - g(x)] \leq -g(x) \]
The limit as \( \beta \to 0^+ \) gives (1.5).

Some of the results obtained by Kaul and Kaur [4] can be adapted to Type I and Type II functions.
THEOREM 3.1. If \( f(x) \) and \( g(x) \) are convex objective and constraint functions, respectively, then \( f(x) \) and \( g(x) \) are Type I.

PROOF. Assume that \( f(x) \) and \( g(x) \) are convex functions, and let \( x_0 \in P \) be fixed. Then, for all \( x \in P \)

\[
f(x) - f(x_0) \geq [ \nabla_x f(x_0) ]^T (x - x_0)
\]

and

\[
g(x) - g(x_0) \geq [ \nabla_x g(x_0) ]^T (x - x_0)
\]

By assumption, \( g(x) \leq 0 \), then

\[-g(x_0) \leq [ \nabla_x g(x_0) ]^T (x - x_0) \]

Hence \( f(x) \) and \( g(x) \) are Type I functions with respect to \( \eta(x) = x - x_0 \) at \( x_0 \).

Type I functions need not be convex, as can be seen from the following example based on Example 2.1 of [4].

EXAMPLE 5.1. The functions \( f: [0, \frac{\pi}{2}] \to \mathbb{R} \) defined by \( f(x) = x + \sin x \) and \( g: [0, \frac{\pi}{2}] \to \mathbb{R} \) defined by \( g(x) = -\sin x \) are Type I functions with respect to

\[
\eta(x) = \frac{2}{\sqrt{3}} (\sin x - \frac{1}{2}) \text{ at } x_0 = \frac{\pi}{6}
\]

but \( f(x) \) is not convex because for \( x = \frac{\pi}{4} \) and \( x_0 = \frac{\pi}{6} \)

\[
f(x) - f(x_0) < [ \nabla_x f(x_0) ]^T (x - x_0)
\]

THEOREM 3.1.a. If \( f(x) \) and \( g(x) \) are convex objective and constraint functions, respectively, then \( f(x) \) and \( g(x) \) are Type II.

PROOF. Assume that \( f(x) \) and \( g(x) \) are convex functions, and let \( x_0 \in P \) be fixed. Then, for all \( x \in D \)
\[ f(x_0) - f(x) \geq [\nabla_x f(x)]^t(x_0 - x) \]

and

\[ g(x_0) - g(x) \geq [\nabla_x g(x)](x_0 - x) \]

By assumption, \( g(x_0) \leq 0 \), then

\[-g(x) \geq [\nabla_x g(x)](x_0 - x) \]

Hence \( f(x) \) and \( g(x) \) are Type II functions with respect to \( \eta(x) = x_0 - x \) at \( x_0 \), where \( x_0 \in P \).

Type II functions need not be convex.

**EXAMPLE 3.1a.** Define \( f(x) \) and \( g(x) \) as in Example 3.1. Then \( f(x) \) and \( g(x) \) are Type II functions with respect to \( \eta(x) = (\frac{1}{2} - \sin x)/\cos x \) at \( x_0 = \frac{\pi}{6} \) but \( f(x) \) is not convex.

We shall say that \( f(x) \) and \( g(x) \) are strictly Type I (strictly Type II) if we have strict inequality in (1.2) and (1.3)((1.4) and (1.5)), respectively.

**THEOREM 3.2.** If \( f(x) \) and \( g(x) \) are strictly convex objective and constraint functions, respectively, then \( f(x) \) and \( g(x) \) are strictly Type I.

**PROOF.** Assume that \( f(x) \) and \( g(x) \) are strictly convex, and let \( x_0 \in P \) be fixed.

Then, for all \( x \in P \)

\[ f(x) - f(x_0) > [\nabla_x f(x_0)]^t(x - x_0) \]

and

\[ g(x) - g(x_0) > [\nabla_x g(x_0)](x - x_0) \]

By assumption, \( g(x) \leq 0 \), then

\[-g(x_0) > [\nabla_x g(x_0)](x - x_0) \]
Hence $f(x)$ and $g(x)$ are strictly Type I with respect to $\eta(x) = x - x_0$ at $x_0$.

Strictly Type I functions need not be strictly convex. The following example is based on Example 2.2 of [4].

**EXAMPLE 3.2.** The functions $f:(0, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = -x + \cos x$ and $g:(0, \frac{\pi}{2}) \to \mathbb{R}$ defined by $g(x) = -\cos x$ are strictly Type I with respect to $\eta(x) = 1 - \frac{2}{\sqrt{2}} \cos x$ at $x_0 = \frac{\pi}{4}$ but $f(x)$ is not strictly convex.

**THEOREM 3.2.a.** If $f(x)$ and $g(x)$ are strictly convex objective and constraint functions, respectively, then $f(x)$ and $g(x)$ are strictly Type II.

**PROOF.** Assume that $f(x)$ and $g(x)$ are strictly convex, and let $x_0 \in P$ be fixed.

Then, for all $x \in D$

$$f(x_0) - f(x) > \left[ \nabla_x f(x) \right]^{t}(x_0 - x)$$

and

$$g(x_0) - g(x) > \left[ \nabla_x g(x) \right](x_0 - x)$$

By assumption, $g(x_0) \leq 0$, then

$$-g(x) > \left[ \nabla_x g(x) \right](x_0 - x)$$

Hence $f(x)$ and $g(x)$ are strictly Type II with respect to $\eta(x) = x_0 - x$ at $x_0$.

Strictly Type II functions need not be strictly convex.

**EXAMPLE 3.2.a.** Define $f(x)$ and $g(x)$ as in Example 3.2 and $\eta(x) = (\cos x - \frac{\sqrt{2}}{2})/\sin x$. Then $f(x)$ and $g(x)$ are strictly Type II with respect to $\eta(x)$ at $x_0 = \frac{\pi}{4}$ but $f(x)$ is not strictly convex.
Hanson [2] noted that invexity can be extended to
\[ \left[ \nabla_x f(x_0) \right]^T \eta(x) \geq 0 \Rightarrow f(x) - f(x_0) \geq 0 \quad (3.1) \]
and
\[ f(x) - f(x_0) \leq 0 \Rightarrow \left[ \nabla_x f(x_0) \right]^T \eta(x) \leq 0 \quad (3.2) \]
Functions satisfying (3.1) and (3.2) are called pseudo-invex and quasi-invex, respectively.

We shall say that \( f(x) \) and \( g(x) \) are pseudo-Type I objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in P \) such that
\[ \left[ \nabla_x f(x_0) \right]^T \eta(x) \geq 0 \Rightarrow f(x) \geq f(x_0) \]
and
\[ \left[ \nabla_x g(x_0) \right] \eta(x) \geq 0 \Rightarrow -g(x_0) \geq 0; \]
and \( f(x) \) and \( g(x) \) are pseudo-Type II objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in D \) such that
\[ \left[ \nabla_x f(x) \right]^T \eta(x) \geq 0 \Rightarrow f(x_0) \geq f(x) \]
and
\[ \left[ \nabla_x g(x) \right] \eta(x) \geq 0 \Rightarrow -g(x) \geq 0 \]

**Theorem 3.3.** If \( f(x) \) and \( g(x) \) are Type I objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are pseudo-Type I for the same \( \eta(x) \).

**Proof.** Assume that \( \left[ \nabla_x f(x_0) \right]^T \eta(x) \geq 0 \) and \( \left[ \nabla_x g(x_0) \right] \eta(x) \geq 0 \) for all \( x \in P \).
Since \( f(x) \) and \( g(x) \) are Type I it follows that \( f(x) \geq f(x_0) \) and \( -g(x_0) \geq 0 \).
Pseudo-Type I functions need not be Type I for the same \( \eta(x) \) as can be seen from the following example based on Example 2.6 of [4].

**EXAMPLE 3.3.** The functions \( f:[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( f(x) = -\cos^2 x \) and \( g:[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( g(x) = -\cos x \) are pseudo-Type I with respect to \( \eta(x) = -\frac{1}{2} + \frac{\sqrt{2}}{2} \cos x \) at \( x_0 = -\frac{\pi}{4} \) but \( f(x) \) and \( g(x) \) are not Type I with respect to the same \( \eta(x) \) at \( x_0 \) as can be seen by taking \( x = 0 \).

**THEOREM 3.3.a.** If \( f(x) \) and \( g(x) \) are Type II objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are pseudo-Type II for the same \( \eta(x) \).

**PROOF.** Assume that \( [ \nabla_x f(x) ]^t \eta(x) \geq 0 \) and \( [ \nabla_x g(x) ]^t \eta(x) \geq 0 \) for all \( x \in D \). Since \( f(x) \) and \( g(x) \) are Type II it follows that \( f(x_0) \geq f(x) \) and \( -g(x) \geq 0 \). Pseudo-Type II functions need not be Type II for the same \( \eta(x) \) as can be seen from the following example.

**EXAMPLE 3.3.a.** The functions \( f:[-\frac{\pi}{2}, \frac{\eta}{2}] \rightarrow \mathbb{R} \) defined by \( f(x) = -\cos^2 x \) and \( g:[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( g(x) = -\cos x \) are pseudo-Type II functions with respect to \( \eta(x) = \sin x (\cos x - \frac{\sqrt{2}}{2}) \) at \( x_0 = -\frac{\pi}{4} \) but \( f(x) \) and \( g(x) \) are not Type II with respect to the same \( \eta(x) \) as can be seen by taking \( x = \frac{\pi}{3} \).

We shall say that \( f(x) \) and \( g(x) \) are quasi-Type I objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in P \) such that
\[ f(x) \leq f(x_0) \Rightarrow [\nabla_x f(x_0)]^t \eta(x) \leq 0 \]

and
\[ -g(x_0) \leq 0 \Rightarrow [\nabla_x g(x_0)]\eta(x) \leq 0; \]

and \( f(x) \) and \( g(x) \) are quasi-Type II objective and constraint functions, respectively, with respect to \( \eta(x) \) at \( x_0 \) if there exists an \( n \)-dimensional vector function \( \eta(x) \) defined for all \( x \in D \) such that
\[ f(x_0) \leq f(x) \Rightarrow [\nabla_x f(x)]^t \eta(x) \leq 0 \]

and
\[ -g(x) \leq 0 \Rightarrow [\nabla_x g(x)]\eta(x) \leq 0 \]

**THEOREM 3.4.** If \( f(x) \) and \( g(x) \) are Type I objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are quasi-Type I for the same \( \eta(x) \).

**PROOF.** Assume that \( f(x) \leq f(x_0) \) for all \( x \in P \) and \( -g(x_0) \leq 0 \).

Since \( f(x) \) and \( g(x) \) are Type I it follows that \([\nabla_x f(x_0)]^t \eta(x) \leq 0 \) and
\([\nabla_x g(x_0)]\eta(x) \leq 0 \).

Quasi-Type I functions need not be Type I with respect to the same \( \eta(x) \).

**EXAMPLE 3.4.** The functions \( f: [0, \pi] \rightarrow R \) defined by \( f(x) = \sin^3 x \) and \( g: [0, \pi] \rightarrow R \) defined by \( g(x) = -\cos x \) are quasi-Type I with respect to \( \eta(x) \equiv -1 \) at \( x_0 = \pi/2 \) but \( f(x) \) and \( g(x) \) are not Type I with respect to \( \eta(x) \) at \( x_0 \) as can be seen by taking \( x = \pi/4 \).
THEOREM 3.4.a. If \( f(x) \) and \( g(x) \) are Type II objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are quasi-Type II for the same \( \eta(x) \).

PROOF. Assume that \( f(x_0) \leq f(x) \) and \(-g(x) \leq 0 \) for all \( x \in D \).

Since \( f(x) \) and \( g(x) \) are Type II it follows that \( [\nabla_x f(x)]^T \eta(x) \leq 0 \) and \( [\nabla_x g(x)] \eta(x) \leq 0 \).

Quasi-Type II functions need not be Type II with respect to the same \( \eta(x) \).

EXAMPLE 3.4.a. The functions \( f:(0, \infty) \rightarrow \mathbb{R} \) defined by \( f(x) = -\frac{1}{x} \) and \( g:(0, \infty) \rightarrow \mathbb{R} \) defined by \( g(x) = -x + 1 \) are quasi-Type II with respect to \( \eta(x) = -x + 1 \) at \( x_0 = 1 \) but \( f(x) \) and \( g(x) \) are not Type II with respect to \( \eta(x) \) at \( x_0 \) as can be seen by taking \( x = 2 \).

THEOREM 3.5. If \( f(x) \) and \( g(x) \) are strictly Type I objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are Type I.

Type I functions need not be strictly Type I with respect to the same \( \eta(x) \) as can be seen from the following example based on Example 2.7 of [4].

EXAMPLE 3.5. The functions \( f:[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( f(x) = -\sin x \) and \( g:[-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( g(x) = -\cos x \) are Type I with respect to \( \eta(x) = \sin x \) at \( x_0 = 0 \) but \( f(x) \) and \( g(x) \) are not strictly Type I with respect to \( \eta(x) \) at \( x_0 \).
**Theorem 3.5.a.** If \( f(x) \) and \( g(x) \) are strictly Type II objective and constraint functions, respectively, with respect to a common \( n(x) \) at \( x_0 \), then \( f(x) \) and \( g(x) \) are Type II.

Type II functions need not be strictly Type II with respect to the same \( n(x) \).

**Example 3.5.a.** The functions \( f: [0, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( f(x) = -\sin x \) and \( g: [0, \frac{\pi}{2}] \rightarrow \mathbb{R} \) defined by \( g(x) = -e^{-x} \) are Type II with respect to \( n(x) \equiv 1 \) at \( x_0 = 0 \) but \( f(x) \) and \( g(x) \) are not strictly Type II with respect to \( n(x) \) at \( x_0 \).

In [4], Kaul and Kaur considered a number of sufficient optimality criteria which do not depend on convexity. Those results can be adapted to the class of Type I functions.

**Theorem 4.1.** Let \( x^* \in X \) and let \( f(x) \) and \( g(x) \) be Type I objective and constraint functions, respectively, with respect to a common \( n(x) \) at \( x^* \).

If there exist \( \mu_0^* \in \mathbb{R} \) and \( \mu^* \in \mathbb{R}^m \) such that \( (x^*, \mu_0^*, \mu^*) \) satisfies the following conditions

\[
\nabla_x (\mu_0^* f(x^*)) + \nabla_x (\mu^* g(x^*)) = 0 \\
g(x^*) \leq 0 \\
\mu^* g(x^*) = 0 \\
(\mu_0^*, \mu^*) \geq 0, \ (\mu_0^*, \mu^*) \neq 0 \\
\mu_0^* > 0
\]

then \( x^* \) is an optimal solution of (1.1).
PROOF. Since \( f(x) \) and \( g(x) \) are Type I with respect to \( \eta(x) \) at \( x^* \), then for any \( x \in P \)

\[
f(x) - f(x^*) \geq \left[ \nabla_x f(x^*) \right]^t \eta(x)
= - \left[ \nabla_x \left( \mu^* \right) \right]^t \eta(x) \quad \text{by} \quad (4.1) \quad \text{and} \quad (4.5)
\]

\[
\times \mu^*_0 \quad g(x^*) = 0 \quad \text{by} \quad (4.3)
\]

Therefore \( f(x) \geq f(x^*) \) for all \( x \in P \).

From \( x^* \in X \) and (4.2) it follows that \( x^* \) is an optimal solution of (1.1).

COROLLARY 4.1. Let \( x^* \in X \) and let \( f(x) \) and \( g(x) \) be Type I objective and constraint functions, respectively, with respect to a common \( \eta(x) \) at \( x^* \).

If there exists \( \mu^* \in \mathbb{R}^m \) such that \( (x^*, \mu^*) \) satisfies the following conditions

\[
\nabla_x f(x^*) + \nabla_x \left( \mu^* \right)^t g(x^*) = 0 \quad (4.6)
\]

\[
g(x^*) \leq 0 \quad (4.7)
\]

\[
\mu^* \geq 0 \quad (4.8)
\]

then \( x^* \) is an optimal solution of (1.1)

REMARK 4.1. Since \( \mu^* \geq 0 \), \( g(x^*) \leq 0 \) and \( \mu^* \). \( g(x^*) = 0 \), then \( \mu^* \). \( g_i(x^*) = 0 \), \( i = 1, 2, \ldots m \). Therefore it is sufficient for \( f(x) \) and the active components of \( g(x) \) at \( x^* \) to be Type I functions with respect to a common \( \eta(x) \) at \( x^* \), instead of \( f(x) \) and \( g(x) \) as was assumed in Theorem 4.1.
THEOREM 4.2. Let $x^* \in X$ and
\[ f(x) - f(x^*) \geq \left[ \nabla_x f(x^*) \right]^t \eta(x) \quad (4.10) \]
and
\[ -g(x^*) \geq \left[ \nabla_x g(x^*) \right] \eta(x) \quad (4.11) \]
for the same $\eta(x)$. If there exist $\mu^*_0 \in \mathbb{R}$ and $\mu^*_e \in \mathbb{R}^m$ such that $(x^*, \mu^*_0, \mu^*_e)$ satisfies (4.1)-(4.4) of Theorem 4.1, then $x^*$ is an optimal solution of (1.1).

PROOF. The proof of this theorem is the same as that of Theorem 3.2 of [4] except that (4.10), (4.11) and $g_I(x^0) \leq 0 = g_I(x^*)$, where $I = \{i \mid g_i(x^*) = 0\}$, imply
\[ 0 \geq f(x^0) - f(x^*) \geq \left[ \nabla_x f(x^*) \right]^t \eta(x^0) \]
and
\[ 0 \geq -g_I(x^*) \geq \left[ \nabla_x g_I(x^*) \right] \eta(x^0) \]

REMARK 4.2. Since $\mu^*_i = 0$ for $i \in J$, where $J = \{i \mid g_i(x^*) < 0\}$, it suffices to assume (4.11) for $g_I$ at $x^*$ instead of $g$ at $x^*$ as was assumed in Theorem 4.2.

THEOREM 4.3. Let $x^* \in X$ and let $I = \{i \mid g_i(x^*) = 0\}$. Let $f(x)$ satisfy
\[ \left[ \nabla_x f(x^*) \right]^t \eta(x) \geq 0 \Rightarrow f(x) \geq f(x^*) \]
and let $g_I$ satisfy $g_I(x^*) \leq 0 \Rightarrow \left[ \nabla_x g_I(x^*) \right] \eta(x) \leq 0$
for the same $\eta(x)$. If there exists $\mu^*_e \in \mathbb{R}^m$ such that $(x^*, \mu^*_e)$ satisfies conditions (4.6)-(4.9) of Corollary 4.1, then $x^*$ is an optimal solution of (1.1).

PROOF. The proof of this theorem is the same as that of Theorem 3.3 of [4]. Note that (3.17) in [4] implies $-g_I(x^*) \leq 0$ and thus $\left[ \nabla_x g_I(x^*) \right] \eta(x) \leq 0$. 
THEOREM 4.4. Let \( x^* \in X \). If there exists \( \mu^* \in \mathbb{R}^m \) such that \((x^*, \mu^*)\) satisfies conditions (4.6)-(4.9) of Corollary 4.1 and if \( f(x) \) satisfies

\[
\left[ \nabla f(x^*) \right]^\top \eta(x) \geq 0 \Rightarrow f(x) \geq f(x^*)
\]

and \( g_1(x) \) satisfies \(-\mu_1^t g_1(x^*) \leq 0 \Rightarrow \left[ \nabla \left( \mu_1^t g_1(x^*) \right) \right] \eta(x) \leq 0 \) for the same \( \eta(x) \), then \( x^* \) is an optimal solution of (1.1).

PROOF. The proof of this theorem is the same as that of Theorem 3.4 of [4] except that (3.19) in [4] now follows from \(-\mu_1^t g_1(x^*) \leq 0\).
5. REFERENCES

[1] Ben-Israel, A. and Mond, B., What is Invexity? Unpublished paper, Department of Mathematics, La Trobe University, Bundoora, Victoria, 3083, Australia.

