Multivariate Arrangement Increasing Functions with Applications in Probability and Statistics

by

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Abstract

A real valued function of s vector arguments in $\mathbb{R}^n$ is said to be arrangement increasing if the function increases in value as the components of the vector arguments become more similarly arranged. Various examples of arrangement increasing functions are given including many joint multivariate densities, measures of concordance between judges and the permanent of a matrix with nonnegative components. Preservation properties of the class of arrangement increasing functions are examined, and applications are given including useful probabilistic inequalities for linear combinations of exchangeable random vectors.
Section 1. Introduction and Summary.

We define a real valued function \( f(x_1, \ldots, x_s) \) of \( n \)-dimensional vector arguments \( x_1, \ldots, x_s \) to be arrangement increasing if the function increases in value as the components of the vectors \( x_1, \ldots, x_s \) become more similarly arranged. Various examples of such functions are given including joint multivariate densities and the well known statistical measures of concordance between \( s \) judges. Preservation properties of this class of functions are discussed, and applications are given which yield probabilistic inequalities for exchangeable random vectors.

Section 2. Definition and Basic Properties of Arrangement Increasing Functions.

For a given vector \( \underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we let \( \underline{x}^+ = (x_{[n]}, \ldots, x_{[1]}) \) and \( \underline{x}^- = (x_{[1]}, \ldots, x_{[n]}) \) be respectively the vectors with the components of \( \underline{x} \) arranged in increasing (decreasing) order. For any permutation \( \pi \) of \( \{1, 2, \ldots, n\} \), we let \( \underline{x}_{\pi} = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \).

**Definition 2.1** We now define an equivalence relation \( \equiv \) and a preordering \( \preceq \) on \((\mathbb{R}^n)^S\). For \((x_1, \ldots, x_s)\) and \((z_1, \ldots, z_s) \in (\mathbb{R}^n)^S\), we define \((x_1, \ldots, x_s) \equiv (z_1, \ldots, z_s)\) if and only if there exists a permutation \( \pi \) of \( \{1, 2, \ldots, n\} \) such that

\[
\underline{x}_{k\pi} = (x_{k\pi(1)}, \ldots, x_{k\pi(n)}) = (z_{k1}, \ldots, z_{kn}) = \underline{z}_k
\]

for all \( k = 1, \ldots, s \).

We define \((x_1, \ldots, x_s) \preceq (z_1, \ldots, z_s) \iff \) there exist a finite number of elements \((y_2^1, \ldots, y_s^1), \ldots, (y_2^p, \ldots, y_s^p)\) in \((\mathbb{R}^n)^{S-1}\) such that

(i) \((x_1, \ldots, x_s) \equiv (x_1^+, y_2^1, \ldots, y_s^1), (z_1, \ldots, z_s) \equiv (x_1^+, y_2^p, \ldots, y_s^p)\)

and
(ii) for each $j = 2, \ldots, p$, there exist a pair of coordinate indices

$\ell, m$ ($\ell < m$) such that $(x_1^j, x_2^j, \ldots, x_s^j)$ may be obtained from

$(x_1^j, x_2^{j-1}, \ldots, x_s^{j-1})$ by interchanging the $\ell$ and $m$ coordinates of any

vector $x_k^{j-1}$ such that $y_k^{j-1} > y_{km}^{j-1}$. (We will call such an operation of

obtaining $(x_1^j, x_2^j, \ldots, x_s^j)$ from $(x_1^j, x_2^{j-1}, \ldots, x_s^{j-1})$ a basic rearrangement.

Example 2.2

\[
[(7,5,3,1),(2,4,6,8),(3,0,9,6)] \overset{a}{=} [(1,3,5,7),(8,6,4,2),(6,9,0,3)] \\
\overset{a}{=} [(1,3,5,7),(2,6,4,8),(3,9,0,6)] \\
\overset{a}{=} [(1,3,5,7),(2,4,6,8),(3,0,9,6)] \\
\overset{a}{=} [(1,3,5,7),(2,4,6,8),(0,3,9,6)] \\
\overset{a}{=} [(1,3,5,7),(2,4,6,8),(0,3,6,9)]
\]

It should be clear that $\overset{a}{\preceq}$ is a preordering on $(\mathbb{R}^n)^s$, and that if

$(x_1, \ldots, x_s) \overset{a}{\preceq} (z_1, \ldots, z_s)$ then the components of the vectors $x_1, \ldots, x_s$ are

relatively less similarly arranged than the components of the vectors $z_1, \ldots, z_s$.

Of course if $(x_1, \ldots, x_s) \overset{a}{=} (z_1, \ldots, z_s)$, then the relative arrangement of the
components in the vectors $x_1, \ldots, x_s$ is equivalent to that of the components in
the vectors $z_1, \ldots, z_s$. For any $(x_1, \ldots, x_s) \in (\mathbb{R}^n)^s$, it follows that

$$(x_1, \ldots, x_s) \overset{a}{\preceq} (x_1^+, \ldots, x_s^+) \overset{a}{=} (x_1^+, \ldots, x_s^+).$$

In Example 2.2, we observed that \([(7,5,3,1),(2,4,6,8),(3,0,9,6)] \overset{a}{=} [(1,3,5,7),(2,4,6,8),(0,3,6,9)]\) and that \([(1,3,5,7),(2,4,6,8),(0,3,6,9)]\) can

be "obtained" from \([(7,5,3,1),(2,4,6,8),(3,0,9,6)]\) as the result of 4 basic

rearrangements. In general, for $(x_1, \ldots, x_s) \in (\mathbb{R}^n)^s$, $(x_1^+, \ldots, x_s^+)$ may be
obtained from \( (x_1, \ldots, x_s) \) in at most \( \binom{n}{2} \) basic rearrangements. This can be seen as follows: we note that \( (x_1, \ldots, x_s) \mapsto (x_{1^\pi}, x_{2^\pi}, \ldots, x_{s^\pi}) \) for some permutation \( \pi \).

By performing at most \( n - 1 \) basic rearrangements, we arrive at \( (x_1^1, x_2^1, \ldots, x_s^1) \) where the first component of \( x_k^1 \) is \( x_k[n] \) for \( k = 2, \ldots, s \). Next by performing at most \( n - 2 \) further basic rearrangements we arrive at \( (x_1^2, x_2^2, \ldots, x_s^2) \) where \( x_{k1}^2 = x_k[n] \) and \( x_{k2}^2 = x_k[n-1] \) for all \( k = 2, \ldots, s \). Proceeding in this way, it is clear that \( (x_1^\pi, x_2^\pi, \ldots, x_s^\pi) \) may be reached in at most \( (n-1) + (n-2) + \ldots + 1 = \binom{n}{2} \) basic rearrangements.

In the case where \( s = 2 \), it should be clear that for any pair of vectors \( x_1 \) and \( x_2 \) one has that
\[
(x_1^\pi, x_2^\pi) \preceq (x_1, x_2) \preceq (x_1^\pi, x_2^\pi).
\]

Moreover \( (x_1^\pi, x_2^\pi) \) may be obtained from \( (x_1, x_2) \) as the result of at most \( n - 1 \) basic rearrangements.

**Definition 2.3** Let \( D_i \subset \mathbb{R}^n \) for \( i = 1, \ldots, s \), and define \( D = D_1 \times \ldots \times D_s \subset (\mathbb{R}^n)^S \).

Normally we will consider sets \( D \) which are permutation invariant in the sense that for any permutation \( \pi \) of \( \{1, 2, \ldots, n\} \), \( (x_1, \ldots, x_s) \in D \Rightarrow (x_{1^\pi}, \ldots, x_{s^\pi}) \in D \). We define a function \( f: D \rightarrow \mathbb{R} \) to be **arrangement increasing (AI)** if it preserves the preordering \( \preceq \), that is \( (x_1, \ldots, x_s) \preceq (z_1, \ldots, z_s) \Rightarrow f(x_1, \ldots, x_s) \leq f(z_1, \ldots, z_s) \).

We will say that \( f \) is arrangement decreasing if \(-f\) is arrangement increasing.

We note that arrangement increasing functions are permutation invariant in the sense that for any permutation \( \pi \),
\[
f(x_1, \ldots, x_s) = f(x_{1^\pi}, \ldots, x_{s^\pi}).
\]

The concept of arrangement increasing functions of two vector arguments is well developed — see for example Marshall and Olkin (1979) and Hollander, Proschan and Sethuraman (1977). Such functions are termed "decreasing in transposition" or DT by Hollander, Proschan and Sethuraman. Our generalization of this concept clearly supports the use of the terminology arrangement increasing.
The next proposition is useful in obtaining many examples of arrangement increasing functions, as well as in relating several basic classes of functions. We preface the proposition with several standard definitions.

**Definition 2.4** If \( x, y \in \mathbb{R}^n \), we say that \( x \) **majorizes** \( y \) (\( x \preceq y \)) if \( \sum_{i=1}^{j} x[i] \geq \sum_{i=1}^{j} y[i] \) holds for all \( j = 1, \ldots, n-1 \), and moreover \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \). A real valued function \( f \) with the property that \( x \preceq y \Rightarrow f(x) \geq f(y) \) (respectively \( f(x) \leq f(y) \)) is called **Schur convex (Schur concave)**.

We consider the lattice \( \mathbb{R}^s \) with componentwise ordering. For \( x, y \in \mathbb{R}^s \), we let \( x \vee y = (\max(x_1, y_1), \ldots, \max(x_s, y_s)) \) and \( x \wedge y = (\min(x_1, y_1), \ldots, \min(x_s, y_s)) \). A real valued function \( f \) with the property that \( f(x \vee y) + f(x \wedge y) \geq f(x) + f(y) \) is called **L-superadditive (or lattice superadditive)**. See for example Marshall and Olkin (1979) for results related to L-superadditive and L-subadditive functions.

A real valued function \( f \) with the property that \( f(x \vee y)f(x \wedge y) \geq f(x)f(y) \) is called **multivariate totally positive of order 2 (MTP₂)**.

**Proposition 2.5** In the following, \( E = \mathbb{R}^n \) or \( (\mathbb{R}^+)^n \)

(a) Let \( f(x_1, \ldots, x_s) = g(x_1^r + \ldots + x_s^r) \). Then \( f \) is arrangement increasing on \( D = E^s \) if and only if \( g \) is Schur convex on \( E \).

(b) Let \( f(x_1, \ldots, x_s) = \sum_{i=1}^{n} g(x_{1i}, x_{2i}, \ldots, x_{si}) \). Then \( f \) is arrangement increasing on \( D = E^s \) if and only if \( g \) is L-superadditive on \( E \).

(c) Let \( f(x_1, \ldots, x_s) = \prod_{i=1}^{n} g(x_{1i}, x_{2i}, \ldots, x_{si}) \) where \( g > 0 \). Then \( f \) is arrangement increasing on \( D = E^s \) if and only if \( g \) is MTP₂.

**Proof.** (a) Let \( g \) be Schur convex on \( E \). It is clear from the definition of \( \preceq \) that if \( (x_1, \ldots, x_s) \preceq (x'_1, \ldots, x'_s) \) then \( x_1 + \ldots + x_s \preceq x'_1 + \ldots + x'_s \). Hence \( f(x_1, \ldots, x_s) = g(x_1^r + \ldots + x_s^r) \preceq g(x'_1^r + \ldots + x'_s^r) = f(x'_1, \ldots, x'_s) \) and \( f \) is arrangement increasing.
Now let us suppose $f$ is arrangement increasing, and that $\gamma \preceq \zeta$ where $\gamma, \zeta \in E$. We want to show that $g(\gamma) \geq g(\zeta)$. Since $g$ is permutation invariant, by the special nature of the majorization preordering, we may without loss of generality assume that $\gamma = (y_1, y_2, \ldots, y_n)$, $\zeta = (y_1 + \varepsilon, y_2 - \varepsilon, y_3, \ldots, y_n)$ where $0 \leq \varepsilon \leq (y_2 - y_1)$. Let us define $x_1 = (y_1, y_2 - \varepsilon, y_3, \ldots, y_n)$, $x_2 = (\varepsilon, 0, 0, \ldots, 0)$ and $x'_2 = (0, \varepsilon, 0, \ldots, 0)$. Then $(x_1, x_2, 0, \ldots, 0) \preceq (x_1, x'_2, 0, \ldots, 0)$ and therefore $g(\zeta) = g(x_1 + x_2) = f(x_1, x_2, 0, \ldots, 0) \leq f(x_1, x'_2, 0, \ldots, 0) = g(\gamma)$.

(b) and (c) may be proved in a similar way by noting the following: Let us suppose that $x_1 = x_1^+$ and that $(x_1, x_2^*, \ldots, x_s^*)$ may be obtained from $(x_1, x_2, \ldots, x_s)$ by the basic rearrangement which interchanges the $\ell$ and $m$ ($\ell < m$) coordinates of any vector $x_k$ such that $x_{k\ell} > x_{km}$. Then

$$\{(x_1, x_2, \ldots, x_s) \vee (x_{1m}, x_{2m}, \ldots, x_{sm}), (x_1, x_2, \ldots, x_s) \wedge (x_{1m}, x_{2m}, \ldots, x_{sm})\}
\quad
= \{(x_{1m}, x_{2m}, \ldots, x_{sm}), (x_1, x_2, \ldots, x_s)\}.$$

**Example 2.6** The function $g$: $[0, +\infty)^n \to \mathbb{R}$ defined by $g(y_1, \ldots, y_n) = \prod_{i=1}^{n} y_i$ is Schur concave and hence $-g$ is Schur convex. It follows from Proposition 2.5a that $f$: $([0, +\infty]^S) \to \mathbb{R}$ defined by $f(x_1, \ldots, x_s) = \prod_{i=1}^{n} \sum_{k=1}^{S} x_{ki}$ is arrangement decreasing. In particular

$$\prod_{i=1}^{n} \sum_{k=1}^{S} x_{ki} \geq \prod_{i=1}^{n} \sum_{k=1}^{S} x_{k[i]}$$

an inequality proved by Ruderman (1952).

**Example 2.7** The function $g$: $[0, +\infty)^s \to \mathbb{R}$ defined by $g(y_1, \ldots, y_s) = y_1 \cdots y_s$ is L-superadditive. It follows from Proposition 2.5b that $\bar{\gamma}(x_1, \ldots, x_s) = \sum_{i=1}^{n} \prod_{k=1}^{s} x_{ki}$
is arrangement increasing on \((0, +\infty)^n\). In particular, as Ruderman (1952) observed, \[ \sum_{i=1}^{n} x_{ki} \leq \sum_{k=1}^{n} x_{k[i]} \]

It is easy to see from this example that for any \(r \leq s\),

\[
f_r(x_1, \ldots, x_s) = \sum_{k_1 < \ldots < k_r} \sum_{i=1}^{n} x_{k_1[i]} \cdots x_{k_r[i]}
\]
is also an arrangement increasing function.

**Remark 2.8** Lorentz (1953) proved that \[ \sum_{i=1}^{n} g(x_{1i}, \ldots, x_{si}) \leq \sum_{i=1}^{n} g(x_{1[i]}, \ldots, x_{s[i]}) \]
for any L superadditive function \(g\) on \(R^S\). Derman, Lieberman and Ross (1972) observed that a cumulative joint distribution function \(F(y_1, \ldots, y_s)\) is L superadditive and derived some implications on the optimal assembly of systems. For example, let us suppose that \(s\) components make up a system, and that associated with each component is a numerical value. If \(a_1, \ldots, a_s\) are the component values, then assume that \(F(a_1, \ldots, a_s)\) is the probability that the system functions properly (\(F(a_1, \ldots, a_s)\) is the reliability of the system). Now let us assume that \(n\) components of each type \(k = 1, \ldots, s\) are available, and hence that \(n\) systems may be assembled in \((n!)^{s-1}\) possible ways. If \(N\) denotes the number of systems that function properly, then \(N\) is a random variable whose distribution depends on the way in which the \(n\) systems are assembled. It follows that \(EN = EN(a_1, \ldots, a_s)\), the expected number of properly functioning systems, is an arrangement increasing function which is maximized when the "best" components of each type are assembled into one system, the "2nd best" of each type into another, etc.

**Example 2.9** The functions \(\min(y_1, \ldots, y_s)\) and \(-\max(y_1, \ldots, y_s)\) are L superadditive on \(R^S\) \((\max(y_1, \ldots, y_s)\) is L subadditive). It follows (from Proposition 2.5 b) that
\[ \sum_{i=1}^{n} \min(x_{1i}, \ldots, x_{si}) \quad \text{and} \quad \sum_{i=1}^{n} \max(x_{1i}, \ldots, x_{si}) \]
are respectively arrangement increasing and arrangement decreasing on \((\mathbb{R}^n)^s\).

In particular, as was proved by Minc (1971),
\[ \sum_{i=1}^{n} \min(x_{1i}, \ldots, x_{si}) \leq \sum_{i=1}^{n} \min(x_{1[i]}, \ldots, x_{s[i]}) \]
and
\[ \sum_{i=1}^{n} \max(x_{1i}, \ldots, x_{si}) \geq \sum_{i=1}^{n} \max(x_{1[i]}, \ldots, x_{s[i]}) . \]

Similarly, we note that as \( \log \min(y_1, \ldots, y_s) \) and \(-\log \max(y_1, \ldots, y_s)\) are \( L \)
superadditive on \((0, +\infty)^s\), it follows that
\[ \prod_{i=1}^{n} \min(x_{1i}, \ldots, x_{si}) \quad \text{and} \quad \prod_{i=1}^{n} \max(x_{1i}, \ldots, x_{si}) \]
are respectively arrangement increasing and arrangement decreasing on \(((0, +\infty)^n)^s\).

In particular, as also proved by Minc (1971),
\[ \prod_{i=1}^{n} \min(x_{1i}, \ldots, x_{si}) \leq \prod_{i=1}^{n} \min(x_{1[i]}, \ldots, x_{s[i]}) \]
and
\[ \prod_{i=1}^{n} \max(x_{1i}, \ldots, x_{si}) \geq \prod_{i=1}^{n} \max(x_{1[i]}, \ldots, x_{s[i]}) \]
when all \( x_{ki} > 0 \).

**Example 2.10** The permanent of an \( n \times n \) matrix with positive elements is an
arrangement decreasing function of its columns (equivalently its rows).

Let \( P(a_1, \ldots, a_n) \) be the permanent of the \( n \times n \) matrix with \( k \)th column =
\( (a_k) = (a_{k1}, \ldots, a_{kn})' \). Then
\[ P(a_1, \ldots, a_n) = \sum_{\pi \in S_n} a_{\pi(1)1} \cdots a_{\pi(n)n} . \]
To show that the permanent is arrangement decreasing, we need only show that 
\[ P(a_1, \ldots, a_n) \geq P(a'_1, \ldots, a'_n) \] where \((a'_1, \ldots, a'_n)\) is obtained from \((a_1, \ldots, a_n)\) by interchanging the \(\ell\) and \(m\) coordinates of any vector \(a_k\) such that \(a_{k\ell} > a_{km}\) (here \(\ell, m\) are arbitrary but fixed and \(\ell < m\)). Without loss of generality we assume that 
\[ a_{k\ell} \leq a_{km} \iff k \leq r \] for some \(r, 1 \leq r < n\). Let us define \(S^\ast\) to be the set of permutations \(\pi\) on \(\{1, \ldots, n\}\) such that \(\pi(\ell) < \pi(m)\). Now for any permutation \(\pi\), define \(\pi^\ast\) by \(\pi^\ast(i) = \pi(i)\) for \(i \neq \ell, m\), and \(\pi^\ast(\ell) = \pi(m), \pi^\ast(m) = \pi(\ell)\). It is clear that if \(\pi\) is such that either \(\max(\pi(\ell), \pi(m)) \leq r\) or \(\min(\pi(\ell), \pi(m)) > r\), then 
\[
a_{\pi(1)}1 \cdots a_{\pi(n)}n + a_{\pi^\ast(1)}1 \cdots a_{\pi^\ast(n)}n
\]
\[= a_{\pi'(1)}1 \cdots a_{\pi'(n)}n + a_{\pi'^\ast(1)}1 \cdots a_{\pi'^\ast(n)}n.\]

On the other hand if \(\pi\) does not fall into either of these categories, it is easy to see that 
\[
a_{\pi(1)}1 \cdots a_{\pi(n)}n + a_{\pi^\ast(1)}1 \cdots a_{\pi^\ast(n)}n \geq a_{\pi'(1)}1 \cdots a_{\pi'(n)}n + a_{\pi'^\ast(1)}1 \cdots a_{\pi'^\ast(n)}n.\]

Hence 
\[
P(a_1, \ldots, a_n) = \sum_{\pi \in S^\ast} [a_{\pi(1)}1 \cdots a_{\pi(n)}n + a_{\pi^\ast(1)}1 \cdots a_{\pi^\ast(n)}n] \]
\[\geq \sum_{\pi \in S'} [a_{\pi'(1)}1 \cdots a_{\pi'(n)}n + a_{\pi'^\ast(1)}1 \cdots a_{\pi'^\ast(n)}n] \]
\[= P(a'_1, \ldots, a'_n).\]

For an example of a probabilistic interpretation of this result, consider the following ball and urn situation. Suppose \(n\) balls are to be thrown (independently) into \(n\) urns. Let \(p_k = (p_{k1}, p_{k2}, \ldots, p_{kn})\) be the probability distribution of the \(k^{th}\) ball for \(k = 1, \ldots, n\). That is \(p_{ki} = \) probability that ball \(k\) ends up in urn \(i\). Then \(P(p_1, p_2, \ldots, p_n)\) is the probability that the \(n\) balls end up in \(n\) different urns. This probability function is arrangement decreasing in the vectors \((p_1, \ldots, p_n)\), and in particular
\[ P(p_1, \ldots, p_n) \geq P(p_1^*, \ldots, p_n^*) \]

where \( p_k^* = (p_k[1], \ldots, p_k[n]) \) for each \( k = 1, \ldots, n \).

Section 3. Preservation and Closure properties of Arrangement Increasing Functions.

The class of arrangement increasing functions is clearly closed under addition and under the operation of taking mixtures (with respect to positive measures). The product of non-negative arrangement increasing functions is arrangement increasing. If \( \phi \) is a non-decreasing function on \( R^m \) and \( f_1, \ldots, f_m \) are arrangement increasing, then so is \( \phi(f_1(\cdot), \ldots, f_m(\cdot)) \) arrangement increasing.

Example 3.1 Let \( E_0, E_1, \ldots, E_s \subset R^n \). For \( i = 1, 2, \ldots, s \), let \( g_i: E_0 \times E_i \to [0, +\infty) \) be arrangement increasing. Then \( f(\lambda, x_1, \ldots, x_s) = \prod_{i=1}^{s} g_i(\lambda, x_i) \) is arrangement increasing in \( \lambda, x_1, \ldots, x_s \). This observation makes it clear that for many classic multivariate densities, the joint density of a random sample \( x_1, \ldots, x_s \) of size \( s \) is AI in the arguments \( \lambda \) (a parameter vector), \( x_1, \ldots, x_s \). Some examples are:

(a) multinomial

\[
f(\lambda, x_1, \ldots, x_s) = \frac{\lambda^x_{ki}}{N!} \prod_{k=1}^{n} \frac{\lambda_i}{x_{ki}!} .
\]

Here \( 0 < \lambda_i, x_{ki} = 0, 1, 2, \ldots, i = 1, \ldots, n, k = 1, \ldots, s \),

\[
\sum_{i=1}^{n} \lambda_i = 1, \sum_{i=1}^{n} x_{ki} = N \text{ for each } k .
\]

(b) multivariate normal distribution with common variance and covariance.

\[
f(\lambda, x_1, \ldots, x_s) = \left| (2\pi)^\sum \right|^{-s/2} \prod_{k=1}^{s} e^{-\frac{1}{2}(x_k - \lambda)^T \sum^{-1} (x_k - \lambda)} .
\]

where \( \sum \) is the positive definite covariance matrix with elements \( \sigma^2 \) along the main diagonal and \( \rho \sigma^2 \) elsewhere, \( \rho > -1/(n-1) \).
For more examples, one need only consult Examples 3.10 of Hollander, Proschan
and Sethuraman (1977).

The following result concerning the composition of AI functions of two vector
variables enables one to construct many interesting examples of AI functions.

**Theorem 3.2** Let \( g_k(x,z) \) be non negative arrangement increasing functions on
\((R^m)^2\) for \( k = 1, \ldots, s \). If \( \mu \) is a permutation invariant Borel measure on \( R^m \)
\( (\int_A \mu(z) = \int_A \mu(z \circ \pi) \) for any permutation \( \pi \)), then the "composition" \( f \) defined by

\[
f(x_1, \ldots, x_s) = \int \prod_{k=1}^s g_k(x_k, z) \, d\mu(z)
\]

is AI on \((R^m)^s\).

**Proof.** Let us suppose that \((x_1, \ldots, x_s)\) and \((x'_1, \ldots, x'_s)\) are given such that
\( x_1 = x'_1 = x_1^\ell \), and \((x'_1, \ldots, x'_s)\) may be obtained from \((x_1, \ldots, x_s)\) by interchanging
the \( \ell \) and \( m \) coordinates \((\ell < m)\) of any \( x_k \) such that \( x_k^{\ell} > x_k^m \). We need only show
that \( f(x_1, \ldots, x_s) \leq f(x'_1, \ldots, x'_s) \). Without loss of generality we may assume that
the indices \( k \) such that \( x_k^{\ell} \leq x_k^m \) are the indices \( k = 1, 2, \ldots, r \) where \( r < s \).

For any vector \( w \in R^m \), let us define \( w^* \) to be the vector obtained from \( w \) by
interchanging its \( \ell \) and \( m \) coordinates.

By breaking up the region of integration into the three regions \( z_1^{\ell} \leq z_m \),
\( z_1^{\ell} = z_m \) and \( z_1^{\ell} > z_m \), one obtains

\[
f(x'_1, \ldots, x'_s) - f(x_1, \ldots, x_s) = \int \left[ \prod_{k=1}^r g_k(x_k^r, z) \prod_{k=r+1}^s g_k(x_k^r, z) + \prod_{k=1}^r g_k(x_k^r, z^*) \prod_{k=r+1}^s g_k(x_k^r, z^*) \right] d\mu(z)
\]

\[
- \int \left[ \prod_{i=1}^r g_k(x_k^i, z) \prod_{i=r+1}^s g_k(x_k^i, z) + \prod_{i=1}^r g_k(x_k^i, z^*) \prod_{i=r+1}^s g_k(x_k^i, z^*) \right] d\mu(z)
\]
\[
\int \left\{ \prod_{k=1}^{r} g_k(x_k^*, z) \left[ \prod_{k=r+1}^{s} g_k(x_k^*, z) - \prod_{k=r+1}^{s} g_k(x_k, z) \right] \right. \\
+ \left. \prod_{k=1}^{r} g_k(x_k^*, z) \left[ \prod_{k=r+1}^{s} g_k(x_k^*, z^*) - \prod_{k=r+1}^{s} g_k(x_k, z^*) \right] \right\} du(z)
\]

\[
\geq 0 \quad \text{since each } g_k \text{ is AI.}
\]

An application of this theorem yields the following extension of Corollary 2.1 in Boland, Proschan and Tong (1987). The proof is similar and thus omitted.

**Corollary 3.3** Let \( X \) be an exchangeable random vector with density or mass function \( f(x) \). For \( k = 1, \ldots, s \), let \( h^k \) be an arrangement increasing function of 2 vectors on \( \mathbb{R}^n \times \mathbb{R}^n \), and let \( \phi_k : \mathbb{R} \to [0, +\infty) \) be non decreasing. Then

\[
\psi(a_1, \ldots, a_s) = E_X \left( \prod_{k=1}^{s} \phi_k (h^k(a_k, x)) \right)
\]

is an arrangement increasing function of \( (a_1, \ldots, a_s) \in (\mathbb{R}^n)^s \).

When choosing \( \phi_k (h^k(a_k, x)) \) to be the indicator function of the set \( \{ x : h^k(a_k, x) \geq c_k \} \) where \( c_k \) is an arbitrary but fixed real number, it follows that

**Corollary 3.4** Under the assumptions of corollary 3.3, one has that

\[
\text{Prob}_{(a_1, \ldots, a_s)} \left[ h^k(a_k, x) \geq c_k : k=1, \ldots, s \right]
\]

is an arrangement increasing function of \( (a_1, \ldots, a_s) \in (\mathbb{R}^n)^s \) for every real vector \( c = (c_1, \ldots, c_s) \).

Note that Corollary 3.4 is an extension (from the bivariate case to the multivariate case) of Corollary 2.2 of Boland, Proschan and Tong (1987).
Remark 3.5 Many useful transformations of the random vector $X$ that are AI functions can be found in Boland, Proschan and Tong (1987). One such transformation that seems to be particularly useful is the linear transformation

$$h_k^*(a_k, X) = \sum_{i=1}^{n} a_k^i x_i$$

For two $s \times n$ real matrices given by

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_s \end{pmatrix}$$

let us define $Y_A = AX$ and $Y_B = BX$. If the joint density or mass function of $X = (X_1, \ldots, X_n)$ is permutation invariant, then by Corollary 3.4 it follows that

$$(a_1, \ldots, a_s) \preceq (b_1, \ldots, b_s)$$

$$\Rightarrow \text{Prob}(Y_A \geq c) \leq \text{Prob}(Y_B \geq c)$$

for all real vectors $c = (c_1, \ldots, c_s)$.

Section 4. Applications to measures of agreement between $s$ judges.

Various measures of concordance have been used to evaluate the degree of agreement between $s$ judges. We consider the situation where each of the $s$ judges rank $n$ objects. Perhaps the most widely used measure of this type is Kendall's coefficient of concordance $W$ (see Kendall (1970)).

For each $k = 1, \ldots, s$, we let $R_k = (R_{k1}, \ldots, R_{kn})$ be the vector of ranks of the $k^{th}$ judge — $R_{ki}$ being the rank assigned by judge $k$ to the $i^{th}$ object. Kendall's $W$ is then defined by

$$W(R_1, \ldots, R_s) = \frac{12 \sum_{i=1}^{n} \sum_{k=1}^{s} (R_{ki} - \frac{n+1}{2})^2}{s^2(n^3-n)}.$$ 

We mention three other measures of concordance:

(a) $\bar{\rho}$ (average Spearman's Rho) defined by

$$\bar{\rho}(R_1, \ldots, R_s) = \frac{1}{\binom{s}{2}} \sum_{k < \ell} \left[ 1 - \frac{6 \sum_{i=1}^{n} (R_{ki} - R_{\ell i})^2}{s(s^2-1)} \right]$$

Spearman's Rho for judges $k$ and $\ell$. 

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\[ = 1 - \frac{2(2s+1)}{s-1} + \frac{1}{\binom{n}{2}s(s+1)(s-1)} \sum_{k<\ell} \left( \sum_{i=1}^{n} R_{ki} R_{\ell i} \right). \]

It should be noted that \( \check{\rho} \) and \( W \) are related through the equation (see Kendall (1970))
\[ \check{\rho} = \frac{sW-1}{s-1} \]

(b) \( \bar{\tau} \) (average Kendall's tau) defined by (see Ehrenberg (1952) and Hays (1960))
\[ \bar{\tau}(R_1, \ldots, R_s) = \frac{1}{\binom{s}{2}} \sum_{k<\ell} \text{Kendall's tau for judges } k \text{ and } \ell \]
\[ = \frac{1}{\binom{s}{2}} \sum_{k<\ell} \left[ 1 - \frac{4Q(k, \ell)}{n(n-1)} \right] \]
\[ = \frac{1}{\binom{s}{2}} \sum_{k<\ell} \left[ 1 - \frac{4}{n(n-1)} \sum_{i<j} [\text{sgn}(R_{ki} - R_{kj})][\text{sgn}(R_{\ell i} - R_{\ell j})] \right] \]
\[ = -1 + \frac{2}{\binom{s}{2}\binom{n}{2}} \sum_{k<\ell} \sum_{i<j} [\text{sgn}(R_{ki} - R_{kj})][\text{sgn}(R_{\ell i} - R_{\ell j})] \]

(c) \( \check{\rho}_F \) (average Spearman's footrule) defined by (see Diaconis and Graham (1977))
\[ \check{\rho}_F(R_1, \ldots, R_s) = \frac{1}{\binom{s}{2}} \sum_{k<\ell} \left[ 1 - \frac{2}{n(n-1)} \sum_{i=1}^{n} |R_{ki} - R_{\ell i}| \right] \]
\[ = 1 - \frac{1}{\binom{s}{2}\binom{n}{2}} \sum_{k<\ell} \sum_{i=1}^{n} |R_{ki} - R_{\ell i}|. \]

It should be noted that for any fixed \( k < \ell \), the functions \( \sum_{i=1}^{n} R_{ki} R_{\ell i}, \sum_{i<j} \text{sgn}(R_{ki} - R_{kj})\text{sgn}(R_{\ell i} - R_{\ell j}), \) and \(-\sum_{i=1}^{n} |R_{ki} - R_{\ell i}|\) of \((R_k, R_\ell)\) are all arrangement increasing. If \( g \) is an arrangement increasing function of \( r \) vectors \((r \leq s)\) which is symmetric in its arguments, it should be clear that
\[ f(x_1, \ldots, x_s) = \sum_{k_1 < \ldots < k_r} g(x_{k_1}, \ldots, x_{k_r}) \]

is an arrangement increasing function of \( s \) vector arguments (which is also symmetric in its vector arguments). It follows that all four of these measures of concordance between judges — \( W, \bar{\rho}, \bar{\tau} \) and \( \bar{\rho}_F \) — are arrangement increasing functions. This is a justification of their use, since certainly we would expect these measures to increase as the judges increase in agreement.

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Bibliography


Multivariate Arrangement Increasing Functions with Applications in Probability and Statistics.

Philip J. Boland and Frank Proschan

FROM August, 1986 TO

A real valued function of $s$-vector arguments in $\mathbb{R}^{n}$ is said to be arrangement increasing if the function increases in value as the components of the vector arguments become more similarly arranged. Various examples of arrangement increasing functions are given including many joint multivariate densities, measures of concordance between judges and the permanent of a matrix with nonnegative components. Preservation properties of the class of arrangement increasing functions are examined, and applications are given including useful probabilistic inequalities for linear combinations of exchangeable random vectors.