OPTIMAL CONSUMPTION AND INVESTMENT POLICIES WITH BANKRUPTCY
MODELLED BY A DIFFUSION WITH DELAYED REFLECTION

BY

SURESH S. SETHI* AND MICHAEL I. TAKSAR**†

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*FACULTY OF MANAGEMENT STUDIES,
UNIVERSITY OF TORONTO,
TORONTO, ONTARIO, CANADA M5S 1V4

**DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY
TALLAHASSEE, FL 32306

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I. Diffusion processes were used for quite a while to model the price fluctuations in continuous-time financial models. In recent years they started being widely used in the consumption/investment models in which an individual needs to maximize total expected utility of consumption, while also maintaining investments whose yields are uncertain. (See Merton [7], [8], Richard [9], Mason [6], Lehoczky, Sethi and Shreve [4], [5], Sethi, Gordon and Ingham [11] and others.)

Diffusion processes appear very natural continuous time limits for the real life discrete processes in which fluctuations are relatively small. However, modeling an individual wealth by such a process often results in a quandry how to proceed in the model when the wealth process reaches zero. Such state is usually referred as the bankruptcy state. A "natural" candidate, the reflected diffusion, does not correspond to reality because it corresponds to the situation in which the investor is not "penalized" for bankruptcy, rather he is "bailed out" immediately each time he is bankrupt.

Until recently there were very few attempts to overcome this problem. In the first papers the parameters of the optimization model were such that the optimal policy does not lead to bankruptcy. In certain cases the "bankruptcy problem" was not considered in the model and the optimal policy derived dealt with negative wealth and negative consumptions, which require additional interpretation (see Merton [8], Sethi and Taksar [12]).

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In the work of Mason [6], the investor reaching bankruptcy is reendowed with the new wealth \( L \) but that can be done only one time.

In Lehoczky, Sethi and Shreve [4], [5], and Karatzas, Lehoczky, Sethi, and Shreve [3] the bankruptcy is terminal but at the time of bankruptcy the agent is given an extra value \( P \) which is added to his cumulative utility. If \( c(t) \) is the consumption at time \( t \) and \( U \) is the utility function of consumption, then the agent's objective in these models is to maximize

\[
E \left\{ \int_0^\tau e^{-\beta t} U(c(t)) \, dt + e^{-\beta \tau} P \right\}
\]

where \( \tau \) is the time of bankruptcy and \( x \) is the initial wealth.

We consider the model in which the bankruptcy is not terminal rather the agent recovers (reorganizes under "Chapter 11") after a certain period of time. Such recovery is modeled by a Brownian motion with delayed reflection at zero.

2. There are \( N + 1 \) possible investments available, one nonrisky with the rate of return \( r \) and \( N \) risky whose prices are modeled as geometric Brownian motion. Let \( P_0(t) \) be the price of riskless asset and \( P(t) = (P_1(t), \ldots, P_N(t)) \) be the prices of risky assets at time \( t \). The equations governing \( P_0 \) and \( P \) are

\[
\frac{dP_0(t)}{P_0(t)} = r \, dt
\]

\[
\frac{dP_i(t)}{P_i(t)} = \alpha_i \, dt + \mathbf{e}_i \mathbf{D} \mathbf{w}^T(t)
\]

where \( \{ \mathbf{w}(t), t \geq 0 \} \) is an \( N \)-dimensional standard Brownian motion, \( \mathbf{e}_i \) in the vector with 1 at the \( i \)th place, \( \alpha_i \) is the average rate of return on the \( i \)th asset, \( \mathbf{D} \) is \( N \times N \) matrix with \( \Sigma = \mathbf{D} \mathbf{D}^T \) positive definite, \( T \) stands for transpose operation.

The investor controls the consumption rate \( c(t) \) and the investment policy vector \( \mathbf{w}(t) = (\pi_1(t), \ldots, \pi_N(t)) \), \( t \geq 0 \) where \( \pi_i(t) \) is the fraction of wealth invested in the \( i \)th risky asset (\( \pi_0(t) = 1 - \sum_{i=1}^{N} \pi_i(t) \) is the fraction of wealth invested in the riskless asset.) There is a strictly increasing, strictly
concave utility function \( U \) on the positive half line which measures the utility of consumption. The objective is to find a policy \((c(t), \pi(t), t \geq 0)\) which maximizes the total reward

\[
V_{c(\cdot), \pi(\cdot)}(x) = E_x \left\{ \int_0^\infty e^{-\beta t} U(c(t)) dt \right\}
\]

(2)

where \( \beta \) is the discount factor and \( x \) is the initial wealth. Under the policy \((c(t), \pi(t))\) the wealth process \( x(t) \) satisfies the following stochastic differential equation

\[
dx(t) = [a - rl] \pi^T(t)x(t) + rx(t) - c(t)] 1_{x(t) > 0} dt
+ \mu 1_{x(t)=0} dt + x(t) \pi(t) D w^T(t),
\]

\( x(0) = x. \)

Here \( a = (a_1, \ldots, a_n) \) is the vector of average returns, \( l \) is the vector whose all components are equal to 1 and \( \mu \) is the exogenous parameter of the model, which we will call the recovery rate.

The equation (3) is the equation for a diffusion process with a delayed reflection at point zero. This process behaves like an ordinary reflected diffusion on the positive half-line, however, in contrast to a regular (instantaneously) reflected process it stays positive amount of time at the boundary. The set of times when this process is at zero does not contain any open interval and has a structure similar to a Cantor set of a positive measure. The coefficient \( \mu \) is the speed of exit from the boundary. It is in inverse proportion to the time the process spends at zero; when \( \mu = 0 \), the boundary becomes absorbing, while \( \mu = \infty \) corresponds to the regular instantaneous reflection (see [1], Ch. 24).

3. We start with an \( N \)-dimensional Weiner process \((w(t), F^x_t, P)\). An admissible policy is a pair \((\pi(t), c(t))\) of adapted \( N \)-dimensional and one-dimensional processes respectively such that for all \( t > 0 \)
\[
\int_{0}^{t} \pi(s) E \pi(s)^{T} ds < \infty,
\]
\[
c(t) = 1_{x(t) > 0} c(t),
\]
where \(x(t)\) is given by (3). The last condition corresponds to the requirement that there is no consumption at the time of bankruptcy. We assume that the function \(U\) is an increasing concave \(C^{3}\) function with
\[
\lim_{c \to \infty} U'(c) = 0.
\]
Both \(r\) and \(\beta\) are assumed to be positive and \(rl \neq \alpha\). Let
\[
(4) \quad \gamma = \frac{1}{2}(\alpha - rl) \Sigma^{-1} (\alpha - rl)^{T}
\]
and \(\lambda_{-}\) be the negative solution of
\[
\gamma \lambda^{2} - (r - \beta - \gamma) \lambda - r = 0.
\]
We assume that for every \(c > 0\)
\[
\int_{c}^{\infty} U'(x)^{-\lambda_{-}} dx < \infty.
\]
The above assumption guarantees that under any policy the total expected return is finite. We define value function
\[
(5) \quad V^{*}(x; \mu) = \sup_{c(\cdot), \pi(\cdot)} \sup_{\pi(\cdot)} V(c(\cdot), \pi(\cdot))(x),
\]
where \(\sup\) is taken over all admissible policies. The second argument \(\mu\) in the left hand side of (5) is to emphasize the dependence of \(V^{*}\) on the recovery rate.

4. Usual argument leads to the Bellman equation for the function \(V^{*}(x; \mu)\)
\[
\beta V(x) = \sup_{c \geq 0, \pi} \left[ (\alpha - r) \pi^T x V'(x) + (rx - c) V'(x) \\
+ \frac{1}{2} \pi \Sigma \pi^T x^2 V''(x) + U(c) \right], x > 0.
\]

However, the equation (6) is not sufficient to determine the value function \(V^*\). The behavior of this function at this point zero should be specified. The integral functionals \(V\) of a delayed diffusion process are subject to

(7) \[
\beta V(0) - \mu V'(0) = U(0),
\]

(see [1] Ch. 24), which we should add to (6) in order to make the Bellman equation complete.

The model with \(N\) risky assets can be always reduced to a model with one risky asset which is governed by a logarithmic Brownian motion with parameters \(\alpha\) and \(\sigma^2\). To this end one needs only to choose \(\alpha\) and \(\sigma^2\) so that for \(\gamma\) given by (2.3)

\[
\gamma = (\alpha - r)^2 / 2\sigma^2
\]

(see [3] for more details). One risky asset can be viewed as a "mutual fund" using self-financing strategy. In the reduced model, \(\pi(t)\) is a one-dimensional process showing proportion of wealth invested in the risky asset. The wealth dynamic and the Bellman equation for the reduced model are respectively

(3') \[
dx(t) = \left[ (\alpha - r) \pi(t) x(t) + rx(t) - c(t) \right] 1_{x(t) > 0} dt \\
+ \mu \left[ \rho 1_{x(t) = 0} dt + x(t) \pi(t) \sigma dw(t) \right]
\]

\[
\beta V(x) = \max_{c \geq 0, \pi} \left[ (\alpha - r) \pi x V'(x) + (rx - c) V'(x) \\
+ \frac{1}{2} \pi \rho^2 x^2 V''(x) + U(c) \right], x > 0.
\]

The boundary condition (7) remains the same for the reduced problem.
5. We solve the Bellman equation (6'), (7) and find the optimal policy by establishing the correspondence with the model with terminal bankruptcy.

Let $V^*_p(x)$ be the value function in the model with terminal bankruptcy, with terminal value $P$, i.e.,

$$V^*_p(x) = \sup_{\pi(\cdot), c(\cdot)} \mathbb{E}_x \left[ \int_0^T e^{-\beta t} u(c(t)) dt + e^{-\beta T} p \right],$$

where the supremum is taken over all admissible policies $(\pi(\cdot), c(\cdot))$.

**Theorem 1.** For each value of recovery rate $\mu (0 \leq \mu \leq +\infty)$, there exists a terminal value $P(\mu)$ ($-\infty \leq P(\mu) \leq +\infty$) such that

$$V^*(x;\mu) = V^*_P(\mu)(x), \ x > 0.$$  

Moreover if $C(x)$ and $\Pi(x)$ are the optimal feedback controls in the model with terminal bankruptcy (i.e., $c(t) = C(x(t)), \pi(t) = \Pi(x(t))$, then $C^1(x) = C(x)1_{x > 0}$ and $\Pi^1(x) = \Pi(x)1_{x > 0}$ are the optimal feedback controls in the model with nonterminal bankruptcy.

The above theorem permits us to find an explicit solution for the model with delayed reflection, using the results of [3]. The following theorem summarizes the qualitative behavior of the optimal policy depending on the parameter of the model.

**Theorem 2.** i) If $\mu = \infty$ then there is no optimal policy and any sequence of policies with $c(t) \uparrow \infty$ is a maximizing sequence. If $\mu < \infty$ then the optimal policy always exists.

ii) If $U'(0) = \infty$ and $\mu = 0$ or if $U(0) = -\infty$ and $\mu > 0$ then for the optimal policy, $q = 0$, where $q$ is the probability of bankruptcy. For this policy $c(t) > 0$ for all $t$.

iii) If $U(0) > -\infty$ and $U'(0) = \infty$, then there exists $a > 0$ (dependent on $\mu$) such that $c(t) > a$ if $x(t) > 0$. Moreover $0 < q < 1$ if $\beta < r + \gamma$ and $q = 1$ if $\beta \geq r + \gamma$. 

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iv) If $U'(0) < \infty$ and $\mu = 0$ then $q = 0$ and there exists $\bar{x}$ such that $c(t) = 0$
if $x(t) \in [0, \bar{x}]$ and $c(t) > 0$ if $x(t) \not\in [0, \bar{x}]$.

v) There exists $\mu^*$ such that if $U'(0) < \infty$ and $0 < \mu < \mu^*$ then there exists $\bar{x}$
dependent on $\mu$ such that if $x(t) \leq \bar{x}$ then $c(t) = 0$ and if $x(t) > \bar{x}$ then $c(t) > 0$. Moreover, $0 < q < 1$ if $\beta < r + \gamma$ and $q = 1$ if $\beta \geq r + \gamma$.

vi) If $U'(0) < \infty$ and $\mu > \mu^*$ then there exists $a$ (dependent on $\mu$) such that
$c(t) > a$ if $x(t) > 0$. Moreover, $0 < q < 1$ if $\beta < r + \gamma$ and $q = 1$ if $\beta \geq r + \gamma$. 
References


A diffusion process with delayed reflection at zero is used to model wealth dynamic in a consumption/investment model. The speed of exit from the boundary corresponds to "recovery rate" from bankruptcy. An optimal behavior in the model is analyzed. Qualitative structure of the optimal feedback controls is described.