A DIFFUSION MODEL FOR OPTIMAL PORTFOLIO SELECTION
. IN THE PRESENCE OF BROKERAGE FEES

BY

Michael Taksar*, Michael J. Klass**, and David Assaf***

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*DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY

**DEPARTMENT OF STATISTICS AND MATHEMATICS
UNIVERSITY OF CALIFORNIA, BERKELEY

***DEPARTMENT OF STATISTICS
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Michael Taksar
Department of Operations Research, Stanford University

Michael J. Klass
Department of Statistics and Mathematics,
University of California, Berkeley

David Assaf
Department of Statistics, Hebrew University, Jerusalem

Abstract

We consider a model with two assets. One has deterministic rate of
growth, while the rate of growth of the second asset is governed by a
Brownian motion with drift. We can shift money from one asset to another;
however, there are losses of money (brokerage fees) involved in shifting
money from risky to the nonrisky asset. We want to maximize the expected
rate of growth of funds.

It is proved that an optimal policy consists of keeping the ratio of
funds in risky and nonrisky assets within a certain interval.
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Michael Taksar*
Department of Operations Research, Stanford University

Michael J. Klass†
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1. Introduction

Problems concerning optimal allocation of wealth among investments have been studied by many authors. In particular, the portfolio problem, where the question is how much should one invest in each of the assets available, attracted much attention in recent years (e.g., [3], [4], [7], [12], [14], [15], [17], [20]).

The goal of this paper is to introduce a continuous time model in which a rate of return or a risky investment fluctuates like a Brownian motion. We think that the investor can choose between two assets: one with deterministic rate of growth \( r \) (called "Bank" in the sequel) and the other (called "Stock" in the sequel) whose rate of return is governed by a \((\mu, \sigma^2)\)-Brownian motion.

At any given time the investor can transfer funds from Bank to Stock and vice versa. However, there is a "penalty" (a brokerage fee, for example) of say 10% for any money withdrawn from Stock. The objective is

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to find a policy which maximizes the expected rate of growth of investors assets.

The specific of our model is in absence of limits on the rate of transfer of funds from Stock or Bank; in other words, it is possible instantaneously to deplete the Bank or Stock account by a finite amount of money. This leads to problems of instantaneous (or in the terminology of [10] "singular") control of Brownian motion, developed in [8], [10], [18], [19] and other publications.

The results obtained in the previous publications cannot be applied directly to our case, because our main quantity, the ratio of money of two assets, is not governed by a Brownian motion with constant drift and variance, but rather by a diffusion process with state-dependent coefficients. However, some of the ideas of [19] can be carried out in our case as well. We describe controls in terms of two increasing functionals, which correspond to the cumulative amount of money transferred up to time $t$ from Stock to Bank and from Bank to Stock respectively. An optimal policy can be formulated qualitatively the same way as in [19]; namely, the ratio of money in Stock to money in Bank must be kept within a certain interval with minimal efforts. This means that there exist two constants $A < B$ such that as soon as the Stock to Bank money ratio drops below $A$ (or exceeds $B$) we must transfer the minimal amount of funds from Bank to Stock (from Stock to Bank) resulting to bring the ratio to $A$ (to $B$).

In Section 2 we give a rigorous mathematical formulation of the problem. We are able to reduce our problem to the problem of minimization of an expected average (per unit of time) cost of a certain integral functional. We are able to write this integral functional only in terms of
the Stock to Bank ratio process \( X(t) \), not invoking any other processes, such as the amount of money in the Bank alone. Then we modify the control functionals in such a way that \( X(t) \) can be described in terms of these new functionals. The cost of control (i.e., the cost of transferring funds) from Stock to Bank is linear with coefficients depending on the state of the system.

In Section 3 we deduce the optimality equation heuristically. Its form is similar to that obtained in other publications dealing with singular control.

In Section 4 we construct what we call control limits policies. These cause the original process to be reflected at the endpoints of an interval \([A,B]\). The average expected cost of such a policy is obtained.

In Section 5 we prove sufficiency of the optimality equation and we show that if the solution exists, then it corresponds to a control limit policy. In Section 6 the existence of a solution of the optimality equation is obtained. It turns out that, given the Bank growth rate \( r \), the expected Stock growth rate \( \mu \) and its variance \( \sigma^2 \), the optimal boundaries depend only on the quantity \((r - \mu)/\sigma^2\).

It should be specially mentioned that our model does not allow borrowing of cash or short selling of stock.

Only a minor technical sophistication is required in order to generalize our results to the case in which there is brokerage commission in buying stock. The optimal policy in this case is similar to the one in the case treated in this paper: the ratio of funds must be kept within a certain interval with minimal efforts.
2. **Formulation of the Problem**

The main exogenous process in our problem is a \((\mu, \sigma^2)\)-Brownian motion \((\Omega, F, F_t, B(\cdot), \theta_t, P)\), where \((\Omega, F)\) is a measurable space, \(B(\cdot, \omega)\) is a stochastic process on \((\Omega, F)\), \(F_t = \sigma(B(s), s \leq t), \theta_t\) is a shift operator in \(\Omega\) and \(P\) is a probability measure on \((\Omega, F)\) such that under \(P\) the process \(B(t)\) is a Brownian motion with drift \(\mu\), variance \(\sigma^2\) and initial state 0.

One dollar invested in the Stock at time 0 becomes worth \(\exp(B(t))\) dollars at time 1. Now and in the sequel we represent \(B(t)\) as \(\mu t + \sigma w(t)\) where \(w(t)\) is a standard Wiener process. One dollar invested in the Bank becomes \(e^{rt}\) dollars after time 1. It is assumed that no loss results from transferring money from the Bank to the Stock. However, if we take \(m\) dollars from the Stock and want to put it in the Bank, then the Bank account will increase by only \(\lambda m\) dollars, where \(\lambda < 1\), the difference being paid out as a commission and therefore lost to the investor.

Our next step is to write differential equations describing the evolution of the Bank and Stock accounts. Let \(U(t)\) be the total amount of money transferred from the Stock to the Bank up to time 1 and \(Z(t)\) be the total amount of money transferred up to time 1 from the Bank to the Stock. Let \(S_0\) and \(B_0\) be the initial amounts of money in the Stock and Bank respectively and let \(S(t)\) be the amount of money in the Stock at time 1 and \(B(t)\) be the amount of money in the Bank at time 1. Then
(2.1) \[ B(t+dt) = B(t) \exp(rdt) + \lambda(U(t+dt) - U(t)) - (Z(t+dt) - Z(t)) \],

(2.2) \[ S(t+dt) = S(t) \exp(\mu dt + \sigma dw(t)) + Z(t+dt) - Z(t) - (U(t+dt) - U(t)) \] .

Or in another form

(2.3) \[ dB(t) = rB(t)dt + \lambda dU(t) - dZ(t) \]

(2.4) \[ dS(t) = S(t)(\mu + \frac{1}{2} \sigma^2)dt + \sigma S(t)dw(t) + dZ(t) - dU(t) \] .

(We refer a reader to [5] or any other book on stochastic integration for explanation why (2.2) and (2.4) are equivalent.) The equations (2.3) and (2.4) become rigorously defined after we integrate both parts from 0- to t (assuming Z(0-) = U(0-) = 0).

(2.5) \[ B(t) = B_0 + \int_0^t rB(s)ds + \lambda U(t) - Z(t) \],

(2.6) \[ S(t) = S_0 + \int_0^t (\mu + \sigma^2/2)S(s)ds + \int_0^t \sigma S(s)dw(s) + Z(t) - U(t) \] .

(We do not assume that U(0) or Z(0) are equal to 0, therefore B(0) and S(0) are not necessarily equal to B_0 and S_0 respectively.)

We define a control (or policy) as a pair of right-continuous, nondecreasing processes \((U(t), Z(t))\), \(t \geq 0\) such that

(2.7) \(U\) and \(Z\) are adapted, i.e., for each \(t\) \(U(t)\) and \(Z(t)\) are \(F_t\)-measurable .
A policy \((U,Z)\) is feasible if

\[(2.8) \ E\{U(t)\} \text{ and } E\{Z(t)\} \text{ are finite for each } t; \text{ and the system of integral equations (2.5), (2.6) has a unique solution and this solution is nonnegative, i.e., for } B(t) \text{ and } S(t) \text{ defined by (2.5), (2.6) for all } t \geq 0, B(t), S(t) \geq 0.\]

Let \(Y(t) \triangleq S(t) + B(t)\) be the total amount of money in both assets. Our objective is to find a policy \((U^*,Z^*)\) which maximizes the expected rate of growth, i.e., maximizes the expression.

\[(2.9) \quad E\{\liminf_{t \to \infty} [\log Y(t)]/t\}.\]

(It is obvious that the base of the logarithm in (2.9) does not alter the optimal policy.) The rest of this section is devoted to reformulation of the problem in more convenient terms. First, by Fatou's lemma \(E\{\liminf \ldots\} \leq \liminf E\{\ldots\}\) and if we find a policy which maximizes

\[(2.10) \quad \liminf E\{[\log Y(t)]/t\}\]

and in addition for this policy \(\lim [\log Y(t)]/t\) exists and

\[(2.11) \quad E\{\lim \log Y(t)/t\} = \lim_{t \to \infty} E\{\log Y(t)/t\},\]

then this policy will maximize (2.9). In the sequel we will work with the criterion (2.10). We will also exclude from consideration the policies for
which \( U(\cdot) \) and \( Z(\cdot) \) have discontinuities at the same points (thereby resulting in simultaneous transfer of funds in both directions). Such policies are easily "improved."

It can be shown that any discontinuous policy \((U, Z)\) can be approximated by a sequence of continuous policies \((U_n, Z_n)\) such that if \( Y_n(t) \) is a value of both assets corresponding to \((U_n, Z_n)\) then

\[
\lim_{t \to \infty} \inf \frac{\log Y(t)}{t} \leq \lim_{n \to \infty} \lim_{t \to \infty} \inf \frac{\log Y_n(t)}{t} \leq \lim_{n \to \infty} \inf \frac{\log Y_n(t)}{t}.
\]

This shows that we can restrict ourselves to consideration of only continuous policies. In the sequel we will always assume that \((U, Z)\) is a pair of continuous increasing functionals such that

\[ U(0) = Z(0) = 0. \]

Adding (2.3) and (2.4), we get the equation for \( Y(t) \)

\[
dY(t) = [rB(t) + (\mu + \sigma^2/2)S(t)]dt + \sigma S(t)dw(t) - (1-\lambda)dU(t).
\]

or in the integral form

\[
(2.12) \quad Y(t) = Y_0 + \int_0^t [rB(t) + (\mu + \sigma^2/2)S(t)]dt + \sigma S(t)dw(t) - (1-\lambda)U(t).
\]

For any twice continuously differentiable function \( f \) we can use a generalized Itô formula (see [15]), and get

\[
f(Y(t)) = f(Y_0) + \int_0^t f'(Y(s-)) \, dY(s) + \frac{1}{2} \int_0^t f''(Y(s-)) \, c <Y, Y>_s.
\]

Here and in the sequel \(<Y, Y>_s\) means the quadratic variation of the process \( Y \).
Taking in the above formula \( f(x) = \ln(x) \), and using (2.12), we get (note that \( d\langle Y, Y \rangle_t = \sigma^2 S^2(t) dt \))

\[
(2.13) \quad \ln Y(t) = \ln Y_0 + \int_0^t \left[ rB(t)/Y(t) + (\mu + \sigma^2/2)S(t)/Y(t) \\
- \sigma^2 S(t)^2/2Y(t)^2 \right] dt \\
+ \int_0^t \sigma S(t)/Y(t)dw(t) - \int_0^t (1-\lambda)Y(t)^{-1} dU(t).
\]

Next define a new variable \( X(t) = S(t)/B(t) \) \((X_0 = S_0/B_0)\) and new control functionals

\[
(2.14) \quad L(t) = \int_0^t S(s)^{-1} dU(s)
\]

\[
(2.15) \quad R(t) = \int_0^t B(s)^{-1} dZ(s)
\]

"Physically", \( L(t) \) is the cumulative percentage of Stock withdrawn up to time \( t \), rather than cumulative amount of money. The same for \( R(t) \). Now we can rewrite (2.13) in terms of the new variable \( X \) and new functional \( L \).

\[
(2.16) \quad \ln Y(s) = \ln Y_0 + \int_0^s \sigma X(t)/(X(t) + 1)dw(t) \\
+ \int_0^s [r/(X(t) + 1) + (\mu + \sigma^2/2)X(t)/(X(t) + 1) \\
- (\sigma^2/2)X^2(t)/(X(t) + 1)^2] dt \\
- \int_0^s (1-\lambda)X(t)/(X(t) + 1)dL(t).
\]
The first integral on the right-hand side of (2.16) is a martingale. Therefore its expectation vanishes. After a trivial transformation, we have

\[
(2.17) \quad t^{-1} E\{ \ln Y(t) \} = t^{-1} \ln Y_0 + r - \frac{1}{t} \int_0^t E\left\{ h(X(u))du + \int_0^t g(X(u))dL(u) \right\}
\]

where, putting \( a = \mu + \sigma^2/2 - r \),

\[
(2.18) \quad h(x) = \frac{\sigma^2}{2} x^2/(x+1)^2 - ax/(x+1),
\]

\[
(2.19) \quad g(x) = (1-\lambda)x/(x+1)
\]

We have now posed the problem in terms of the process \( X(t) \), for which we need to minimize the following limiting expected "cost" per unit time:

\[
(2.20) \quad \limsup_{t \to \infty} \frac{1}{t} E\{ \int_0^t h(X(t))dt + \int_0^t g(X(t))dL(t) \}.
\]

To complete the formulation we must find the equation governing the process \( X(t) \). The process \( B(t) \), defined by (2.5) is a finite variation (VF in notations of [15]) process. Hence, applying the integration by parts formula (see [15], p. 303) for \( B(t)^{-1} S(t) \), we get (using substitutions (2.14), (2.15))
\[ (2.21) \quad X(s) = X_0 + \int_0^s \sigma S(t)/B(t) dw(t) + \int_0^s (\mu + \sigma^2/2 - r) S(t)/B(t) dt \\
- \int_0^s [S(t)/B(t) + \lambda S(t)^2/B(t)^2] dL(t) \\
+ \int_0^s (1 + S(t)/B(t)) dR(t) \]

\[ = X_0 + \int_0^s \sigma X(t) dw(t) + \int_0^s aX(t) dt - \int_0^s j(X(t)) dL(t) \]

\[ + \int_0^s k(X(t)) dR(t) \]

where \( j(x) = x + \lambda x^2 \) and \( k(x) = 1 + x \).

Thus \( X(*) \) can be expressed in terms of \( w(*) \), \( L(*) \) and \( R(*) \).

Using (2.16), the same is true of \( Y(*) \). Specifically, we obtain

\[ (2.23) \quad Y(t) = Y_0 \exp\left[ \int_0^t (1-\lambda)^{-1} \sigma g(X(s)) dw(s) \right. \]

\[ \left. - \int_0^t h(X(s)) ds - \int_0^t g(X(s)) dL(s) \right] \]

Of course (2.21) and (2.23) determine the values of \( B(*) \) and \( S(*) \) since

\[ (2.24) \quad S(t) = Y(t)X(t)/(1 + X(t)), \quad B(t) = Y(t)/(1 + X(t)) \].

(The process \( X(t) \) can take on infinite values when \( B(t) = 0 \). When this happens special care must be taken to define precisely what all the parameters mean at this time \( t \) and later. This can be made very accurate by means of transformation of the state space and a slight modification of the functional \( L \). However we skip this rigorous construction, which is not needed in the sequel anyway.)
Now we can forget about the original formulation and set up the problem in the following way. A policy is a pair \((L, R)\) of two increasing continuous adapted processes. A policy is said to be feasible if the equation
\[
dX = \sigma Xdw + aXdtd - j(X)dL + k(X)dR, \tag{2.25}
\]
\[
X(0) = x_0
\]
has a unique nonnegative solution and if for each \( t \)
\[
E\left[ \int_0^t S(s)dL(s) \right] + E\left[ \int_0^t B(s)dR(s) \right] < \infty,
\]
where \( S(t) \) and \( R(t) \) are found from (2.23) and (2.24).

The objective is to find a policy \((L, R)\) which minimizes the limiting expected cost per unit time given in (2.20). The quantity in the braces in (2.20) will be called the cost up to time \( t \), the first integral being holding cost and the second integral is the cost of control up to time \( t \). The function \( h(x) \) is the holding cost function and \( g(x) \) is the unit cost of control \( L \) at the point \( x \).

To see when the problem becomes trivial some further analysis is necessary. The function \( h(x) \) is a quadratic polynomial in \( x/(x+1) \). This function is a strictly increasing (decreasing) function of \( x \) on \([0, \infty)\) iff \( a < 0 \) (iff \( a \geq \sigma^2 \)). In this case an optimal policy is to move \( X \) to 0 (to \( \infty \)) and then exercise no control. This corresponds to transferring all the money to the Bank (to the Stock) and then doing nothing. Therefore the limiting expected cost per unit of time is reduced to zero. The problem is not trivial if
\[
0 < a < \sigma^2, \tag{2.26}
\]
which is equivalent to \( |\mu - r| < \sigma^2/2 \). In the sequel we will always assume
that (2.26) holds. In this case \( h(x) \) attains its minimum at the point \( x^* \) such that \( x^*/(x^*+1) = a/\sigma^2 \) or

\[
(2.27) \quad x^* = (\mu - r + \sigma^2/2)/(r - \mu + \sigma^2/2),
\]

and this minimum equals \(-a^2/2\sigma^2 \) (\( = h(x^*) \)).

Suppose there are no brokerage fees. In this case \( \lambda = 1 \) and the second integral in the right-hand side of (2.17) vanishes. Then the best policy is to keep \( X(u) \) equal to the optimal proportion \( x^* \), that is, within an interval reduced to a single point. The expected average rate of growth of funds per unit time in this case will be \( r + a^2/2\sigma^2 \equiv \mu + (r - \mu + \sigma^2/2)^2/2\sigma^2 \), which is strictly greater than that given by the Stock (\( \equiv \mu \)) or Bank (\( \equiv r \)) alone.

3. Derivation of the Optimality Equation

Suppose the optimal policy is found and the optimal value of (2.27) for this policy is \( d \). Let the initial state of the process \( X(\cdot) \) be \( x \), then the cumulative expected cost up to time \( t \) under this policy can be expressed as \( td + V(x) \) (such a representation of the expected cost is pretty standard, e.g., see [2] for a more detailed discussion). Let us start with the initial point \( x \) for \( X(\cdot) \) and for the amount of time \( \delta \) do nothing and then start to use an optimal policy. This policy will be suboptimal and we can write

\[
V(x) + td \leq h(x)\delta + (t-\delta)d + E\{V(X(\delta))|X(0) = x\}.
\]

Subtracting \( V(x) + td \) from both sides, dividing everything by \( \delta \) and taking the limit as \( \delta \to 0 \), we obtain
(3.1) \[ \Gamma V(x) + h(x) - d \geq 0 \]

where \( \Gamma \) is the infinitesimal generator of a diffusion process with variance \( \sigma(x) = \sigma x \) and drift \( a(x) = ax \):

(3.2) \[ \Gamma = \frac{1}{2} \sigma^2 x^2 \frac{d^2}{dx^2} + ax \frac{d}{dx} . \]

Next assume that we instantaneously transfer a small amount from the Stock to the Bank, thereby changing \( X \) from \( x \) to \( x - \delta \), and then proceed by acting optimally. (Although we agreed upon considering only continuous policies, we can still use discontinuous policies while deriving optimality equation, viewing instantaneous transfer of funds as a limiting case of continuous policies with increasing rate of transfer of funds.)

Equation (2.25) shows that the amount of control \( \Delta L \) needed to do that equals to \( \delta/j(x) + o(\delta) \). Formula (2.20) shows that the cost of this control is \( g(x) \Delta L + o(\Delta L) = \delta g(x)/j(x) + o(\delta) \). Therefore we can write

\[ V(x) + td \leq td + V(x-\delta) + \delta g(x)/j(x) + o(\delta) \]

or (putting \( G(x) \equiv g(x)/j(x) \))

(3.3) \[ V'(x) \leq G(x) \equiv (1-\lambda)/(x+1)^{-1}(1 + \lambda x)^{-1} . \]

Similar reasoning, applied to the case when \( X \) is moved from \( x \) to \( x + \delta \), yields

(3.4) \[ V'(x) \geq 0 . \]

If we restrict our set of controls to functionals \( (R,L) \) such that
\[ dR = r(t)dt \quad \text{and} \quad dL = \lambda(t)dt \quad \text{with} \quad 0 \leq r(t), \lambda(t) \leq \theta \quad \text{for some fixed} \quad \theta > 0, \]
then we can write the classical Bellman equation for determining the
optimal rates \( r(t) \) and \( \lambda(t) \). Since the cost of control is linear, the Bellman equation shows that the optimal policy is of a "bang-bang" type: the rate \( r(t) \) is always either 0 or \( \theta \), and the same is true for \( \lambda(t) \). We can conjecture that the bang-bang policy is also optimal in this more general situation. If so, at least one of the inequalities (3.1), (3.3), (3.4) must be tight, which results in the (optimality) equation

\[
\min \{TV(x) + h(x) - d, G(x) - V'(x), V'(x)\} = 0.
\]

(Cf., (2.13) in [19].) The solution of (3.5) is invariant under changing \( V(x) \) to \( V(x) + c \). Therefore, we can add the condition

\[
(3.6) \quad V(0) = 0.
\]

In Section 5 we show that if there exists a solution of (3.5) and (3.6) subject to minor technical conditions, then there exists an optimal policy \((L^*, R^*)\) which minimizes (2.20) and for which the value of (2.22) is equal to \( d \). This optimal policy can be determined through the function \( V \).

4. Control Limit Policies

There is a special class of policies defined in [8] and called control limits policies. A policy of this type keeps the controlled process within a certain interval \([A, B]\) exerting minimal control necessary for that. For our future goals we need a construction and an analytical description of these policies. This is done in [6]. It follows from section 23 of [6] that for each \( 0 < A < B < \infty \) and each \( A \leq X_0 \leq B \) there exist two increasing continuous adapted to \( \mathbf{F}_t \) functionals \( \ell \) and \( r \) and a process \( X(t) \) such that
(4.1) \[ A \leq X(t) \leq B \quad \text{for all} \quad t \geq 0 \]

(4.2) \[ X(t) = X_0 + \int_0^t \sigma X(s) \, dw(s) + \int_0^t a X(s) \, ds + r(t) - \ell(t) \]

and

(4.3) \[ r(t) = \int_0^t 1_A(X(s)) \, dr(s) \]

(4.4) \[ \ell(t) = \int_0^t 1_B(X(s)) \, d\ell(s) \]

(To apply the results of [6] we need the functions \( \sigma(x) = \sigma x \) and \( a(x) = ax \) to be Lipschitz and \( \sigma(x) \geq \varepsilon > 0 \) on \([A,B] \), which is trivially satisfied if \( 0 < A \).

Put

(4.5) \[ R(t) = k(A)^{-1} r(t) \]

(4.6) \[ L(t) = j(B)^{-1} \ell(t) \]

Then by virtue of (4.3) and (4.4) we can rewrite (4.2) as

(4.6) \[ X(t) = X_0 + \int_0^t \sigma X(s) \, dw(s) + \int_0^t a X(s) \, ds \]

\[ + \int_0^t k(X(s)) \, dR(s) - \int_0^t j(X(s)) \, dL(s) \, . \]

The policy \((L,R)\) for which (4.6) is true will be called control limit policy with control limits \(A\) and \(B\).

Our next step is to find a limiting expected per unit time cost under a control limit policy.
(4.7) **Theorem.** Let \((L,R)\) be a control limit policy with control limits \(A\) and \(B\). Suppose there exists a twice continuously differentiable function \(f(x)\) and a constant \(e\) subject to

\[(4.8)\quad \Gamma f(x) + h(x) - e = 0, \quad A \leq x \leq B\]

\[(4.9)\quad f'(A) = 0\]

\[(4.10)\quad f'(B) = G(B),\]

where \(\Gamma, h(\cdot)\) and \(G(\cdot)\) are as defined in (3.2), (2.13) and (3.3) respectively. Then for the policy \((L,R)\) the limit (2.20) exists and is equal to \(e\) (independent of \(X_0\)).

**Proof.** We assume \(X_0 \in [A,B]\). (For \(X_0\) outside the interval \([A,B]\), the proof is the same.)

Note that in the representation (4.6) the first integral in the right hand side is a continuous martingale and all the other integrals, considered as functions of \(t\), are processes of bounded variation. Therefore the quadratic variation process \(<X,X>_t\) (as defined in [16]) is equal to the quadratic variation of the first integral in (4.6), namely \(\int_0^t x(s)^2 ds\).

Applying the generalized Ito's formula (see [16], p. 301) to \(f(x(t))\), we get
(4.11) \[ f(X(t)) - f(X_0) = \int_0^t f'(X(s))dX(s) + \frac{1}{2} \int_0^t f''(X(s))d\langle X, X \rangle_s \]

\[ = f(X_0) + \int_0^t f'(X(s))\sigma X(s)dw(s) \]

\[ + \int_0^t \left[ \frac{1}{2} \sigma^2 X(s)^2 f''(X(s)) + aX(s)f'(X(s)) \right]ds \]

\[ + \int_0^t f'(X(s))k(X(s)dR(s) - \int_0^t f'(X(s))j(X(s))dL(s). \]

Take expectations of both sides of (4.11). The integral with respect to \( dw(s) \) is a martingale and so its expectation vanishes. By virtue of (4.8), the integrand in the integral associated with \( ds \) on the right-hand side of (4.11) can be replaced by \( e^{-h(X(s))} \). By virtue of (4.3) the integrand in the third term can be replaced by \( f'(A)k(A) \). Therefore, due to (4.9) the third term vanishes. Similarly, due to (4.4) and (4.10), the integrand in the last term can be replaced by \( f'(B)j(B) = G(B)j(B) \).

Dividing both sides of (4.11) by \( t \), we get

(4.12) \[ t^{-1}E[f(X(t)) - f(X_0)] \]

\[ = t^{-1}[et - \int_0^t h(X(s))ds - \int_0^t G(B)j(B)dL(s)]]. \]

Since by definition \( G(x) = g(x)/j(x) \), the integrand in the last term equals \( g(B) \). By virtue of (4.4), we can replace \( g(B) \) by \( g(X(s)) \). By the construction, the process \( X(t) \in [A,B] \) for all \( t \). Therefore \( f(X(t)) \) is uniformly bounded in \( t \) and \( E(f(X(t)))/t \to 0 \) as \( t \to \infty \). Now, letting \( t \) in (4.12) go to infinity, we get the statement of the theorem.
5. **Sufficiency of the Optimality Equation**

Suppose that the solution \((V(x), d)\) of (3.5) and (3.6) exists. We want to show that \(d\) is the optimal limiting expected cost per unit time and that the optimal policy is a control limit one.

First we show that the constant \(d\) minorizes the limiting expected cost per unit time of any feasible policy \((L, R)\).

Denote by \(C^k(I)\) the set of \(k\) times continuously differentiable functions on the interval \(I\), and by \(C^{k-1}(I)\) the set of \(f \in C^{k-1}(I)\) for which the \(k\)-th derivative exists except a finite or a discrete countable number of points at which it has discontinuities of the first order. We denote by \(R_+\) the positive half line.

\[(5.1)\] **Theorem.** If \((V(x), d)\) is a solution of (3.5) and (3.6) such that \(V(x) \in C^2(R_+)\), and \(V\) is bounded, then for any feasible policy \((L, R)\)

\[d \leq \lim_{t \to \infty} \sup t^{-1}E\left\{ \int_0^t h(X(s))ds + \int_0^t g(X(s))dL(s) \right\}.

**Proof.** First observe that the equation (3.5) implies (3.1), (3.3) and (3.4).

Apply a generalized Ito formula (see [16], p. 301) to \(V(X(t))\).

Recalling formula (2.21) for \(X(t)\), we obtain

\[(5.2)\] \(V(X(t)) - V(X_0) = \int_0^t \left[ aX(s)V'(X(s)) + \frac{1}{2} \sigma^2 X^2(s)V''(X(s)) \right]ds + \int_0^t \sigma X(s)V'(X(s))dW(s) + \int_0^t V'(X(s))k(X(s))dR(s)

\[- \int_0^t V'(X(s))j(X(s))dL(s).\]
To apply the results of [16] in order to obtain (5.2), we must have $V \in C^2(R_+)$. However, given that (5.2) is valid for all $V \in C^2(R_+)$, we can obtain the same equality for any $V \in C^2(R_+)$ by approximating with appropriate functions from $C^2(R_+)$. We will denote the right-hand side of (5.2) as $I_1 + I_2 + I_3 - I_4$.

Take expectations of both sides of (5.2). The expectation of $I_2$ vanishes; by virtue of (3.1) the integrand in $I_1$ is not smaller than $d - h(X(s))$; inequality (3.4) implies that $I_3$ is always nonnegative, while (3.3) implies that integrand in $I_4$ is at most $G(X(s))j(X(s)) = g(X(s))$. Therefore

$$
(5.3) \quad E\{V(X(t)) - V(X_0)\} \geq E\left[ \int_0^t [d - h(X(s))]ds \right. \\
- \left. \int_0^t g(X(s))dL(s) \right] \\
= td - E\left[ \int_0^t h(X(s))ds + \int_0^t g(X(s))dL(s) \right].
$$

Divide both sides of the inequality (5.3) by $t$ and let $t \to \infty$. The function $V$ is bounded; hence $E\{V(X(t))\}/t \to 0$. Now, taking lim sup as $t \to \infty$ of both sides of the inequality (5.3), we get the statement of the theorem.

(5.4) **Theorem.** Let $(V(x), d)$ be a solution of (3.5) and (3.6) subject to the conditions of Theorem (5.1). Suppose that there exists an interval $[A, B]$ such that
\[(5.5) \quad TV(x) + h(x) - d = 0; \quad A \leq x \leq B\]

\[(5.6) \quad V'(x) = 0; \quad x \leq A\]

\[(5.7) \quad V'(x) = G(x), \quad x \geq B.\]

Then there exists an optimal policy. This policy is a control limit one with control limits \(A\) and \(B\).

**Proof.** The requirement \(V \in C^2(\mathbb{R}_+)\) means that \(V''(x+); \) and \(V''(x-)\) exist and have at most finite number of discontinuities on \([A,B]\).

From (5.5)

\[(5.8) \quad V''(x+) = \left(\frac{1}{2} \sigma^2 x^2\right)^{-1}[d - h(x+)-a x V'(x+)'], \quad A \leq x < B.\]

The right-hand side of (5.8) is continuous and hence so is the left-hand side. Therefore \(V''(x+)\) is continuous on \([A,B]\) and \(V''\) exists everywhere. By the same token, \(V''(x-)\) exists everywhere on \((A,B]\) and is continuous. Hence \(V(x) \in C^2([A,B]).\)

Conditions (5.6) and (5.7) imply \(V'(A) = 0\) and \(V'(B) = G(B).\) Therefore we can apply Theorem (4.7) to \((V(x),d), x \in [A,B],\) which states that for the control limit policy with control limits \(A\) and \(B\) the limit (2.20) exists and is equal to \(d.\)

By Theorem (5.1) \(d\) minorizes the limiting expected cost per unit time for any feasible policy. Therefore the policy described above is optimal.

To tie up the loose ends we need to verify the two following facts: first, we need to show that the integral equation (4.18) (or (2.25)) has a unique solution \(X(\cdot),\) where \((L,R)\) is a fixed control limit policy. Uniqueness is proved in the same way as for a classical stochastic differential equation (e.g., see [6] or [13], Section 3.3, or [1], Section 6)
using continuity of the derivatives of the functions $a(x) \equiv ax$, $a(x) \equiv ax$, $k(x) \equiv 1 + x$ and $j(x) \equiv x + \lambda x^2$. The proof in [13] goes through without changes in our case. The only additional fact which is necessary for the proof is the inequality

$$E_x \{L(s)\}, E_x \{R(s)\} \leq Cs + D, \quad s \geq 0$$

for some constants $C$ and $D$. The above inequality can be trivially verified by standard analytic techniques and we omit the verification.

Next, we need to verify that under any control limit policy $(L,R)$, (2.11) is true. (Recall that in Section 2 we replaced our original criterion with (2.10), conjecturing that the optimal policy must be subject to (2.11).)

Divide both parts of (2.16) by $s$ and let $s \to \infty$. Without loss of generality we may assume $A \leq X_0 \leq B$. The process $X(t)$ under the control $(L,R)$ is a uniformly bounded positive recurrent strong Markov process. Therefore, for any bounded continuous function $f$

$$\frac{1}{S} \int_0^S f(X(t)) dt \to \int_A^B f(x) \phi(dx) \quad \text{a.s.}$$

(5.9)

where $\phi$ is the (unique) invariant measure for $X(\cdot)$ (see [6] Ch. 23). Moreover since this convergence is bounded, the expectation of the left-hand side of (5.9) converges to the same quantity. Using Kolmogorov's inequality for martingales, and uniform boundedness of $X(t)/(X(t)+1)$, we can show that

$$\frac{1}{S} \int_0^S \sigma X(t)/(X(t) + 1) dw(t) \to 0 \quad \text{a.s. and in } L^2.$$
Hence, dividing both sides of (2.16) by $s$ and letting $s \to \infty$, the sum of the first two integrals on the right-hand side of (2.16) converges almost surely and in $L^1$ to the same constant. On the other hand

$$s^{-1} \int_0^s (1-\lambda)X(t)/(X(t) + 1) \, dL(t) = [(1-\lambda)B/(B+1)]L(s)s^{-1}.$$ 

The functional $L$ is the local time of a strong Markov process at the point $B$. Hence $L$ is inverse to a process with independent increments. Using usual arguments for processes with independent increments, we can show that $L(s)/s \to E_{\phi}[L(1)]$ where $E_{\phi}$ is the expectation associated with the process $X(\cdot)$ having stationary initial distribution $\phi$; moreover $E[L(s)/s]$ converges to the same constant. In view of (2.16) this implies (2.11).

**Remark.** The above arguments show that for the optimal policy $\lim_{t \to \infty} (t^{-1} \ln Y(t))$ exists and is almost surely equal to a (nonrandom) constant.

6. **Solution of the Optimality Equation**

In this section we will show that there exists a solution of (3.5) and (3.6) subject to the conditions of the Theorem (5.4).

(6.1) **Proposition.** If $(V(x), d)$ is a solution of (3.5) and (3.6) subject to (5.5)-(5.7), then

(6.2) $V''(A) = 0$,

(6.3) $V''(B) = G'(B)$.

The proof of this proposition is the same as the proof of the necessity part of Theorem (4.7) in [19].
We cannot infer that (6.2) and (6.3) together with (5.5-5.7) are also sufficient conditions for (3.5) to be satisfied. In our case \( h(x) \) is not convex and we cannot apply the same reasonings as in the sufficiency part of Theorem (4.7) of [19]. The next theorem identifies supplementary conditions under which (6.2) and (6.3) are sufficient for optimality.

(6.4) **Theorem.** Let \( A < x^* < B \), where \( x^* \) is given by (2.27). Let \( V(x) \in C^2(R_+) \) and let \( (V(x),d) \) satisfy (5.5)-(5.7), (6.2) and (6.3). Suppose in addition (3.3), (3.4) and (6.5) below hold.

(6.5) If \( f(x) \) is the solution of the equation \( \Gamma f(x) + h(x) - d = 0 \) with initial conditions \( f(B) = V(B), f'(B) = V'(B) \), then for each sufficiently small \( \delta > 0 \), \( f(B+\delta) < V(B+\delta) \).

Then \( (V(x),d) \) satisfies (3.5) and \( A \) and \( B \) are the optimal control limits.

**Proof.** 1°. To verify (3.5) we need to show that (3.1) is valid for each \( x \geq 0 \). For \( x \leq A \)

(6.6) \[ \Gamma V(x) + h(x) - d \equiv h(x) - d. \]

Since \( A < x^* \), the right hand side of (6.6) is decreasing function of \( x \) on \([0,A]\) and it attains its minimum at point \( A \). By virtue of (6.2) and (5.5) \( h(A)-d = 0 \).

Direct computation shows that for \( x \geq B \)
(6.7) \[ \Gamma V(x) + h(x) - d = \frac{1}{2} \sigma x^2 G'(x) + axG(x) + h(x) - d \]

\[ = \frac{1}{2} \sigma^2 (\lambda x)^2 / (1 + \lambda x)^2 \quad \text{for} \quad \lambda \lambda < x^* \quad \text{where} \quad \lambda x < x^*. \]

By (5.5) and (6.3) the value of (6.7) at the point \( B \) equals 0. The function \( h(\lambda x) \) is increasing (decreasing) iff \( \lambda x > x^* \) (iff \( \lambda x < x^* \)). Therefore, if we show that \( h(\lambda x) - d \) is increasing in a small neighborhood of \( B \), then \( h(\lambda x) \) is increasing for all \( x \geq B \) and (6.7) is nonnegative on \([B, \infty)\).

Assume \( h(\lambda x) - d \) is decreasing on \([B, B+\delta)\), hence is negative on this interval. Let \( f \) be the function of the condition (6.5). Put \( F(x) = f(x) - V(x) \). It is obvious that \( f \) and \( V \) are analytic functions in a right neighborhood of \( B \) and so is \( F \). Due to (6.5) \( F(B) = F'(B) = 0 \) and \( F(x) \) is negative on \([B, B+\varepsilon)\). On the other hand, \( \Gamma F(x) = (\Gamma f(x) + h(x) - d) - (\Gamma V(x) + h(x) - d) = -h(\lambda x) + d \), which is positive on \((B, B+\delta)\). The latter contradicts to following lemma.

(6.8) Lemma. If \( F \) is an analytic function in a right neighborhood of \( B \) such that \( F(B) = F'(B) = 0 \) and \( \Gamma F(x) > 0 \), \( x \neq B \), then \( F(x) \) is increasing in a right neighborhood of \( B \).

Proof. If \( \Gamma F > 0 \), then \( F \neq 0 \); therefore there exists \( k \geq 2 \) such that \( F^{(k)}(B) \neq 0 \), and \( F^{(m)}(B) = 0 \) for all \( m < k \). If \( F^{(k)}(B) < 0 \), then \( F^{(k-1)}(x) < 0 \) in the neighborhood of \( B \). This implies \( F^{(k-2)}(x) < 0 \) in the neighborhood of \( B \) and so on. Eventually we get that \( F'(x) \) and \( F''(x) \) are negative in the neighborhood of \( B \) and \( \Gamma F(x) < 0 \) because \( \Gamma \) is a differential operator with positive coefficients.
Our aim is now to construct \((V(x), d)\) satisfying the conditions of the Theorem (6.4). However, it is easier to deal with the derivative of \(V\). Let \(W = V'\), then, differentiating (5.8), we get

\[
\frac{1}{2} \sigma^2 x^2 W''(x) + (\sigma^2 + a)xW'(x) + aW(x) + H(x) = 0, \quad A \leq x \leq B
\]

(6.9)

\[
W(x) = 0, \quad x \leq A;
\]

(6.10)

\[
W(x) = G(x) \equiv (1-\lambda)/[(x+1)(1+\lambda x)], \quad x \geq B.
\]

(6.11)

where \(H(x) = (x+1)^2(\sigma^2 x/(x+1) - a)\) is the derivative of \(h(x)\). Conditions (6.2) and (6.3) become

\[
W'(A) = 0,
\]

(6.12)

\[
W'(B) = G'(B).
\]

(6.13)

If we can find \(W\), then we can easily recover \(V\) and \(d\) through the formulae

\[
V(x) = \int_{0}^{x} W(y)dy
\]

(6.14)

\[
d = h(A).
\]

(6.15)

Our objective now is to find points \(A < x^* < B\) and a solution \(W\) of (6.9) which is positive and lies below the function \(G\). This solution must be tangent to the \(x\)-axis at point \(A\) and tangent to the function \(G(x)\) at \(B\) (see Fig. 1), in such a way that if we extend the solution to a small
right neighborhood of B, then it will still lie below the function $G$ (the latter condition as we will see implies (6.5)).

\[ G(x) = \frac{(1 - \lambda)}{(1+x)(1+\lambda x)} \]

$W(x)$ satisfies the ordinary differential equation

\[ \frac{1}{2} \sigma^2 x^2 W'' + (\sigma^2 + a) x W' + aW + H = 0 \]

\[ A \quad x^* \quad B \]

Fig. 1

\((6.16)\) Theorem. There exist points $A \ x^* \ B$ and a function $W$ subject to (6.9)-(6.13) such that

\[(6.17) \quad 0 \leq W(x) \leq G(x), \quad A \leq x \leq B \]
and the solution \( f \) of

\[
(6.18) \quad \frac{1}{2} \sigma^2 x^2 f''(x) + \left( \sigma^2 + \alpha \right) x f'(x) + \alpha f(x) + H(x) = 0 ,
\]

\[
(6.19) \quad f(B) = G(B), \quad f'(B) = G'(B)
\]
satisfies \( f(x) \leq G(x) \) in the right neighborhood of \( B \). The pair \((V(x),d)\) given by (6.14), (6.15) satisfies the conditions of Theorem (6.4).

Proof. 1°. We will consider the case when \( \alpha \triangleq 2a/\sigma^2 < 1 \); that is, \( \mu < r \). The other cases are treated similarly.

Consider two fundamental solutions of the homogeneous part of (6.18). They are \( \phi_1(x) = x^{-1} \) and \( \phi_2(x) = x^{-\alpha} \), by inspection. The solution of (6.9) with initial conditions \( W(A) = 0 \) and \( W'(A) = 0 \) can be written as (we use subscript \( A \) to emphasize dependence on initial point \( A \))

\[
(6.20) \quad W_A(x) = \phi_1(x) \int_A^x \phi_2(y) \hat{H}(y)/\psi(y) \, dy - \phi_2(x) \int_A^x \phi_1(y) \hat{H}(y)/\psi(y) \, dy ,
\]

where \( \hat{A}(y) = 2H(y)/\sigma^2 x^2 \) and \( \psi(y) = \phi_1(y) \phi_2'(y) - \phi_2(y) \phi_1'(y) \)
\[= (1-\alpha)y^{-\alpha-2} \] is the Wronskian. (See [9], Ch. VI.25 or any textbook in Ordinary Differential Equations.) Substituting the value of \( \psi(y), \phi_1(y) \) and \( \phi_2(y) \) into (6.20) we get
\[ (6.21) \quad W_A(x) = \frac{2}{\sigma^2(1-\alpha)} \left[ x^{-1} \int_A^x H(y) \, dy - x^{-\alpha} \int_A^x y^{\alpha-1} H(y) \, dy \right] \]

\[ = 2\sigma^{-2} x^{-\alpha} \int_A^x [h(A) - h(y)] y^{\alpha-2} \, dy . \]

The last equality in (6.21) is obtained through integration by parts, using the fact that \( H = h' \).

2°. The idea of the construction of the function \( W \) of the statement of the theorem is the following. We consider the family of functions \( W_A(x) \) with \( 0 < A < x^* \). We gradually move the initial point \( A \) from \( x^* \) to the left until we hit the first point \( A \) for which \( W_A(\cdot) \) touches the function \( G(\cdot) \); see Fig. 2.

We know that \( h \) decreases on \([0,x^*]\) and increases on \([x^*,\infty)\). From this and (6.21) we see that if \( A < x^* \), then

\[ (6.22) \quad W_A(x) > 0, \quad A < x \leq x^* . \]

Let \( z(A) = \sup \{ x : W_A(x) \geq 0 \} \) and \( A = \sup \{ A : z(A) = \infty \} \). From (6.21) and (6.22) and properties of \( h \) we see that if \( A < x^* \) then \( z(A) > x^* \) (trivially \( z(x^*) = x^* \)) and \( z(A) \) is a decreasing function of \( A \) on \([0,x^*]\). From (6.21) we also see that \( W_A(x) < 0 \) for all \( x > z(A) \), therefore \( W_A(x) \) has at most one zero on \((A,\infty)\).

Monotonicity of the function \( z(\cdot) \) shows that \( z(A) = \infty \) for all \( A \leq A \).

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G(x) = (1 - \lambda)/(x+1)(1+\lambda x)

Family of functions \( W_A(x) \),
satisfying
\[
\frac{1}{2} \sigma^2 x^2 W''_A + (\sigma^2 + a) x W'_A + a W_A + H = 0
\]

\[
H(x) = (x+1)^{-(\sigma^2 x/(x+1) - a)}
\]

Fig 2.

Clearly \( W_A(x) \) is jointly continuous in positive \( A \) and \( x \) (by (6.21)). Similarly

\[
\beta(A) \triangleq \inf_{A \leq x \leq z(A)} (G(x) - W_A(x))
\]
is a continuous function of \( A \) on \((\bar{A}, x^*)\). Since \( z(x^*) = x^* \), we have

(6.23) \hspace{1cm} \beta(x^*) = G(x^*) > 0 \, .

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for $A = \bar{A}$

$$\int_0^\infty [h(A) - h(y)] y^{-2} dy = 0$$

(while for any $A > \bar{A}$ the above integral is negative). Therefore

$$W(x) = 2\sigma^{-2} x^{-\alpha} \int_\bar{A}^\infty (h(y) - h(\bar{A})) y^{-2} dy$$

As $y \to \infty$

$$h(y) - h(\bar{A}) \to \frac{\sigma^2}{2} - a - h(\bar{A}) > 0$$

because $\alpha = 2a/\sigma^2 < 1$ by assumption and $h(y) < 0$ for all $y < x^*$. Therefore the integral in (6.25) considered as a function of $x$ converges to zero at a rate $x^{\alpha-1}$, hence $W_A(x) = O(1/x)$. On the other hand, the function $G(x) \equiv (1-\lambda)/[(1+x)(1+\lambda x)] = O(1/x^2)$. Therefore there exists a point $u$ such that $W_A(u) > G(u)$. Since $W_A(u)$ is a continuous function of $A$ there exists $y > \bar{A}$ such that $W_y(u) > G(u)$ which implies (6.24).

Assume that $\bar{A} = 0$. Inspection of (6.21) shows that
Now we show that there exist a point $y > \bar{A}$ for which

$$
(6.24) \quad \beta(y) < 0.
$$

Assume that $\bar{A} > 0$. Inspection of the solution (6.21) shows that for $A = \bar{A}$

$$
\int_{\bar{A}}^{\infty} [h(A) - h(y)] y^{-\alpha} dy = 0
$$

(while for any $A > \bar{A}$ the above integral is negative). Therefore

$$
(6.25) \quad W(x) = 2\sigma^{-2} x^{-\alpha} \int_{\bar{A}}^{\infty} (h(y) - h(\bar{A})) y^{-\alpha} dy.
$$

As $y \to \infty$

$$
h(y) - h(\bar{A}) + \frac{a^2}{2} - a - h(\bar{A}) > 0,
$$

because $\alpha = 2a/\sigma^2 < 1$ by assumption and $h(y) < 0$ for all $y < x^*$. Therefore the integral in (6.25) considered as a function of $x$ converges to zero at a rate $x^{\alpha-1}$, hence $W(x) = O(1/x)$. On the other hand, the function $G(x) \equiv (1-\lambda)/[(1+x)(1+\lambda x)] = O(1/x^2)$. Therefore there exists a point $u$ such that $W_A(u) > G(u)$. Since $W_A(u)$ is a continuous function of $A$ there exists $y > \bar{A}$ such that $W_y(u) > G(u)$ which implies (6.24).

Assume that $\bar{A} = 0$. Inspection of (6.21) shows that
\[
\lim_{x \to 0} \left( \lim_{A \to 0} W_A(x) \right) = 1 ,
\]

while \( G(0) = 1 - \lambda < 1 \). By continuity argument, we see that there exists \( y > 0 \) and \( x > y \) such that \( W_y(x) > G(x) \), whence (6.24) follows.

3°. Continuity of \( \beta(x) \) and (6.23) and (6.24) imply existence of \( A \) such that

\[
(6.26) \quad \beta(A) = 0 .
\]

Since \( G \) and \( W_y \) are continuous, there exists a point \( B < z(A) \) such that \( G(B) = W_A(B) \). Relation (6.26) implies in particular

\[
(6.27) \quad W_A(x) \leq G(x) , \quad A \leq x \leq B + \epsilon
\]

for \( \epsilon = z(A) - B > 0 \). The inequality (6.27) is strict in the neighborhood of \( B \) because both \( W_A \) and \( G \) are analytic. Therefore \( G(x) - W_A(x) \) attains a local minimum at \( B \) whence \( G'(B) = W'_A(B) = 0 \). From \( G'(B) = W'_A(B) \) we see that \( B > x^* \), because \( G'(x) < 0 \) for all \( x \), and from (6.21) we see that \( W_A(x) \) increases for \( x < x^* \). On the other hand, if we consider \( f \) subject to (6.18) and (6.19) it must coincide with \( W_A(x) \) by uniqueness of solutions, whence \( f(x) \leq G(x) \) in a right neighborhood of \( B \). Therefore, the function

\[
W(x) = \begin{cases} 
0 , & x \leq A , \\
W_A(x) , & A \leq x \leq B , \\
G(x) , & x \geq B ,
\end{cases}
\]
satisfies all the conclusions of the theorem except possibly the last one. We still need to check that \((V(x),d)\) given by (6.14) and (6.15) satisfy the condition (6.5) of the Theorem (6.4).

To check the condition (6.5) note that

\[
V(x) = V(B) + \int_B^x W(y)dy = V(B) + \int_B^x G(y)dy, \quad x > B,
\]

while \(f(x)\), which figures in condition (6.5) can be written as

\[
f(x) = V(B) + \int_B^x W_A(y)dy, \quad x > B,
\]

by virtue of uniqueness of the solutions of a differential equation.

Inequality (6.27) implies \(f(x) < V(x)\) for all \(x\) in a right neighborhood of \(B\).

This theorem completes the proof that the optimal policy is a control limit one. The proof is constructive and from it we can observe in addition that the left control limit cannot be equal to 0 and the right limit cannot be equal to \(\infty\). In Stock-Bank language this means that a nontrivial optimal policy (i.e., the one which does not keep all the funds in one asset) cannot consist of only buying stock or only selling stock: it must exercise both transactions at different moments of time.

The proof of Theorem (6.16) shows that function \( f \) subject to (6.18), (6.19) is of the form (6.21). Thus to find the optimal boundaries we need to find \( A \) and \( B \) such that

\[
B^{-\alpha} \int_{A}^{B} [\tilde{h}(A) - \tilde{h}(y)] y^{\alpha-2} dy = (1-\lambda)((1+B)(1+\lambda B))^{-1}
\]

(7.1)

\[
\frac{d}{dx} \left( x^{-\alpha} \int_{A}^{x} [\tilde{h}(A) - \tilde{h}(y)] y^{\alpha-2} dy \right) \bigg|_{x=B} = \frac{d}{dx} \left( \frac{1-\lambda}{(1+x)(1+\lambda x)} \right) \bigg|_{x=B}
\]

(7.2)

where \( \tilde{h}(y) = 2\sigma^{-2}h(y) \equiv (x/(x+1))^2 - \alpha x/(x+1) \). Elementary transformation show that if (7.1)-(7.2) hold then

\[
\tilde{h}(A) = \tilde{h}(\lambda B)
\]

whereas either

\[
B = B_1(A) \triangleq \lambda^{-1} A
\]

(7.3)

or

\[
B = B_2(A) \triangleq \lambda^{-1}((\alpha-1)A+\alpha)/((2-\alpha)A+1-\alpha)
\]

(7.4)

Thus the numerical calculations are reduced to one-parametric search of \( A \) such that (7.1) is true for \( B \) given either by (7.3) or by (7.4). It is interesting to see that the optimal boundaries \( A \) and \( B \) depend only on \( \lambda \) and \( \alpha \). There is an a priori theoretical justification why the optimal policy depends only on \( \lambda \) and \( \beta \triangleq (\mu-r)/\sigma^2 \) (recall that \( \alpha = (\mu-r)/\sigma^2 + 1 \)) but we will not outline them here. From section 2 we know that the optimal policy is not trivial iff \(-1 < (\mu-r)/\sigma^2 < +1\). (Otherwise the optimal policy is to put all funds into the asset with highest expected rate of
return.) On the graph below we depict $A \triangle A\vert(A+1)$ and $B \triangle B\vert(B+1)$ as function of $\beta$. The quantities $A$ and $B$ represent the optimal barriers for the fraction of the total assets which should be kept in Stock.

These boundaries are calculated (as a function of $\beta$) for three different values of $\lambda$. The linear function $y^*(\beta) = (\beta+1)/2$ is equal to $x^*/(x^*+1)$. This represents the optimal fraction of funds kept in stock when $\lambda = 1$. Three curves above $y^*(\beta)$ are the values for upper boundary $B$, while three curves below $y^*(\beta)$ show the values of lower boundary $A$ for $\lambda$ equal to .99, .95 and .92 respectively.

Figure 3
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References


