INFINITE HORIZON INVESTMENT CONSUMPTION MODEL
WITH A NONTERMINAL BANKRUPTCY

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I. Introduction

This paper deal with the optimal behavior of an individual whose aim is to maximize total (over an infinite time interval) expected discounted utility of consumption. The models similar to the one studied in this paper were done by Lehoczky, Sethi and Shreve [7], [8] and Karatzas, Lehoczky, Sethi and Shreve [6] (referred to as KLSS hereafter). The latter paper is of the closest relevance to ours and we will rely on the results of this paper and use their notations.

We start with the classical model in which an investor makes consumption and investment decisions continuously in time. He has an initial wealth \( x \) and utility function of consumption \( U \). In addition there are \( N + 1 \) possible investments available; one nonrisky with rate of return \( r \) and \( N \) risky whose prices are modeled as geometric Brownian motions.

In particular, let \( P_0(t) \) be the price of the riskless asset and \( P(t) = (P_1(t), \ldots, P_N(t)) \) be the prices at time \( t \) of the risky assets. The equations governing \( P_0 \) and \( P \) are

\[
\frac{dP_0(t)}{P_0(t)} = rd t
\]

\[
\frac{dP_i(t)}{P_i(t)} = \alpha_i dt + \epsilon_i D e^T (t)
\]

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Here \( \{w(t), t \geq 0\} \) is an \( N \)-dimensional standard Brownian motion, \( e_i \) is the vector with 1 at the \( i \)-th place and 0 at all other places, \( \alpha_i \) is the average rate of return on the \( i \)-th asset, \( D \) is an \( N \times N \) matrix with \( \sum = DD^T \) positive definite; \( T \) stands for the transpose operation.

The control variables are the consumption rate \( c(t) \) and the investment-policy vector \( \pi(t) = (\pi_1(t), \ldots, \pi_N(t)) \), \( t \geq 0 \), where \( \pi_i(t) \) is a fraction of the investor's wealth in the \( i \)-th risky asset \( (\pi_0(t) \triangleq 1 - \sum_{i=1}^n \pi_i(t) \) is a fraction of the wealth invested in the nonrisky asset). We allow short selling and unlimited borrowing, therefore there are no restrictions on the vector \( \pi(t) \).

The quality of consumption is measured by a strictly increasing, strictly concave utility function \( U \) on the positive half line. The objective is to find a policy \( (c(t), \pi(t), t \geq 0) \) so as to maximize the total reward

\[
V_{c(\cdot), \pi(\cdot)}(x) = \mathbb{E}_x \{ \int_0^\infty e^{-\beta t} U(c(t)) dt \}. \tag{1.3}
\]

Here \( x \) is the initial wealth, \( \beta \) is the discount factor.

Given policy \( (c(t), \pi(t), t \geq 0) \) the investor's wealth process \( x(t) \) satisfies the following stochastic differential equation.

\[
x(0) = x, \tag{1.4}
\]

\[
dx(t) = [(\alpha - r1)\pi^T(t)x(t) + rx(t) - c(t)]dt
+ x(t)\pi(t)Ddw(t), \quad \text{if } x(t) > 0, \tag{1.5}
\]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) is the vector of the average returns of \( N \) assets and \( 1 \) is the vector with all coordinates equal to one. To make a complete specification of the model one must specify the options available to the investor and
equations governing the wealth development when his wealth reaches zero. The state with zero wealth will be referred to as bankruptcy. In the bankruptcy state the rate of consumption \( c(t) \) must be equal to zero. This will be discussed in more details in Section 3.

There have been many consumption-investment models in the finance literature (e.g., Samuelson [13], Merton [10], [11], etc.). However, it was not until recently that the importance of taking special care of bankruptcy was fully recognized. In many cases the parameters of the model were such that the optimal policy did not lead to bankruptcy. In the classical paper of Merton [11], on the other hand, a certain combination of parameters would bring bankruptcy in the optimal policy although such possibility is ignored. As a result, the optimal policy which was constructed in [11] brings a solution with negative consumption and negative wealth (see [15]).

In the work of Lehoczky, Sethi and Shreve [8] and Karatzas, Lehoczky, Sethi and Shreve [6] another approach was developed (see also [12] and [14]). There is a terminal value \( P \) assigned to the bankruptcy state and at the time \( \tau \) when the wealth reaches zero the investor quits the game. The objective in these models is to maximize

\[
E_x \left\{ \int_0^\tau e^{-\beta t} U(c(t)) \, dt + e^{-\beta \tau} P \right\}.
\]

We introduce a completely different bankruptcy model in which an investor does not quit the game. Rather upon going bankrupt he may recover from it after a temporary but random sojourn in bankruptcy. Such recovery may be brought about in a number of ways. The individual may receive a bequest or he may generate an innovative idea having a commercial value. The rate of such recovery reflects essentially the innate ability or resourcefulness. We should, however,
note that such a recovery is not instantaneous. The individual must stay at
the bankruptcy state for a positive amount of time and during this time, his
consumption rate must be zero. This type of bankruptcy can be modelled by a
continuous diffusion process with a delayed reflection.

To understand intuitively, why a delayed reflection process is a proper
model of the bankruptcy with recovery, consider the following situation. Suppose
that each time when the investor's wealth process \( x(t) \) reaches zero it stays at 0
for an exponentially distributed time with parameter \( \delta \), then the investor gets a
wealth of the size \( \varepsilon \) and he starts his optimization process anew. When both
\( \varepsilon \) and \( \delta \) are small (and \( \varepsilon = \text{const} \, \delta^2 \)) the wealth process can be approximated by
a continuous diffusion process with a delayed reflection at zero or - in another
terminology - with a sticky boundary at zero (see Feller [4]). A diffusion
process with delayed reflection spends a positive amount of time at the point
zero although there does not exist an open interval during which the process
stays continuously at zero. In fact the set of times when such a process visits
zero is a perfect set (or generalized Cantor set) of a positive Lebesgue measure.
The measure of this set is in inverse proportion to an exogenous parameter \( \mu \geq 0 \),
which we call rate of recovery. Thus the wealth equation (1.5) should take the
form

\[
dx(t) = \left[(a-r1)\pi^T(t)x(t) + rx(t) - c(t)\right]1_{x(t)>0}dt
\]

\[
+ \mu1_{x(t)=0}dt + x(t)\pi(t)Dw^T(t).
\]

Equation (1.5') shows that the recovery rate \( \mu \) can be viewed as a rate of wealth
accumulation at the time when \( x(t) = 0 \); this permits the investor to leave the
bankruptcy state (c.f. [1] Sec. 24).
We will use the terms bankruptcy with recovery (or nonterminal bankruptcy) and terminal bankruptcy respectively, to describe the behavior considered in our model and KLSS model.

In this paper we will establish a one-to-one correspondence between all recovery rates $\mu$ in our model and terminal values $P$ in KLSS model in such a way that the value functions and the forms of the optimal feedback policies in both models coincide.

In our opinion the use of a delayed reflection process as a model for bankruptcy behavior is very natural, however it appears that the previous papers in finance did not employ it. In fact, we are aware of only one paper of Harrison and Lemoine [3] which uses sticky Brownian motion for modeling of applied phenomena in operations research.

This paper is structured as follows: In Section 2 we state our main assumptions for the parameters of the model. In Section 3 we introduce the notion of admissible policies and describe the dynamics of the system. In Section 4 we analyze the Bellman equation of the problem. In the fifth Section we establish the link between value functions in the models with terminal and nonterminal bankruptcies. In Section 6, 7 and 8 we prove that for each recovery rate $0 \leq \mu \leq \infty$ there exists a model with terminal bankruptcy with the same value function. In the last Section we give a tabulation of the results.

2. Main Assumptions and Notations.

We assume that the utility function $U(c)$ is an increasing concave $C^3$ function $[0,\infty)$ such that
\[
\lim_{c \to \infty} \frac{U(c)}{c} = 0, \quad (2.1)
\]

which is equivalent to

\[
\lim_{c \to \infty} U'(c) = 0. \quad (2.2)
\]

We assume both \( r \) and \( \beta \) to be positive, but no restrictions is imposed on the vector \((\alpha_1, \alpha_2, \ldots, \alpha_N)\) of average returns of risky assets.

Let

\[
\gamma \triangleq \frac{1}{2}(\alpha - r \mathbf{l}) B^{-1}(\alpha - r \mathbf{l})^T \quad (2.3)
\]

We assume

\[
\alpha \neq r \mathbf{l}, \quad (2.4)
\]

which implies

\[
\gamma > 0. \quad (2.5)
\]

Let \( \lambda_- \) and \( \lambda_+ \) be respectively negative and positive solutions of the following equation

\[
\gamma \lambda^2 - (r - \beta - \gamma) \lambda - r = 0. \quad (2.6)
\]

We assume that for every \( c > 0 \)

\[
\int_{c}^{\infty} \frac{dx}{U'(x)} < \infty. \quad (2.7)
\]

The condition (2.7) is necessary to ensure that the value function is finite. (see KLSS for more details). The value function is defined as
\[ V^*(x) = \sup_{c(\cdot), \pi(\cdot)} V_{c(\cdot), \pi(\cdot)}(x), \quad x \geq 0. \quad (2.8) \]

To emphasize the dependence of the value function \( V^* \) on the recovery rate \( \mu \) we will in certain cases write (2.8) as \( V^*(x; \mu) \).

We will also need in the sequel the value function of the KLSS terminal bankruptcy model. We define

\[ V^*_p(x) = \sup_{c(\cdot), \pi(\cdot)} E_x^\tau \left[ e^{-\beta \tau} U(c(t)) dt + e^{-\beta \tau} p \right], \quad (2.9) \]

where \( \tau \) is the first hitting time of 0 by the process \( x(t) \) governed by (1.4), (1.5).

In Sections 6, 7 and 8, we will establish a relation between \( V^*_p(x) \) and \( V^*(x; \mu) \), namely we will find a correspondence

\[ P \leftrightarrow \mu(P), \]

such that

\[ V^*_p(x) = V^*(x; \mu(P)), \quad x \geq 0. \quad (2.10) \]

By virtue of this correspondence we will be able to use the results of KLSS and describe the optimal solution.

3. Admissible Policies.

In this section we will give a heuristic explanation of how the price equations (1.2), (1.3) result in the wealth equation (1.5') and we also rigorously formulate the required notion of admissible policy.

Suppose that \( c(t) \) be the consumption rate at time \( t \), \( P_i(t) \) and \( n_i(t) \) be the price and the number of shares of the \( i \)-th asset held at time \( t \).
Then the agent's wealth at time $t$ is

$$x(t) = \sum_{i=0}^{N} n_i(t)P_i(t).$$

(3.1)

Suppose that the trading of shares is done at discrete time points, $t - h$, $t$, and $t + h$ with the consumption levels being constant over these intervals. We assume that the trading of shares at time $t$ is used to finance consumption during the interval $[t, t+h)$. This leads us to the following equation

$$x(t-h) = \sum_{i=0}^{N} n_i(t-h)P_i(t) = \sum_{i=0}^{N} n_i(t)P_i(t) + c(t)h = x(t) + c(t)h$$

(3.2)

From (3.1) and (3.2) we obtain

$$x(t) - x(t-h) = \sum_{i=0}^{N} n_i(t-h)[P_i(t) - P_i(t-h)] - c(t)h.$$

(3.3)

The continuous analog of (3.3) becomes

$$dx(t) = \sum_{i=0}^{N} n_i(t)dP_i(t) - c(t)dt.$$  

(3.4)

The equations (3.2), (3.3) and (3.4), however, were derived for the case in which the wealth level $x(t)$ is positive and the agent is allowed to make consumption and investment decisions. When the investor reaches bankruptcy, he does not own any assets, he is not allowed to trade and he is not allowed any positive consumption. However, if $x(t-h) = 0$ then with probability $ph$ at time $t$ the agent received a wealth $\varepsilon$ and with probability $1 - ph$ he remains bankrupt. Thus we may write

$$E\{x(t) - x(t-h)\} = \varepsilon ph, \quad \text{if } x(t-h) = 0.$$  

(3.5)
If we put \( \pi_i(t) = n_i(t)P_i(t)/x(t) \), which is the fraction of the wealth invested in the \( i \)-th asset then (3.4), (1.2) and (1.3) yield (1.5) for \( x(t) > 0 \). Equation (1.5) together with a continuous analogue of (3.5) gives us (1.5'). A more rigorous derivation of (1.5') requires an analysis which is beyond the scope of the present paper. An interested reader may look Harrison and Pliska [4] and Harrison and Lemoine [3] for more details.

Now we formulate rigorously the mathematical model. We start with an \( N \)-dimensional Wiener process \( \{w(t), F_t, t \geq 0\} \) on a probability space \( \{\Omega, F, P\} \) where \( F_t \) is a increasing family of right-continuous \( \sigma \)-fields to which \( w(t) \) is adapted.

An \( N \)-dimensional \( F_t \)-adapted process \( \{\pi(t), t \geq 0\} \) is called admissible if a.s.

\[
\int_0^t \pi(s)^T \pi(s) \, ds < \infty \quad \text{for all } t > 0. \tag{3.6}
\]

Let \( \{c(t), t \geq 0\} \) be any adapted process and \( \pi \) as above. For any such pair and any \( x \geq 0 \) we can construct a solution \( x(t) \) of (1.4), (1.5'). The pair \( \{c(t), \pi(t), t \geq 0\} \) is called admissible at \( x \) if \( \pi(\cdot) \) is admissible and

\[
c(t)1_{x(t)=0} = 0 \quad \text{for all } t \text{ a.s.}. \tag{3.7}
\]

Condition (3.7) specifies that only those polices are considered which have zero consumption in the bankruptcy state. Note that for any admissible policy the solution \( x(t) \) of (1.4), (1.5') is nonnegative.

4. The Bellman Equation and Reduction to One Risky Asset.

We are interested in an equation that must be satisfied by \( V^* \) in (2.8) with the supremum in the right hand side being taken over admissible pairs. Usual arguments lead to the Bellman equation
\[ \beta V(x) = \max_{c \geq 0, \pi} \left\{ [(\alpha - r\mathbf{1})^T x V'(x) + (r x - c) V'(x) + \frac{1}{2} \sum_{\pi} x^T \pi x^2 V''(x) + U(c)] \right\}, \quad x > 0. \] (4.1)

To find \( V^*(x) \), however, the equation (4.1) is not sufficient. We need to specify the behavior of \( V^* \) in the neighborhood of the point 0. In KLSS the value function \( V_p^*(x) \) satisfies an obvious relation

\[ V_p^*(0) = p \] (4.2)

To see what kind of relation the value function in our case must satisfy at the origin, consider \( V_{c(*)}, \pi(*) \) with \( c(*) \), \( \pi(*) \) being a feedback (or Markov) policy \( \pi(t) = \Pi(x(t)) \) and \( c(t) = C(x(t)) \) where \( \Pi \) and \( C \) are measurable functions with \( C(0) = 0 \). Then \( V(x) \equiv V_{c(*)}, \pi(*) (x) \) will satisfy the differential equation (4.3) with boundary conditions (4.4) below (see [1] ch. 24, Lemma2).

\[ \beta V(x) = \left\{ [(\alpha - r\mathbf{1})^T x \Pi(x) + r x - C(x)] V'(x) + \frac{1}{2} \sum_{\pi} \Pi(x)^T x^2 \Pi(x) + U(C(x)) \right\} \] (4.3)

\[ \beta V(0) - \mu V'(0) = U(C(0)) \] (4.4)

The latter suggests the boundary condition for the Bellman equation (4.1).

(4.5) **Theorem** Suppose \( V \) is a nonnegative \( C^2 \) function satisfying the Bellman equation (4.1) and

\[ \beta V(0) - \mu V'(0) = U(0). \] (4.6)

If \( U(0) \) is finite then \( V(x) \geq V^*(x), \ x > 0. \)

Proof. Let \( x > 0 \) and \((c(*)\), \( \pi(*) \)) be an admissible policy, and \( x(t) \) be the corresponding wealth trajectory. Let \( T(N) \) be the first time at which \( x(t) \) hits \( N. \)
Using Ito's formula and (1.5'), we may write

\[ E_x\{V(x(0)) - e^{-T(N)}V(x(T(N))}\} = E_x\{ \int_0^{T(N)} (\beta e^{-\beta t}V(x(t)) - \]

\[ e^{-\beta t}v'(x(t))[(a-r)\pi(t)x(t) + r x(t) - c(t)]1_{x(t)>0} \]

\[ -e^{-\beta t}v'(x(t))\mu 1_{x(t)=0} - \frac{1}{2} e^{-\beta t}v''(x(t)) x(t)^2\pi(t)\sum\pi(t)dt \]  \( (4.7) \)

\[ + \int_0^{T(N)} e^{-\beta t}v'(x(t))x(t)\pi(t)Dd\omega(t) \} \]

The last integral in the right hand side of (4.7) is a martingale and its expectation vanishes. By virtue of (4.1) and (4.6) the integrand in the first integral in the right hand side of (4.7) does not exceed

\[ e^{-\beta t}U(c(t))1_{x(t)>0} + e^{-\beta t}U(0)1_{x(t)=0} \]  \( (4.8) \)

In view of (3.7) the expression (4.8) equals \( e^{-\beta t}U(c(t)) \) and (4.7) yields

\[ V(x) - E_x\{e^{-T(N)}V(x(T(N))}\} \geq \]

\[ E_x\{ \int_0^{T(N)} e^{-\beta t}U(c(t))dt\} \]  \( (4.9) \)

Letting \( N \to \infty \) and using Fatou's lemma, we get

\[ V(x) \geq V_{c(\cdot),\pi(\cdot)}(x) \]  \( (4.10) \)

Maximizing over all \( c(\cdot), \pi(\cdot) \) yields the required result.

(4.11) Remark. When \( U(0) = -\infty \) the theorem (4.5) is true without condition (4.6). In the latter case \( V_{c(\cdot),\pi(\cdot)}(x) = -\infty \) for any feasible policy which brings bank-
ruptcy with positive probability and (4.10) holds. If the policy \( c(\cdot), \pi(\cdot) \) a.s. does not lead to bankruptcy, then we can use the results for the model with terminal bankruptcy to get (4.10) (see KLSS Theorem 4.1 and Remark 4.2).

Next, we will consider a "reduced" consumption/investment model with a single risky asset.

Choose \( a \) and \( \sigma > 0 \) such that

\[
\gamma = \frac{(a-r)^2}{2\sigma^2}
\]

(4.12)

where \( \gamma \) is given by (2.3). Consider a model with one risky asset with parameters \( a \) and \( \sigma^2 \). (One risky asset can be viewed as a "mutual fund" using self-financing strategy. For more details see KLSS and Harrison and Pliska [4]).

In the reduced model \( \pi(t) \) is a one-dimensional process whose meaning is the proportion of wealth invested in the risky asset. The reduced wealth equation is

\[
dx(t) = [(a-r)\pi(t)x(t) + rx(t) - c(t)]1_{x(t)>0}dt + \mu 1_{x(t)=0}dt + x(t)\pi(t)dw(t)
\]

(4.13)

where \( w(t) \) is a standard one-dimensional Wiener process adapted to \( F_t \). The corresponding Bellman equation for the value function \( V \) is

\[
\delta V(x) = \max_{c \geq 0, \pi} \left[ (\sigma - r)\pi x V'(x) + (r - c) V'(c) + \frac{1}{2} \pi^2 \sigma^2 x^2 V''(x) + U(c) \right], \ x > 0
\]

(4.14)

The "reduced" consumption/investment problem is equivalent to the original one in the following sense: If the \( c(\cdot) \) and \( \pi(\cdot) \) are the optimal feedback policies given by

\[
c(t) = C(x(t)), \ \pi(t) = \Pi(x(t))
\]
with \( C \) and \( \Pi \) being measurable functions then the optimal policy in the original model is the feedback policy with the same feedback function \( C \) for the consumption rate and
\[
\Pi(x) = \Pi(x) \frac{\sigma^2}{\alpha - r} (\alpha - r) \Sigma^{-1}
\]

The proof of equivalency of the "reduced" and "nonreduced" models is similar to the one carried out in KLSS. We therefore omit the proof.

In the sequel we will consider only the reduced model. We will seek the solution of the reduced Bellman equation (4.14) subject to the boundary condition (4.6).

5. Reduction to the problem with terminal bankruptcy.

The main idea of the proof is that the Bellman equation (4.14) with the boundary condition (4.6) can be solved by comparing it to the Bellman equation of the terminal bankruptcy problem in which the boundary condition (4.6) is replaced by
\[
V(0) = P.
\]
In KLSS it was shown that \( V_p(x) \) is the solution of (4.14), (5.1). (Here and in the sequel we write \( V_p(x) \) and \( V(x; \mu) \) in lieu of \( V_p^*(x) \) and \( V^*(x; \mu) \).) If we could find \( P(\mu) \), for which \( V_p(\mu)(\cdot) \) satisfies (4.6), then \( V_p(\mu) \) is the solution of (4.14), (4.6). Namely, the following theorem is true.

(5.2) **Theorem.** Suppose \( V_p(x) \) is the value for the problem with terminal bankruptcy and \( C(x), \Pi(x) \) are the corresponding optimal feedback controls. Let
\[
\mu = \left[ \theta V_p(0) - U(0) \right] / V_p'(0)
\]
and let \( x(t) \) be the solution of the stochastic differential equation (4.13) with \( \mu \) given by (5.5), \( \pi(t) = \Pi(x(t)) \) and \( c(t) = C(x(t)) \). If
\[ \mathbb{E}\left\{ \int_0^t x^2(t)V_p(x(t))dt \right\} < \infty \quad \text{a.s.,} \quad (5.4) \]

\[ e^{-\beta t}\mathbb{E}\{V_p(x(t))\} \to 0, \quad \text{as } t \to \infty. \quad (5.5) \]

then \( V_p(x) \) is the value function in the model with nonterminal bankruptcy with the recovery rate \( \mu \) and the optimal policy in this model is

\[ c(t) = C(x(t)) \mathbb{1}_{x(t) > 0}, \quad \pi(t) = \Pi(x(t)) \mathbb{1}_{x(t) > 0}. \quad (5.6) \]

**Proof.** Let \( V = V_p \). Since \( C(\cdot) \) and \( \Pi(\cdot) \) represent the optimal feedback controls for the problem with terminal bankruptcy, they are the argmaxima of the right hand side of (4.14) and

\[ \beta V(x) = (\alpha - r)\Pi(x)xV'(x) + (rx - C(x))V'(x) + \frac{1}{2} \Pi(x)^2 \sigma^2 x^2 V''(x) + U(C(x)), \quad x > 0 \quad (5.7) \]

Repeating the argument used for deriving (4.9) from (4.1), and (4.6) we get from (5.7) and (4.6)

\[ V(x) = \mathbb{E}_x\left\{ e^{-t}V(x(t)) \right\} \]

\[ = \mathbb{E}_x\left\{ \int_0^t V(C(s))e^{-\beta s}ds \right\}. \quad (5.8) \]

As \( t \to \infty \) the right hand side of (5.8) converges to \( V(x) \) by virtue of (5.5).

Theorem (5.2) shows that in order to solve the model with nonterminal bankruptcy with recovery rate \( \mu \), it is enough to find \( P(\mu) \) such that for the value function \( V_p \) the relation (5.3) holds. Then

\[ V(x; \mu) = V_{P(\mu)}(x). \quad (5.9) \]

We will establish such a correspondence by looking at the function \( \mu(P) \), given by (5.3). We will show that there exist \( U(0)/\beta < \bar{P} < \beta \) such that \( \mu(P) \) is a continuous strictly increasing function on \((\bar{P}, \beta)\) such that
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\[ \mu(\mathcal{P}^+) = 0 \]  
(5.10)

\[ \mu(\mathcal{P}^-) = \infty \]  
(5.11)

Then it is possible to get inverse function \( P(\mu) \) and get (5.9).

6. Correspondence between \( P \) and \( \mu \): The cases of \( U(0) = -\infty \) and \( \mu = 0 \).

As was noticed in the previous section, if \( U(0) = -\infty \) then the optimal policy must stay away from bankruptcy with probability one whatever the recovery rate. Thus, for any \( \mu \)

\[ V(x; \mu) = V_{-\infty}(x) \equiv V_{U(0)/\beta}(x) \]

(see KLSS Sec. 12).

Another extreme case which should be treated separately is \( \mu = 0 \). In this case there is no recovery and once the bankruptcy is reached, the agent remains bankrupt forever. Simple arguments show that in this case the model with non-terminal bankruptcy is identical to the one with terminal bankruptcy with \( P = U(0)/\beta \). Thus we can use the results of Sections 11 and 13 of KLSS.

We denote by \( q \) the probability of bankruptcy under the optimal policy and let \( c(t), x(t) \) be the consumption and wealth process under the optimal policy.

(6.1) Theorem. If \( \mu = 0 \), then \( q = 0 \) and

a) If \( U'(0) = \infty \) then \( c(t) > 0 \) for all \( t \).

b) If \( U'(0) < \infty \), then there exists \( \bar{x} \) such that \( c(t) = 0 \)

\[ \text{iff } x(t) \leq \bar{x} \text{ and } c(t) > 0 \text{ iff } x(t) > \bar{x}. \]

7. Correspondence between \( P \) and \( \mu \) when \( U(0) > -\infty \) and \( U'(0) = \infty \).

We say that the consumption constraints are inactive if for the solution \( V^* \) of (4.14) for any \( x > 0 \), the maximum at the right hand side of (4.14) is attained at the point \( (c, \pi) \) with \( c > 0 \).
In KLSS it was shown that for (4.14) with the boundary condition (4.2) the consumption constrains are inactive if \( U'(0) = -\infty \) or if \( U'(0) < -\infty \) and \( P \geq P^* \) where \( P^* \) is given by (12.1) of KLSS. In this section we investigate the correspondence (5.3) between \( P \) and \( \mu \) in these two cases.

Formula (6.5) of KLSS shows

\[
V'(x) = U'(C(x)) \tag{7.1}
\]

where \( C(x) \) is the maximizer of (4.14). Combining (7.1) with (4.2) we get

\[
\mu = [\beta P - U(0)] / U'(C(0^+)) \tag{7.2}
\]

Put \( C(0^+) = a \equiv a(P) \), where as shown in Section 11 of KLSS \( a(P) \) is the (unique) solution of

\[
F(c) = -(\lambda_+ + 1) P \tag{7.3}
\]

where \( \lambda_+ \) is the positive solution of (2.6) and

\[
F(c) = -\left( \frac{U'(c)}{\gamma \lambda_-(\lambda_+ + 1)} \int \frac{d\theta}{c (U'(\theta))^\lambda_-} - \frac{\lambda_+ + 1}{\beta} U(c) + \frac{\lambda_+}{f} c U'(c) \right), \tag{7.4}
\]

\( c > 0 \).

Lemma 11.1 of KLSS shows that \( F(c) \) is a continuous strictly decreasing function of \( c, c > 0 \), and the solution of (7.3) exists for \( P > U(0)/\beta \). Therefore the solution \( a(P) \) of (7.3) is a continuous increasing function of \( P, P > U(0)/\beta \).

(7.5) **Theorem.** If \( U(0) > -\infty \) and \( U'(0) = -\infty \) then \( \mu = \mu(P) \) given by (7.2) is a continuous increasing function which maps \( (U(0)/\beta, U(\infty)/\beta) \) onto \( (0, \infty) \).

**Proof.** Since \( U'(x) \) is a decreasing function of \( x \) and \( a(P) \) is an increasing function, \( U'(C(0^+)) \equiv U'(a(P)) \) is a decreasing function of \( P \). Thus \( \mu(P) \) is an
increasing function. The continuity of \( \mu(\cdot) \) follows from continuity of \( U'(\cdot) \) and \( a(\cdot) \).

In the proof of Lemma 11.1 in KLSS it was shown that \( F(0^+) = -U(0)(\lambda_+ + 1) / \beta \).

Therefore

\[
a(P) \to 0 \quad \text{as} \quad P \to U(0)/\beta.
\]

Thus \( U'(a(P)) \to \infty \) as \( P \to U(0)/\beta \) and therefore \( \mu(P) \to 0 \).

Note that the negative solution \( \lambda_- \) of (2.6) is less than \(-1\). Therefore, (2.7) implies

\[
\liminf_{c \to \infty} c U'(c) = 0.
\]  

Taking the limit of (7.4) as \( c \to \infty \) and using (7.6), we obtain

\[
F(\infty) = -(\lambda_+ + 1) U(\infty) / \beta.
\]

Therefore \( a(P) \to \infty \) as \( P \to U(\infty)/\beta \) and \( U'(a(P)) \to 0 \) by virtue of (2.2). Accordingly, \( \mu(P) \to \infty \) as \( P \to U(\infty)/\beta \).

(7.7) Corollary. Set \( P = U(0)/\beta \) and \( \bar{P} = U(\infty)/\beta \). Then for each \( \mu \in [0, \infty) \) there exists \( P(\mu) \in [P, \bar{P}] \) such that (5.9) holds.

Proof. For \( 0 < \mu < \infty \) the statement follows from Theorem (7.5) and relations (5.10 and (5.11) proved above. If \( \mu = 0 \), the result follows from Theorem (6.1). If \( \mu = \infty \) (the case of instantaneous reflection) then one can easily show that there exists no optimal policy and any sequence of policies \( c_n(t) \to \infty \) uniformly as \( n \to \infty \) will be a minimizing sequence for (1.3). In the latter case \( V(x; \infty) = U(\infty)/\beta = V_P(x) \).
8. Correspondence between \( P \) and \( \mu \) when \( U'(0) < \infty \).

Let

\[
p^* = \frac{1}{\beta} U(0) - \frac{(U'(0))^{\lambda_- + 1}}{\beta \lambda_-} \int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_-}}
\]

By virtue of Section 12 of KLSS the consumption constraints are inactive if \( U'(0) < \infty \) and \( P \geq P^* \). The same arguments as in section 7 show

\[
\mu(P) = (\beta P - U(0))/U'(a(P)), \quad P > P^*
\]

where \( a(P) \) is the solution of (7.3). Likewise, \( \mu(P) \) is a continuous increasing function on \((P^*, U(\infty)/\beta)\) with

\[
\lim_{P \to U(\infty)/\beta} \mu(P) = \infty.
\]

On the other hand, we can see by inspection that \( a(P) \to 0 \) as \( P \to P^* \). Thus

\[
\lim_{P \to P^*} \mu(P) = (\beta P - U(0))/U'(0)
\]

In section 13 of KLSS it was shown that if \( U(0)/\beta \leq P < P^* \) then

\[
V_P'(0) = \tilde{y},
\]

where

\[
\tilde{y} \equiv \tilde{y}(P) = (-\beta \lambda_- (P - U(0)/\beta) \int_0^\infty \frac{d\theta}{(U'(\theta))^{\lambda_- - 1}} \frac{1}{\lambda_- + 1}
\]

One can see by inspection that \( \tilde{y}(P) \) is a continuous decreasing function and

\[
\lim_{P \to U(0)/\beta} \tilde{y}(P) = \infty
\]

\[
\tilde{y}(P^*) = U'(0)
\]

By virtue of (5.3) and (8.3)
\[ \mu(P) = (\beta P - U(0))/\tilde{\gamma}, \quad U(0)/\beta < P \leq P^*. \]

Thus, in view of (8.4) and (8.5)

\[
\lim_{P+U(0)/\beta} \mu(P) = 0 \quad (8.6)
\]

\[
\mu(P^*) = (\beta P - U(0))/U'(0). \quad (8.7)
\]

Relations (8.1), (8.2) and (8.6), (8.7) yield the following theorem.

(8.8) **Theorem.** If \( U'(0) < \infty \) then the function \( \mu(P) \) is a continuous increasing mapping of \( (U(0)/\beta, U(\infty)/\beta) \) onto \( (0, \infty) \).

(8.9) **Corollary.** If \( U'(0) < \infty \) then for each \( \mu \in [0, \infty] \) there exists \( P(\mu) \in [P, \bar{P}] \) such that

\[
V(x; \mu) = V_{P(\mu)}(x)
\]

where \( P \) and \( \bar{P} \) are the same as in Corollary (7.7). The function \( P(\mu) \) is a continuous increasing function.

The proof of this corollary is similar to that of Corollary (7.7).

To tie up loose ends we need to show that (5.4) and (5.5) are valid for the solution (4.13). This is a straightforward verification, using the explicit expression for \( V \) obtained in KLSS.

9. Tabulated results.

Theorem (5.2) and the correspondence between \( P \) and \( \mu \) established in Sections 6, 7, and 8 enable us to describe the nature of the optimal policy and the character of the optimal process, using the results of KLSS.
In the table below, \( q \) stands for the probability of bankruptcy under optimal policy, \( \beta \) is the discount factor, \( r \) is the rate of return on a riskless asset, \( \gamma \) is given by (2.3) and

\[
\mu^* = \mu(P^*) \equiv -\frac{U'(0)^{\lambda-}}{\beta^{\lambda-}} \int_0^\infty \frac{d\theta}{U'(\theta)^{\lambda-}}.
\]

<table>
<thead>
<tr>
<th>( \mu = 0 ) (absorption)</th>
<th>( 0 &lt; \mu \leq \mu^* )</th>
<th>( \mu^* &lt; \mu &lt; \infty )</th>
<th>( \mu = \infty ) (instantaneous reflection)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U(0) = -\infty )</td>
<td>( U'(0) = \infty )</td>
<td>( c(t) &gt; 0, \quad q = 0 )</td>
<td>No optimal policy; consume quickly to bankruptcy ( V(x; \infty) = U(\infty) / \beta )</td>
</tr>
<tr>
<td>( U(0) &gt; -\infty )</td>
<td>( U'(0) = \infty )</td>
<td>( c(t) &gt; 0 )</td>
<td>( c(t) &gt; a &gt; 0 ) ( 0 &lt; q &lt; 1 ) if ( \beta &lt; r + \gamma ) ( q = 1 ) if ( \beta \geq r + \gamma )</td>
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<td>( q = 0 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U(0) &gt; -\infty )</td>
<td>( U'(0) &lt; \infty )</td>
<td>( c(t) = 0, \text{ if } x(t) \leq \bar{x} )</td>
<td>( c(t) = 0, \text{ if } x(t) \leq \bar{x} )</td>
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<td>( q = 0 )</td>
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<tr>
<td>( \bar{x} = 0 \text{ if } \mu = \mu^* )</td>
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<td>( 0 &lt; q &lt; 1 ), if ( \beta &lt; r + \gamma )</td>
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<tr>
<td>( q = 1 ), if ( \beta \geq r + \gamma )</td>
<td>( q = 1 ), if ( \beta \geq r + \gamma )</td>
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Table 1.
REFERENCES


