The $X+Y, X/Y$ Characterization of the Gamma Distribution

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Abstract

We prove, by elementary methods, that if \( X \) and \( Y \) are independent random variables, not constant, such that \( X + Y \) is independent of \( X/Y \) then either \( X, Y \) or \( -X, -Y \) have gamma distributions with common scale parameter. This extends the result of Lukacs, who proved it for positive random variables, using differential equations for the characteristic functions. The aim here is to use more elementary methods for the \( X, Y \) positive case as well as elementary methods for proving that the restriction to positive \( X, Y \) may be removed.
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Summary: We prove, by elementary methods, that if $X$ and $Y$ are independent random variables, not constant, such that $X + Y$ is independent of $X/Y$ then either $X, Y$ or $-X, -Y$ have gamma distributions with common scale parameter. This extends the result of Lukacs, who proved it for positive random variables, using differential equations for the characteristic functions. The aim here is to use more elementary methods for the $X,Y$ positive case as well as elementary methods for proving that the restriction to positive $X,Y$ may be removed.

Introduction

We say that $X$ is a gamma-a variate if $X$ has the standard gamma density $\frac{x^{a-1}e^{-x}}{\Gamma(a)}, x > 0$. If $X$ and $Y$ are independent gamma-a and gamma-b variates then $X + Y$ is independent of $X/Y$. This article is concerned with the converse: if $X + Y$ is independent of $X/Y$ for independent $X, Y$, what can be said about $X$ and $Y$? The general result is this: ruling out the case where $X$ and $Y$ are constants, in which case any function of $X$ and $Y$ is independent of any other, if $X + Y$ is independent of $X/Y$ then there is a constant $c$ and positive constants $a$ and $b$ such that $cX$ is gamma-a and $cY$ is gamma-b. The constant $c$ may be negative, but neither $X$ nor $Y$ can take both positive and negative values; either $X, Y$ or $-X, -Y$ are pairs of positive gamma variates with a common scale parameter and possibly different gamma parameters.

In 1955, Lukacs [3] proved the basic result under the assumption that $X$ and $Y$ were positive. He showed that the characteristic functions of $X$ and $Y$ satisfied differential equations whose only solutions were characteristic functions of gamma variates with a common scale parameter. A few years after that, in the late 1950's, I needed the $X + Y, X/Y$ characterization in developing methods for generating random points on surfaces by means of projections of points with independent coordinates. But the coordinates could be negative as well as positive, so I set out to extend Lukacs' result by removing the restriction that $X$ and $Y$ be positive. I was able to do this, using elementary methods, but I still needed Lukacs' result for the positive case, so I put the matter aside until I could find an elementary argument that established that case as well, perhaps motivated by a sentiment attributed to Herman Rubin: If you have to use characteristic functions you don't really understand what is going on.

I was not able to find an elementary proof of the $X, Y$ positive case, and the matter sat for years, until I was sent a manuscript, by Findeisen, which contained a clever device that might be used to establish Lukacs' result without resorting to characteristic functions. In the form that Findeisen's result was published, [2], there is a disclaimer suggested by the referees, to the effect that characteristic function results are implicit in parts of Findeisen's arguments.

And there the matter rests today, the point of departure for this article. In it, I will use a variation of Findeisen's device, together with my earlier proof that the $X,Y$ positive restriction can be removed, to provide a complete treatment of the $X + Y, X/Y$ characterization of the gamma distribution by elementary methods. Opinions differ on what is elementary, of course. In the development below, the most advanced result that I need is the fact that a distribution on $[0,1]$ is determined by its moments. This was a deep result when first proved by Hausdorff, but it may now, thanks to Feller, be considered elementary, as the elegant proof in [1] shows, using basic probability and limit arguments.
The Unrestricted Theorem

**THEOREM.** If $X$ and $Y$ are independent, non-degenerate (i.e. not constant) random variables such that $X + Y$ is independent of $X/Y$, then there are constants $a, b$ and $c$ such that $cX$ has the gamma density $z^{a-1}e^{-z}/\Gamma(a)$ and $cY$ has the gamma density $y^{b-1}e^{-y}/\Gamma(b)$.

This is the most general form of the $X/Y, X+Y$ characterization of the gamma distribution. It does not require that the variates be positive. Our proof depends on four propositions, each of which will be proved by elementary methods below. Two of the propositions depend on what we call the exponential moments of a non-negative random variable $Z$, defined as the sequence of values

$$E(Z^n e^{-Z}) / E(e^{-Z}) \quad \text{for } n = 1, 2, 3, \ldots$$

Evidently the exponential moments all exist, since $z^n e^{-z}$ is bounded for $z \geq 0$.

Four Propositions

**Proposition 1.** If $X$ and $Y$ are independent random variables, not constant, such that $X + Y$ is independent of $X/Y$ then either $Pr(X > 0, Y > 0) = 1$ or $Pr(X < 0, Y < 0) = 1$.

**Proposition 2.** If $X$ and $Y$ are independent, positive, not-constant random variables such that $X/Y$ is independent of $X + Y$, then there are positive constants $a, b$ and $k$ such that the exponential moments of $X$ and $Y$ are those of gamma distributions with common scale parameter: for $n = 1, 2, 3, \ldots$,

$$E(X^n e^{-X}) / E(e^{-X}) = k^n \Gamma(a + n) / \Gamma(a) \quad \text{and} \quad E(Y^n e^{-Y}) / E(e^{-Y}) = k^n \Gamma(b + n) / \Gamma(b).$$

**Proposition 3.** Every distribution on $[0, \infty)$ is determined by its exponential moments.

**Proposition 4.** If $X$ and $Y$ are independent, positive random variables such that $X/Y$ is independent of $X - Y$, then $X$ and $Y$ are both constants.

These four propositions will be proved below.

Proof of the Main Theorem

We now have independent $X$ and $Y$, not constant, with $X/Y$ independent of $X + Y$. Assume the four Propositions. Then Proposition 1 ensures that either $X, Y$ or $-X, -Y$ are pairs of positive variates. Proposition 2 then provides the exponential moments of $X$ and $Y$, or $-X$ and $-Y$, and Proposition 3 ensures that, with the resulting gamma exponential moments, $cX$ and $cY$ are gamma-a and gamma-b for some constant $c$, possibly negative. Proposition 4 is not used directly, but is required for the proof of Proposition 1.

Proof of Proposition 1.

We have independent $X$ and $Y$, not constant but otherwise unrestricted, such that $X + Y$ is independent of $X/Y$. We must prove that either $Pr(X > 0, Y > 0) = 1$ or $Pr(X < 0, Y < 0) = 1$. Let $p_x = Pr(X > 0)$ and $p_y = Pr(Y > 0)$. If $p_x p_y > 0$, let $(X_+, Y_+)$ be the point $(X, Y)$ conditioned by $X > 0$ and $Y > 0$:

$$Pr(X_+ < z, Y_+ < y) = \frac{Pr(0 < X < z, 0 < Y < y)}{p_x p_y}.$$

Evidently $X_+$ is independent of $Y_+$ (a product measure is still a product measure when restricted to a product set), and, in fact, $X_+ + Y_+$ is independent of $X_+ / Y_+$ because

$$Pr(X_+ + Y_+ < r, X_+/Y_+ < s) = \frac{Pr(0 < X + Y < r, 0 < X/Y < s)}{p_x p_y} = \frac{Pr(0 < X + Y < r) Pr(0 < X/Y < s)}{p_x p_y}.$$

Thus $X_+ + Y_+$ is independent of $X_+/Y_+$, since their joint distribution is a product. Propositions 2 and 3 apply: there is a positive constant $c$ such that $cX_+$ is gamma-a and $cY_+$ is gamma-b.

This takes care of the positive quadrant, with measure $p_x p_y$. If $(1 - p_x)(1 - p_y) > 0$ then $(X_-, Y_-)$ is well-defined and an argument similar to that for $(X_+, Y_+)$ shows that $cX_-$ and $cY_-$ must be standard gamma variates for some negative constant $c$. Thus $0 < p_x < 1$ and $0 < p_y < 1$ and $X + Y$ independent of
require that \(X\) and \(Y\) each have densities that are proper mixtures of scaled "negative" and "positive" gamma densities. But it is elementary to verify that such mixtures for \(X\) and \(Y\) do not render \(X + Y\) independent of \(X/Y\).

Thus four possibilities remain:

- \((a)\) \(p_x = 1, p_y = 1\)
- \((b)\) \(p_x = 0, p_y = 1\)
- \((c)\) \(p_x = 0, p_y = 0\)
- \((d)\) \(p_x = 1, p_y = 0\).

If conditions \((a)\) or \((c)\) hold, then \(X, Y\) or \(-X, -Y\) are independent pairs of positive variates and Propositions 2 and 3 apply. If \((b)\) holds, then \(-X\) and \(Y\) satisfy the conditions of Proposition 4, so they must be constant; if \((d)\) holds, then Proposition 4 shows that \(X\) and \(-Y\) must be constant.

Thus the \(X/Y, X + Y\) characterization of the gamma distribution is established, with no restrictions except that \(X\) and \(Y\) are not constants.

**Proof of Proposition 2.**

We have independent, positive non-constant \(X\) and \(Y\) with \(X/Y\) independent of \(X + Y\). Consider the exponential moments of \(X, Y\) and \(X + Y\):

\[
R_n = E[X^n e^{-X}] / E[e^{-X}]
\]
\[
S_n = E[Y^n e^{-Y}] / E[e^{-Y}]
\]
\[
T_n = E[(X + Y)^n e^{-X-Y}] / E[e^{-X-Y}] = \sum_{i=0}^{n} \binom{n}{i} R_i S_{n-i}.
\]

If \(X\) and \(Y\) were gamma variates with common scale parameter, then \(R_n, S_n, T_n\) would have the form, for some positive constants \(a, b\) and \(k\):

\[
R_n = k^n \Gamma(a + n)/\Gamma(a), S_n = k^n \Gamma(n + b)/\Gamma(b), T_n = k^n \Gamma(a + b + n)/\Gamma(a + b).
\]

The independence of \(X/(X + Y)\) and \(X + Y\) will be used to provide a pair of recurrence equations for \(R_{n+1}\) and \(S_{n+1}\) that will have a unique solution: the exponential moments of expression (1). Then, because by Proposition 3 the exponential moments determine the distribution, we will be led to gamma distributions.

The recursions may be derived by dividing, side for side, the relation

\[
E[X^n (X + Y) e^{-X-Y}] = E[(\frac{X}{X + Y})^n] E[(X + Y)^{n+1} e^{-X-Y}]
\]

by the sides of

\[
E[X^n e^{-X-Y}] = E[(\frac{X}{X + Y})^n e^{-X-Y}].
\]

These relation follow easily from the independence of \(X, Y\) and of \(X/Y, X + Y\). Upon division, side for side, we get the relation

\[
R_{n+1}/R_n + S_1 = T_{n+1}/T_n.
\]

Reversing the roles of \(X\) and \(Y\) then provides

\[
S_{n+1}/S_n + R_1 = T_{n+1}/T_n.
\]

Note that the gamma exponential moments in (1) satisfy (2) and (3). We must show that no others do. When \(n = 1\), (2) and (3) lead to

\[
(R_2 - R_1^2)/R_1 = (S_2 - S_1^2)/S_1.
\]
Now $R_2 - R_1^2$ is the variance of a non-degenerate random variable, (The $W$ defined by

$$\Pr(W \leq w) = \int_0^w e^{-x} dF(x)/\int_0^\infty e^{-x} dF(x),$$

with $F$ the distribution of $X$). Thus there is a positive value $k$ such that

$$R_2 = kR_1 + R_1^2, \quad \text{and} \quad S_2 = kS_1 + S_1^2.$$  

A little algebra will verify that (2) and (3) give $R_{n+1}$ and $S_{n+1}$ in terms of $R_1, S_1, R_2, S_2, \ldots, R_n, S_n$, and since (5) provides $R_2$ and $S_2$ in terms of $R_1$ and $S_1$ and the common parameter $k$, the two sets of exponential moments for $X$ and $Y$ are determined by $R_1$ and $S_1$ and the constant $k$. Specifically, given $R_1, S_1$ and the common value $k$ required by (4), define $a$ and $b$ by the conditions $R_1 = ka, S_1 = kb$. Then $R_2 = k^2a(a+1), S_2 = k^2b(b+1), R_3 = k^3a(a+1)(a+2), S_3 = k^3b(b+1)(b+2)$ and, in general,

$$R_n = k^n\Gamma(a+n)/\Gamma(a) \quad \text{and} \quad S_n = k^n\Gamma(b+n)/\Gamma(b)$$

provides the unique solution to conditions (2), (3) and (4) derived from the assumption of independent pairs $X, Y$ and $X/Y, X + Y$.

**Proof of Proposition 3**

Let $X$ be non-negative with distribution $F$ and exponential moments

$$R_n = E(x^n e^{-X})/E(e^{-x}) = \int_0^\infty z^n e^{-z} dF(z)/\int_0^\infty e^{-z} dF(z), \quad n = 1, 2, 3, \ldots.$$  

We must show that the $R$'s determine $F$. To do this, let $W$ be the random variable with distribution $G$ defined by

$$G(w) = \Pr(W \leq w) = \int_0^w e^{-x} dF(x)/\int_0^\infty e^{-x} dF(x).$$

Then the exponential moments of $X$ are the regular moments of $W$:

$$E(W^n) = \int_0^\infty w^n dG(w) = \int_0^\infty z^n e^{-z} dF(z)/\int_0^\infty e^{-z} dF(z).$$

The distribution of $X$ determines that of $W$, and vice versa; indeed, $F(x) = \int_0^x e^y dG(y) = \int_0^\infty e^y dG(y)$.

It turns out that the moments of $W$ determine its distribution, but that result requires analytic function theory, violating our proposed goal that proofs be elementary. We overcome this problem by converting $W$ to a random variable $Z$ on the unit interval. For such, an elementary proof that the moments determine the distribution is available—see Feller [1], pages 225-227 for a beautiful elementary proof that for points of continuity $z$,

$$\Pr(Z \leq z) = \lim_{n \to \infty} \sum_{j \leq nz} \binom{n}{j} (-1)^{n-j} E[(Z^j(1-Z))^{n-j}].$$

So, let $Z = e^{-W}$. Then the distribution of $Z$ is determined by its moments. To see that the moments of $W$, (the exponential moments of $X$), determine the moments of $Z$, write

$$E(Z^k) = E(e^{-kW}) = \int_0^\infty e^{-kz} e^{-x} dF(x)/\int_0^\infty e^{-x} dF(x)$$

$$= \int_0^\infty e^{-x}(1-kz + \frac{(kz)^2}{2!} - \frac{(kz)^3}{3!} + \ldots) dF(x)/\int_0^\infty e^{-x} dF(x).$$

We may exchange the integral and summation operations to get

$$E(Z^k) = \sum_{j=0}^\infty \frac{(-k)^j}{j!} \int_0^\infty (kz)^j e^{-x} dF(x)/\int_0^\infty e^{-x} dF(x) = \sum_{j=0}^\infty \frac{(-k)^j}{j!} R_j.$$
Thus the exponential moments of $X$ determine the moments of $Z$, which determine the distribution of $Z$, which determines the distribution of $W = -\ln(Z)$, which determines the distribution of $X$, and that sequence of implications provides proof of the proposition:

\[ \text{exp. moments of } X \Rightarrow \text{moments of } Z \Rightarrow \text{dist. of } Z \Rightarrow \text{dist. of } W \Rightarrow \text{dist. of } X. \]

**Proof of Proposition 4.**

We have $X$ and $Y$ independent, positive and $X/Y$ independent of $X - Y$. Evidently this cannot hold if only one of $X$ or $Y$ is constant, so assume that neither is constant. We will develop a contradiction.

Let $p = \Pr(Y > X > 0) = \Pr(Y/X > 1)$. Then

\[ p = \Pr(Y > 0, Y/X > 1) = \Pr(Y > 0) \Pr(Y/X > 1) = p^2. \]

Thus $p$ is idempotent, $p^2 = p$, and $p$ must be 0 or 1. Interchanging the roles of $X$ and $Y$ if necessary, we may assume that $p = \Pr(Y > X) = 1$. Then $X$ must be bounded, and $Y$, not constant, will have two points of increase $x_1 < x_2$ such that $\Pr(X > x_2) = 0$.

Since $Y$ is not constant, it (or its distribution) has two points of increase $y_1 < y_2$, and we may define these four sets of points $(x, y)$:

\[ A = \{ (x, y) : y/x < y_2/x_2 \text{ and } y - x > y_2 - x_2 \} \]
\[ B = \{ (x, y) : y/x > y_2/x_2 \text{ and } y - x < y_2 - x_2 \} \]
\[ C = \{ (x, y) : y/x < y_2/x_2 \text{ and } y - x < y_2 - x_2 \} \]
\[ D = \{ (x, y) : y/x > y_2/x_2 \text{ and } y - x > y_2 - x_2 \} \]

Because every point $(x, y)$ in $B$ satisfies $z > x_2$, we must have $\Pr(B) = 0$. From the independence of $Y/X$ and $Y - X$ we also have

\[ \Pr(A) \Pr(B) = \Pr(C) \Pr(D). \]

Thus $\Pr(C) \Pr(D) = 0$, and we get a contradiction from the fact that sets $C$ and $D$ each contain a point of increase of $(X, Y)$: $C$ contains $(x_1, y_2)$ and $D$ contains $(x_2, y_1)$.

The contradiction arose from assuming neither $X$ nor $Y$ was constant. Since $Y - c$ independent of $Y/c$ requires that $Y$ be constant, and $c - X$ independent of $c/X$ requires that $X$ be constant, Proposition 3 is proved: $Y - X$ independent of $Y/X$ for positive independent $X$ and $Y$ requires that both be constant.

Proof of the four Propositions, and hence an elementary proof of the unrestricted $X + Y, X/Y$ characterization of the gamma distribution, is now complete.

**References**

