FINITE HORIZON SINGULAR CONTROL AND A RELATED TWO-PERSON-GAME

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FSU Technical Report No. M779
AFOSR Technical Report No. 87-221

March, 1988

1Research sponsored by the Universidade de Sao Paulo, CNPq (Brasil) and the United States Air Force Office of Scientific Research, Contract No. F49260-85-C-0007, Grant AFOSR-87-0278, Grant AFOSR-88-0040.

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FINITE HORIZON SINGULAR CONTROL AND A RELATED TWO-PERSON-GAME

P. R. Santana and M. Taksar

Abstract

We consider the finite horizon problem of tracking a Brownian Motion, with possibly non zero drift, by a process of bounded variation, in such a way as to minimize total expected cost of "action" and "deviation from a target state." The cost of "action" is given by two functions (of time), which represent price per unit of increase and decrease in the state process; the cost of "deviation" is incurred continuously at a rate given by a function convex in the state variable and a terminal cost function. We obtain the optimal cost function for this problem, as well an $\varepsilon$-optimal strategy, through the solution of a system of variational inequalities, which has a stochastic representation as the value function for an appropriate two-person game.
1. INTRODUCTION

In this study we wish to control a Brownian motion by using additive strategies, i.e. the evolution of the state process is subjected to

\[ X_s = x + W_{s-t} + R_{s-t} - L_{s-t}, \quad 0 \leq t \leq s \leq T < \infty, \]

where \( x \) is the initial state at initial time \( t \),

\( T \), the finite horizon, is a positive constant,

\( \{W_z, 0 \leq z \leq T-t\} \) is a \((\mu, \sigma^2)\) Brownian motion, \( W_0 = 0 \),

\( R_{s-t} \) and \( L_{s-t} \) represent the control action; they are respectively the total amount of increase and decrease in the state process between \( t \) and \( s \). We require \( R \) and \( L \) to be adapted, non-decreasing left continuous processes with \( R_0 = L_0 = 0 \). We call the pair \((R, L)\) a policy.

This model and some variations of it are relevant, for instance, to study the automotive cruise control of an aircraft under an uncertain wind condition, as in Chow et al (1985). We can think of \( X \) representing the speed, \( W \) representing random cumulative disturbance due to shifting wind conditions and \((dR, dL)\) standing for corrective thrust force.
A structure of competing costs is involved in our problem: the cost of control, and running and terminal costs. The latter express our desire to keep the state process close to our target state, the origin, both during the mission and at its conclusion. So we are given a control cost of \( r(s)[\ell(s)] \) per unit of increase [decrease] at time \( s \), a running cost incurred continuously at rate \( h(s, X_s) \), with \( h(s, \cdot) \) convex, and a terminal cost \( g(X_T) \), with \( g(\cdot) \) convex. Hence, given a policy \( S = (R, L) \), the expected cost takes the form:

\[
J(t, x; S) = E \left[ \int_t^T h(s, X_s) ds + \int_{[t,T]} r(s) dR_s - t + \int_{[t,T]} \ell(s) dL_{s-t} + g(X_T) \right].
\]

The optimal cost function or the value function \( V \) for this problem is defined as the infimum of \( J \) over all possible policies: \( V(t, x) = \inf_S J(t, x; S) \). A policy \( S^\varepsilon \) is said to be \( \varepsilon \)-optimal (for a point \( (t, x) \) and for a given \( \varepsilon \geq 0 \)) if

\[
J(t, x; S^\varepsilon) \leq V(t, x) + \varepsilon.
\]

A \( 0 \)-optimal policy is called optimal.

As main results of this study, we determine the function \( V \) under suitable restrictions on the cost functions, and present an \( \varepsilon \)-optimal policy (actually, a family of policies \( S^\varepsilon(t, x) \) with \( J(t, x; S^\varepsilon) \leq V(t, x) + \varepsilon \) uniformly in compact subsets of the \( (t, x) \)-plane).

Karatzas (1983) investigated this problem in the symmetric case, i.e. when \( W \) is a \((0,1)\) Brownian motion, \( h(s, x) = h(x) \) is an even convex function, \( r(s) = \ell(s) = 1 \), and \( g(x) = 0 \).
(Baldrsson (1985) studied a more general problem, but still in the symmetric case.) It is shown in his work that the optimal policy \((R^*, L^*)\) takes the form: "do not act as long as the state process is in the interior of a certain region; when on the boundary of this region, exert control only in order not to exit; if you happen to start outside the region, use as much control as necessary so that you get immediately to the nearest point of the boundary, and then continue as before." In other words, he determines the existence of a positive function \(b(s), s \in [0,T]\) such that the controller does nothing at time \(s\) if \(-b(s) < X_s < b(s)\), and the control process \(R^*(L^*)\), except for a possible initial jump, acts as the local time at the boundary \(-b(s)(b(s))\). It will be seen that our \(\epsilon\)-optimal policy clearly resembles this one.

It was first conjectured by Bather and Chernoff\(^\circ\) (1966), and then proved rigorously by several authors (see, for instance, Karatzas and Shreve (1985) for a proof by probabilistic arguments), that this control problem, in its symmetric form, is equivalent to a better understood problem of optimal stopping with absorption at the origin. The optimal stopping rule \(\sigma\) for this problem is described by two regions in the \((t,x)\)-plane: a closed stopping set, and its complement, the continuation set. \(\sigma\) is then the first time the process enters the stopping set.
The equivalence anticipated by Bather and Chernoff is that the regions of inaction/action (in the control problem) and continuation/stopping (in the stopping problem) should be the same. Moreover, denoting by $U$ the optimal risk for the stopping problem and $V_x$ the space gradient for the control problem, the identity $V_x(t,x) = U(t,x)$, $x \geq 0$, should hold. These authors used this relationship by posing and "solving" more or less favorable stopping problems, then obtaining bounds on the continuation region in the stopping problem, which immediately translate into bounds on the inaction region in the control problem. For other uses of this equivalence in the literature, the reader is referred to Karatzas and Shreve (1984).

We recall that both papers by Karatzas (1983) and Bather and Chernoff (1966) treat the symmetric case (Brownian motion with zero drift, $h$ and $g$ even, $r = L$), an essential feature that permits the reduction of a "two-sided" control problem to a "one-sided" stopping problem. In this case, one first finds $V$ and the bound for the continuation region in the half space $[0,T] \times \mathbb{R}^+$ and then extends then to the whole strip $[0,T] \times \mathbb{R}$ by symmetry. This technique cannot be used when we allow a non-zero drift and the cost functions to be more general, as we do in this study.

In the non-symmetric setting, Taksar (1985) studied the similar problem of finding $V(x)$, the minimum expected average cost per time unit given the initial state $x$ (the stationary problem):
\[ V(x) = \inf_{(R,L)} \left\{ \lim_{T \to \infty} \frac{1}{T} \left[ \int_0^T h(x+W_t+R_t-L_t) dt + rR_T + \ell L_T \right] \right\}, \]

where \( W \) is a \((\mu, \sigma^2)\) Brownian motion, \( h(\cdot) \) is a general convex function and the cost functions \( r(s) \) and \( \ell(s) \) are the nonnegative constants \( r \) and \( \ell \).

For this control problem, Taksar developed a two-person game version of the optimal stopping problem related to a \((\mu, \sigma^2)\) Brownian motion, determined its value function \( U \) and stopping regions \( A \) and \( B \) for each player, and showed an equivalence between the two problems in the same sense as before: the action region for the control problem is \( A \cup B \), and \( \frac{d}{dx} V(x) = U(x), \ x \in \mathbb{R} \). His work involves the solution for a one-dimensional free boundary problem.

In a similar way, we show that the solution for a two-dimensional free boundary problem, which is known to exist and to be unique, can be represented stochastically as the value function \( U(t,x) \) for a certain two-person game problem. Knowing its relationship to the control problem, this entails a stochastic representation for \( V(t,x) \).
2. SUMMARY

In section 3 we formulate the control problem in its probabilistic context. To clarify exposition, some heavy restrictions are imposed on the cost functions. We proceed then to develop, sometimes heuristically, the properties that the value function $V$ must satisfy.

These properties are put in terms of a parabolic Free boundary problem in section 4, that is, a minimum set of inequalities on the partial derivatives of $V$ which is necessary and sufficient for optimality. A second Free boundary problem is proposed, and it is shown that its solution induces a solution to the former.

We begin section 5 posing a two-person stochastic game problem, and describing some aspects of game theory that are needed in this work. We end this section by identifying the value function $U$ of the game with the solution of the second Free boundary problem of section 4, which enables us to give a stochastic representation for $V$.

In section 6 we show that the control problem admits an $\epsilon$-optimal strategy if we know the solution to the Free boundary problem.
Finally, in section 7, we drop some restrictions on the cost functions and mention some possible generalizations to our problems.
3. STATEMENT OF THE PROBLEM AND OPTIMALITY EQUATION

We start with a \((\mu, \sigma^2)\) Brownian motion (BM) on a real line \(\mathbb{R}\), i.e., we are given \((\Omega, F, F_s, W, P_X)\), where \((\Omega, F)\) is a measurable space, \(W = W_\omega(\omega)\) is a measurable mapping of \(\Omega\) into \(C([0,T])\), the space of continuous functions on \([0,T]\), \(T\) a fixed finite positive constant called the finite horizon, \(F_s = \sigma(\{W_u, 0 \leq u \leq s\})\) and \(P_X, x \in \mathbb{R}\), is a family of measures on \(\Omega\) such that, under \(P_X\), the process \(\{W_u, 0 \leq u \leq T\}\) is a BM with variance \(\sigma^2\), drift \(\mu\) and initial state \(x\).

A policy is defined as a pair \(S = (R_s, L_s), 0 \leq s \leq T\) of non-decreasing, left-continuous, \(F_s\)-adapted processes with \(R_0 = L_0 = 0\). The process \(R_s(L_s)\) is interpreted as the cumulative input (output) in the system up to time \(s\). Given a policy \(S = (R, L)\), and an initial time \(t\), the controlled process under \(P_0\) is

\[
(3.1) \quad x_s = x + W_{s-t} + R_{s-t} - L_{s-t}, \quad 0 \leq t \leq s \leq T.
\]

We consider the cost functions

\[
(3.2.1) \quad h: [0,T] \times \mathbb{R} \to [0,\infty), \text{ continuous, and for fixed } t, \text{ } \\
\text{h}(t, \cdot) \text{ is strictly convex and twice continuously differentiable with } h_{xx}(t, \cdot) > c > 0; \text{ h}(t, x) \text{ represents }
\]

8
the rate at which costs are continuously accumulated at time \( t \) and state \( x \). For simplicity, we impose \( h \) with a minimum 0 at the origin:

\[
h(t,0) = 0, \quad 0 \leq t \leq T.
\]

(3.2.ii) \( r: [0,T] \rightarrow [c,C] \), where \( c, C \) are constants with \( 0 < c < C < \infty \), Lipschitz continuous, differentiable; \( r(t) \) represents the price of one unit of increase in the system at time \( t \). For simplicity, we assume \( r''(t) = \frac{d}{dt} r(t) \leq 0, \quad 0 < t < T \).

(3.2.iii) \( \ell: [0,T] \rightarrow [c,C] \), \( c \) and \( C \) as in (3.2.ii), Lipschitz continuous, differentiable; \( \ell(t) \) represents the price of one unit of decrease in the system at time \( t \). For simplicity, we assume \( \ell''(t) = \frac{d}{dt} \ell(t) \leq 0, \quad 0 < t < T \).

(3.2.iv) \( g: \mathbb{R} \rightarrow [0,\infty) \) is convex and continuously differentiable; \( g(x) \) represents a terminal cost at state \( x \). For simplicity, we impose \( g \) with a minimum at the origin.

For this section, we impose at most polynomial growth in the space argument, that is, for some \( m \geq 1 \), and some \( K > 0 \),

\[
(3.3) \quad \left| \frac{\partial}{\partial x} h(t,x) \right| + \left| \frac{d}{dx} g(x) \right| \leq K(1 + |x|^m)
\]

on \([0,T] \times \mathbb{R}\). Further restrictions will be later imposed in the cost functions for the solution of the game problem.
For each policy $S = (R, L)$ and initial state $x$ at time $t$ we associate the expected cost

\begin{equation}
J(t, x; S) = E_x \left[ \int_t^T h(s, X_s) ds + \int_{[t,T)} r(s) dR_s - t \right. \\
\left. + \int_{[t,T)} L(s) dL_s - t + g(X_T) \right],
\end{equation}

where $E_x$ is the expectation operator under $P_x$.

Our purpose is to characterize the optimal cost $V$:

\begin{equation}
V(t, x) = \inf_{S} J(t, x; S).
\end{equation}

If there is a policy $S^* = S^*(t, x)$ such that

\begin{equation}
J(t, x; S^*) = V(t, x),
\end{equation}

$S^*$ is called an optimal policy for $(t, x)$. If, for $\varepsilon > 0$, there exists a policy $S^\varepsilon = S^\varepsilon(t, x)$ such that

\begin{equation}
J(t, x; S^\varepsilon) \leq V(t, x) + \varepsilon,
\end{equation}

$S^\varepsilon$ is called an $\varepsilon$-optimal policy for $(t, x)$.

The first property we can deduce for $V$ is that it inherits convexity from the cost functions $h$ and $g$. The proof of the next lemma follows the same steps of Baldursson (1985).
Lemma (3.8): Under assumptions (3.2), $V(t, \cdot)$ is convex.

Proof: Consider $X_s = x + W_{s-t} + R_{s-t} - L_{s-t}$ and $X'_s = y + W'_{s-t} + R'_{s-t} - L'_{s-t}$, two controlled processes starting at x and y subjected to arbitrary policies $S = (R, L)$ and $S' = (R', L')$ respectively. We have, for $\theta \in [0,1],$

$$h(s, \theta X_s + (1-\theta)X'_s) \leq \theta h(s, X_s) + (1-\theta)h(s, X'_s),$$

$$g(\theta X_t + (1-\theta)X'_t) \leq \theta g(X_t) + (1-\theta)g(X'_t),$$

and noticing that the set of policies is a convex set,

$$V(t, \theta x + (1-\theta)y) \leq J(t, \theta x + (1-\theta)y; \theta R + (1-\theta)R'; \theta L + (1-\theta)L')$$

$$\leq \theta J(t, x; S) + (1-\theta)J(t, y; S')$$

after applying the former inequalities and the definition of $J$. The result follows taking infima in $S$ and $S'.$

We now look for the optimality equation for $V(t, x)$. From (3.4) it is apparent that, for $t = T$, $V$ must satisfy the boundary condition

$$V(T, x) = g(x), x \in \mathbb{R}. \quad (3.9)$$

Proceeding heuristically for a while, let us assume that $V$ is twice continuously differentiable in the state variable $x$ and continuously differentiable in the time variable $t$. Using the notation $V_x$ for $\frac{\partial V}{\partial x}$, $V_{xx}$ for $\frac{\partial^2 V}{\partial x^2}$, etc., one can derive the necessary conditions.
(3.10.i) \[ V_x(t,x) + r(t) \geq 0 \]

(3.10.ii) \[ -V_x(t,x) + \xi(t) \geq 0 \]

(3.10.iii) \[ \frac{1}{2} \sigma^2 V_{xx}(t,x) + \mu V_x(t,x) + V_t(t,x) + h(t,x) \geq 0 \]

To check (3.10.i), observe that, for \( y > x \),

\[ V(t,x) \leq r(t)(y-x) + V(t,y), \]

otherwise an immediate jump from \( x \) to \( y \) would reduce expected cost. (3.10.ii) is obtained similarly. To verify (3.10.iii) notice that the policy "do nothing for a short interval \([t, t+\delta]\) and then continue optimally" yields the inequality

\[ V(t,x) \leq E_x \left[ \int_t^{t+\delta} h(s, X_s) ds + V(t+\delta, W_\delta) \right] \]

\[ = h(t,x) \delta + o(\delta) + E_0 V(t+\delta, x+W_\delta). \]

Expanding \( V(t+\delta, x+W_\delta) = V(t+\delta, x) + V_x(t+\delta, x) W_\delta + \frac{1}{2} V_{xx}(t+\delta, x) W_\delta^2 + o(W_\delta^2) \) and taking expectations,

\[ V(t,x) \leq h(t,x) \delta + o(\delta) + V(t+\delta, x) + V_x(t+\delta, x) \mu \delta \]

\[ + \frac{1}{2} V_{xx}(t+\delta, x) [\sigma^2 \delta + \mu^2 \delta^2] + E_0 [o(W_\delta^2)], \]

or

\[ \frac{V(t,x) - V(t+\delta, x)}{\delta} \leq h(t,x) + \frac{o(\delta)}{\delta} + \]

\[ + V_x(t+\delta, x) \mu + \frac{1}{2} V_{xx}(t+\delta, x) [\sigma^2 + \mu^2 \delta] \]

\[ + \frac{1}{\delta} E_0 [o(W_\delta^2)]. \]
Letting $\delta \to 0$, we get (3.10.iii).

It will be shown in the next section that, in order to get a set of sufficient conditions for optimality, we have to adjoin the following identity to the inequalities (3.10):

$$\min(V_x(t,x) + r(t); -V_x(t,x) + \ell(t));$$

(3.11)

$$\frac{1}{2} \sigma^2 V_{xx}(t,x) + \mu V_{xx}(t,x) + V_t(t,x) + h(t,x)) = 0$$

on $[0,T] \times \mathbb{R}$, that is, for each $(t,x)$, one of the three inequalities (3.10) attains the lower bound 0. This expresses (see, for instance, Karatzas (1983)) the heuristic principle of optimality, already alluded to in the introduction: "either exert full force (i.e., jump immediately to the appropriate place), or else do nothing." Harrison and Taksar (1983) justify this principle for instantaneous control problems (i.e., when the controller can instantaneously change the state process, as in our case) on the grounds that it is the appropriate action that one could reasonably expect, as $\theta \to \infty$, for the control problem where the controller can only increase or decrease the state at a rate which is not to exceed $\theta$.

With the definition of the linear operator

(3.12) \[ L = \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \mu \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \]
our optimality equation for $V$ becomes

$$\min(LV+h, r+V_x, e^{-V} \ell - V_x) = 0 \text{ on } [0,T] \times \mathbb{R};$$

(3.13)

$$V(T,x) = g(x), \ x \in \mathbb{R}.$$
4. FREE BOUNDARY PROBLEMS

We present the properties that \( V \) should satisfy, found in (3.8) and (3.10), plus the optimality equation (3.13), in terms of a free-boundary problem (or Stefan problem). We write \( C^{K, \ell}(A \times B) \) for the space of functions which are \( K \) times \( \ell \) times continuously differentiable in the first variable (second variable) on \( A \times B \).

(4.1) Free boundary problem I:

Construct a function \( f(t, x) \) in \( C^{1,2}([0,T] \times \mathbb{R}) \cap C([0,T] \times \mathbb{R}) \), as well as \( C([0,T]) \) functions \( a(t), b(t), \) with \( a(t) < b(t) \) \( \forall t \), such that

\[
\text{(4.2) } Lf(t,x) + h(t,x) = 0 \text{ on } \partial \text{ab } \overset{\text{def}}{=} \{(t,x): 0 \leq t < T, a(t) \leq x \leq b(t)\}
\]

where \( L \) is as in (3.12);

\[
\text{(4.3) } Lf(t,x) + h(t,x) \geq 0 \text{ on } [0,T) \times \mathbb{R} - \partial \text{ab;}
\]

\[
\text{(4.4) } f_X(t,x) = \ell(t) \text{ on } E \overset{\text{def}}{=} \{(t,x): 0 \leq t < T, x \geq b(t)\} \text{ and } f_X(t,x) \leq \ell(t) \text{ on } [0,T) \times \mathbb{R} - E;
\]

\[
\text{(4.5) } f_X(t,x) = -r(t) \text{ on } F \overset{\text{def}}{=} \{(t,x): 0 \leq t < T, x \leq a(t)\} \text{ and } f_X(t,x) \geq -r(t) \text{ on } [0,T) \times \mathbb{R} - F;
\]
(4.6) \( f_{xx}(t,x) \geq 0 \) on \([0,T] \times \mathbb{R}\);

(4.7) \( f(T,x) = g(x) \), \( x \in \mathbb{R} \).

The functions \( h, \ell, r \) and \( g \) for this problem are as defined in section 3, and relations (4.2) - (4.7) are supposed to hold almost everywhere.

To show the connection between this problem and our optimization question, we have next a verification result.

Proposition (4.8): Suppose \((V,a,b)\) solves (4.1). Then, for any policy \( S \), and for every \((t,x)\) on \([0,T] \times \mathbb{R}\), the inequality \( V(t,x) \leq J(t,x;S) \) holds.

Proof: Adopting the representation \( X_S = x + aB_{s-t} + \mu(s-t) + R_{s-t} + L_{s-t}, \ t \leq s \leq T \) for the state process, where \( B \) is a \((0,1)\) BM starting at 0, and applying Doleáns Dade-Meyer (1970) formula to \( V(s,X_S) \), we have, for any policy \( S = (R,L) \):

\[
V(T,X_T) = V(t,x) + \sigma \int^T_t V_x(s,X_S) dB_{s-t} \\
+ \int^T_t \left[ \frac{1}{2} \sigma^2 V_{xx}(s,X_S) + \mu V_x(s,X_S) + V_t(s,X_S) \right] ds \\
+ \int_{[t,T]} V_x(s,X_S) dR_{s-t} - \int_{[t,T]} V_x(s,X_S) dL_{s-t} \\
+ \sum_{t < s < T} [V(s,X_{S+}) - V(s,X_S) - V_x(s,X_S) \Delta X_S]
\]

where \( \Delta X_S = X_{S+} - X_S \).
Since, by (4.7), \( V(T, X_T) = g(X_T) \) and recalling the definition of \( L \), we get, after rearranging terms and applying expectation to both sides (note that the expectation of the stochastic integral is 0):

\[
V(t, x) = E_x \left[ -\int_t^T L(s, X_s) ds - \int_{[t,T]} V_x(s, X_s) dR_{s-t} \right.
\]

\[
+ \int_{[t,T]} V_x(s, X_s) dL_{s-t} - \sum_{t \leq s < T} \left[ V(s, X_{s-}) - (V(s, X_s) + V_x(s, X_s) \Delta X_s) + g(X_T) \right].
\]

By (4.2) and (4.3), \(-\int_t^T L(s, X_s) ds \leq \int_t^T h(s, X_s) ds,\)
by (4.4), \(\int_{[t,T]} V_x(s, X_s) dL_{s-t} \leq \int_{[t,T]} \ell(s) dL_{s-t},\)
by (4.5), \(-\int_{[t,T]} V_x(s, X_s) dR_{s-t} \leq \int_{[t,T]} r(s) dR_{s-t},\)
and by (4.6), the sum term (fourth term in the brackets) is nonnegative; thus

\[
V(t, x) \leq E_x \left[ \int_t^T h(s, X_s) ds + \int_{[t,T]} r(s) dR_{s-t} + \int_{[t,T]} \ell(s) dL_{s-t} + g(X_T) \right]
\]

\[= J(t, x; S), \text{ proving the proposition.} \]

Assuming that the function \( f \) in (4.1) is once more differentiable in the state variable, we derive a second free-boundary problem by formally differentiating the equations of (4.1). Here we put \( f_x = q \) and the free boundaries \( a(\cdot) \) and \( b(\cdot) \) are the same
as in (4.1). It is the solution of this second system that we look for in the next section.

\[(4.9)\] Free boundary problem II:

Construct a function \( q(t,x) \) in \( C^1([0,T] \times \mathbb{R}) \cap C([0,T] \times \mathbb{R}) \), as well as \( C([0,T]) \) functions \( a(t), b(t) \), with \( a(t) < b(t) \) \( \forall t \), such that \( q_{xx}(t,x) \) is continuous on \( [0,T] \times \mathbb{R} \) \( \setminus \{(t,x): 0 \leq t < T, x = a(t), x = b(t)\} \), and

\[(4.10)\] \( Lq(t,x) + h_x(t,x) = 0 \) on \( \partial\Omega \) \( \text{def} \{ (t,x): 0 \leq t < T, a(t) < x < b(t) \} \),

\[(4.11)\] \( Lq(t,x) + h_x(t,x) \geq 0 \) on \( E \) (see (4.4)),

\[(4.12)\] \( Lq(t,x) + h_x(t,x) \leq 0 \) on \( F \) (see (4.5)),

\[(4.13)\] \( q(t,x) = \ell(t) \) on \( E \)

\[(4.14)\] \( q(t,x) = -r(t) \) on \( F \)

\[(4.15)\] \( -r(t) < q(t,x) < \ell(t) \) on \( \partial\Omega \),

\[(4.16)\] \( q(T,x) = g^-(x), \ x \in \mathbb{R} \).

In what follows we will need the next elementary lemma, where \( U = U(t,x) \).

**Lemma (4.17):** If \((U,a,b)\) is a solution to (4.9), then \( a(t) \leq 0 \leq b(t), t \in [0,T) \).

(Of course, for each \( t \), one of the inequalities is strict.)
Proof: Suppose not; for instance, suppose \( b(t_0) < 0 \) for some \( t_0 \).

By continuity of \( b \), there exists a subset \( A \) of \( E = \{(t,x): 0 \leq t < T, \ x \geq b(t)\} \) with positive measure such that \( x < 0 \) on \( A \). By (4.13)

\[
(4.18) \quad L U(t,x) + h_x(t,x) = \mathcal{L}^-(t) + h_x(t,x) \quad \text{on } A.
\]

Recalling assumptions (3.2.i) and (3.2.iii) on \( \mathcal{L} \) and \( h \), respectively, the right hand side of (4.18) is strictly negative, contradicting (4.11). Thus \( b(t) \geq 0 \), \( t \in [0,T) \). The relation \( a(t) \leq 0 \), \( t \in [0,T) \) is proved in the same way.

The next lemma shows how to construct a solution to Free boundary problem I when we have a solution to Free boundary problem II:

Lemma (4.19): Let \((U,a,b)\) be a solution to (4.9). Then \((V,a,b)\) is a solution to (4.1), where \( V \) is defined by

\[
(4.20) \quad V(t,x) = \int_0^T \left[ \frac{\sigma^2}{2} U_x(s,0) + \mu U(x,0) \right] ds + \int_0^x U(t,y) dy + g(0).
\]

Proof: We begin by showing that \( V \) so constructed satisfies (4.3). Let \((t,x) \in F \). In view of Lemma (4.17), we know that \( x \leq a(t) \leq 0 \). We consider the case where \( x < a(t) \). We have

\[
(4.21) \quad V_x(t,x) = U(t,x)
\]

\[
(4.22) \quad V_{xx}(t,x) = U_x(t,x)
\]

\[
(4.23) \quad V_t(t,x) = -\frac{\sigma^2}{2} U_x(t,0) - \mu U(t,0) + \int_0^x U_t(t,y) dy.
\]
By (4.12), (4.10), for \( y \in [x,0] \), the relation

\[ (4.24) \quad -U_t(t,y) \geq \frac{\sigma^2}{2} U_{xx}(t,y) + \nu U_x(t,y) + h_x(t,y) \]

holds. Substituting (4.24) in the integral term of (4.23) and performing the integration, we get (recall \( h(t,0) = 0 \)):

\[ (4.25) \quad V(t,x) \geq -\frac{\sigma^2}{2} U_x(t,x) - \nu U(t,x) - h(t,x). \]

Now add (4.21), (4.22), (4.25) to get \( LV \geq -h \), that is, (4.3) holds on \( F \). For \( (t,x) \in E \), the proof is similar, as well as for \( (t,x) \in \tilde{D}ab \), which proves (4.2).

(4.4) and (4.5) are just another way of writing (4.13), (4.14) and (4.15) in view of (4.21). Also (4.7) is immediate by (4.20), the definition of \( V \), and (4.16).

It remains to show (4.6). By (4.4) and (4.5), it holds trivially in \( E \cup F \). Now let \( (t,x) \) be a point in \( \tilde{D}ab \), and put \( t_A = \inf \{ s : s \geq t ; (s,W_s) \in A \} \), where \( A \) is a closed set and \( W \) is a \((\nu,\sigma^2)BM\). In Dynkin (1963), we see that \( U_{xxx}, U_{xx}, U_{xt} \) are continuous in \( \tilde{D}ab \), and using a mollifier technique in the same way that will be used in Theorem 5.19, we can justify the use of Ito's formula in \( \tilde{D}ab \) for the function \( U_x \). We write, with \( \tau = t_E \wedge t_F \wedge T \):

\[
U_x(\tau, W_\tau) - U_x(t,x) = \int_t^\tau \left[ U_{xt} + \nu U_{xx} + \frac{\sigma^2}{2} U_{xxx} \right](s,W_s)ds \\
+ \sigma \int_t^\tau U_{xx}(s,W_s)dB_s,
\]
where $B$ is a $(0,1)$BM. Taking expectations, rearranging terms, we get, in view of (4.10), (4.16):

$$U_x(t,x) = E_x \left[ \int_0^\tau h_x(s, W_s) ds + U_x(\tau, W_\tau) \right].$$

Since $h_{xx}(t,x) > 0$, $\tau > t$ w.p.1, and $U_x(\tau, W_\tau) = 0$ when $\tau = t_E$, $t_F$, and $U_x(\tau, W_\tau) = g'(X_\tau) \geq 0$ when $\tau = T$, we get $U_x(t,x) = V_{xx}(t,x) > 0$ in $D_1$, and the proof is complete.
5. THE GAME PROBLEM

In this section we pose a particular two-person stochastic game which eventually gives us the solution to the system (4.9), the Free-boundary problem II. In what follows, the probabilistic set up is the same as in section 3, except for the fact that there are no controls involved. The cost functions \( h, g, \ell, r \) are also the same, but some additional restrictions will be imposed as long as needed. The main course of this section is suggested by Friedman (1973) and Taksar (1985).

We proceed to describe the game. Two players \( P_1 \) and \( P_2 \) observe a \((\mu, \sigma^2)\)BM \( \{W_s, t \leq s \leq T\} \) starting at \( x \) at time \( t \). \( P_1 \) has the right to stop the process if \( W > 0 \) and \( P_2 \) has the right to stop the process if \( W < 0 \). There is accumulated income associated with the process

\[
(5.1) \quad m(t, z) = \int_t^Z h_x(s, W_s) ds.
\]

If the process is stopped by \( P_1 \) at time \( z, t \leq z < T \), \( P_1 \) pays to \( P_2 \) the quantity \( m(t, z) + \ell(z) \); if \( P_2 \) stops the process, he receives \( m(t, z) - r(z) \) from \( P_1 \). If neither player stops the process before \( T \), \( P_2 \) receives \( m(t, T) + g^C(W_T) \) from \( P_1 \). (Recall that \( h_x < 0 \) on \([0, T] \times (-\infty, 0)\), \( h_x > 0 \) on \([0, T] \times (0, \infty)\), \( h_x(s, 0) = 0 \).
Negative amounts means that P₂ pays to P₁.) In other words, whoever stops the game at time \( z < T \) pays additional fee of \( \ell(z) \) for \( P₁ \) and \( r(z) \) for \( P₂ \). The game is over whenever \( P₁ \) or \( P₂ \) stops the game or time \( T \) is reached.

Let \( A_{t,x} \) be the class of \( W \)-stopping times \( \sigma \) with \( \sigma \geq t \) and \( W_\sigma > 0 \); \( B_{t,x} \) the class of \( W \)-stopping times \( \tau \) with \( \tau \geq t \) and \( W_\tau < 0 \). For a starting point \((t,x)\) and stopping times \( \sigma, \tau \), we associate the payoff function \( G \) given by

\[
(5.2) \quad G(t,x;\sigma,\tau) = E_x [m(t,\sigma \wedge t) + \ell(\sigma)I(\sigma < \tau \wedge T) - r(\tau)I(\tau < \sigma \wedge T) + g^-(W_T)I(\sigma \wedge T)]
\]

where \( I(\cdot) \) stands for indicator function of a set \( \{\cdot\} \) with argument \( \omega \in \Omega \), and \( m \) is defined in (5.1).

It is clear that \( P₁ \) wants to minimize \( G \) by choosing \( \sigma \in A_{t,x} \), and \( P₂ \) wants to maximize \( G \) by choosing \( \tau \in B_{t,x} \).

Hereafter, we refer to this scheme as "the game problem," or "the game."

If

\[
(5.3) \quad \inf_{\sigma \in A_{t,x}} \sup_{\tau \in B_{t,x}} G(t,x;\sigma,\tau) = \sup_{\tau \in B_{t,x}} \inf_{\sigma \in A_{t,x}} G(t,x;\sigma,\tau)
\]

we say that the game has value \( U(t,x) \), where \( U(t,x) \) is the common number in (5.3).

Let \( t_A = \inf\{s: s \geq t, (s, W_s) \in A\} \), for a closed set \( A \). If there exists closed subsets \( E \) and \( F \) of \([0,T] \times \mathbb{R}\) such that \( t_E \in A_{t,x} \)
and \( t_F \in B_{t,x} \) for all \( (t,x) \in [0,T] \times \mathbb{R} \), and

\[
(5.4) \quad G(t,x;\tau;t_E,\sigma,t_F) \leq G(t,x;\tau;\sigma,t_F) \leq G(t,x;\sigma,t_F)
\]

for all \( \sigma \in A_{t,x}, \tau \in B_{t,x}, (t,x) \in [0,T] \times \mathbb{R} \), then we say that \( (t_E,t_F) \) is a saddle point for the game and that \((E,F)\) forms a saddle point of sets, or optimal stopping sets. Of course, if there is such a pair \((E,F)\), then

\[
(5.5) \quad U(t,x) = G(t,x;\tau;\sigma,t_F), \text{ for all } (t,x).
\]

We impose the additional restriction on the terminal function \( g \), given in (3.2):

\[
(5.6) \quad -r(T) \leq g^-(x) \leq \ell(T).
\]

Condition (5.6) avoids the well known phenomenon of existence of a sequence \( \{\sigma_n\} \) of stopping rules, say, for player \( P_1 \), such that \( \sigma_n \) approaches optimality as \( n \) increases, but the limit \( \sigma \) is sub-optimal. In this case, an optimal stopping rule may fail to exist. For a nice explanation of (5.6) in our context, see Karatzas and Shreve (1984).

We begin to characterize the relations between \( U \) and saddle point of sets.

**Theorem (5.7)** Let the conditions (3.2), (3.3) and (5.6) hold and suppose \((E,F)\) is a saddle point of sets for the game problem. Then the value \( U \) has the following properties:
\[(5.8)\] \(U(t,x) \leq \ell(t)\) if \((t,x) \in [0,T] \times (0,\infty),\)

\[(5.9)\] \(U(t,x) \geq -r(t)\) if \((t,x) \in [0,T] \times (-\infty,0),\)

\[(5.10)\] \(U(t,x) = \ell(t)\) if \((t,x) \in E,\)

\[(5.11)\] \(U(t,x) = -r(t)\) if \((t,x) \in F,\)

\[(5.12)\] \(U(T,x) = g^-(x),\)

\[(5.13)\] \(U(t,x) \leq E_x^t [m(t,\lambda) + U(\lambda,W(\lambda))]\) for any \(W\)-stopping time \(\lambda\) such that \(t \leq \lambda \leq t_F \wedge T,\)

\[(5.14)\] \(U(t,x) \geq E_x^t [m(t,\xi) + U(\xi,W(\xi))]\) for any \(W\)-stopping time \(\xi\) such that \(t \leq \xi \leq t_E \wedge T.\)

(Here \(m\) is as in (5.1) and we write \(W(\cdot)\) instead of \(W\) for typographical reasons.)

**Proof:** From the definition of \(U(t,x)\) we have

\[G(t,x;t_E,\tau) \leq U(t,x) \leq G(t,x;\sigma,t_F)\]

for all \(\sigma \in \mathcal{A}_{t,x}\), \(\tau \in \mathcal{B}_{t,x}\). If \((t,x) \in [0,T] \times (0,\infty),\) the region where \(P_{1}\) can stop the process, then \(\sigma = t \in \mathcal{A}_{t,x}\). But \(G(t,x;t,t_F) = \ell(t),\) and this applied to the right hand side of the inequality above leads to (5.8). (5.9) is proved similarly. (5.10) and (5.11) come from (5.5) and the definitions of \(t_E, t_F\). The boundary condition (5.12) comes from the definitions of \(U, G.\)
To prove (5.13), observe that

\[
U(t,x) = \inf_{\sigma \in A_{t,x}} G(t,x;\sigma, t_F)
\leq \inf_{\sigma \in A_{t,x}} \mathbb{E}_x [m(t, \sigma \land t_F \land T) + \ell(\sigma) I(\sigma < t_F \land T) - r(t_F) I(t_F < \sigma \land T) + g^{-}(W_T) I(\sigma \land t_F \geq T)]
\]

\[
= \inf_{\sigma \in A_{t,x}} \mathbb{E}_x [m(t, \sigma \land t_F \land T) + \ell(\sigma) I(\sigma < t_F \land T) + r(t(t_F < \sigma \land T) + g^{-}(W_T) I(\sigma \land t_F \geq T)])/F_{\lambda},
\]

where $F_{\lambda} = \sigma(W_s, t \leq s \leq \lambda)$. By the strong Markov property, and using the fact that $t_F \geq \lambda$, the right hand side is equal to

\[
\inf_{\sigma \in A_{t,x}} m(t, \lambda) + \mathbb{E}_W(\lambda) [m(\lambda, \sigma \land t_F \land T) + \ell(\sigma) I(\sigma < t_F \land T) - r(t(t_F < \sigma \land T) + g^{-}(W_T) I(\sigma \land t_F \geq T)].
\]

But $\inf_{\sigma \in A_{t,x}} \mathbb{E}_W(\lambda) [m(\lambda, \sigma \land t_F \land T) + \ell(\sigma) I(\sigma < t_F \land T) - r(t(t_F < \sigma \land T) + g^{-}(W_T) I(\sigma \land t_F \geq T)] = U(\lambda, W(\lambda))$ for any $\lambda = \lambda(\omega)$ with $t \leq \lambda \leq t_F \land T$, so (5.13) follows. The proof of (5.14) is similar.

For our purposes, the following converse of theorem (5.7) is more useful.

Theorem (5.15): Let (3.2), (3.3) and (5.6) hold. Suppose $U(t,x)$ is a Borel measurable function on $[0,T] \times \mathbb{R}$, and $E, F$ are closed
subsets of $[0,T] \times \mathbb{R}$ such that $t_E \in A_{t,x}$, $t_F \in B_{t,x}$, and suppose (5.8) - (5.14) of theorem (5.7) hold. Then $U(t,x)$ is the value of the game problem, and $(E,F)$ is a saddle point of sets.

Proof: We have to show that

(5.16) $G(t,x;\tau) \leq U(t,x)$, $\tau \in B_{t,x}$,

(5.17) $G(t,x;\sigma) \geq U(t,x)$, $\sigma \in A_{t,x}$.

To show (5.16), let $\tau \in B_{t,x}$, and consider the random variable $U(t_E^{\tau}T, W(t_E^{\tau}T))$:

(i) on the set $\{t_E^{<\tau}T\}$, $U(t_E^{\tau}T, W(t_E^{\tau}T)) \geq -r(\tau)$ by (5.9), since $(\tau, W(\tau)) \in [0,T] \times (-\infty, 0)$,

(ii) on the set $\{t_E^{\tau}T\}$, $U(t_E^{\tau}T, W(t_E^{\tau}T)) = \ell(t_E)$ by (5.10), since $(t_E, W(t_E)) \in E$,

(iii) on the set $\{t_E^{\tau}T\}$, $U(t_E^{\tau}T, W(t_E^{\tau}T)) = g^{-1}(W_T)$ by (5.12).

Inserting (i), (ii), (iii) into the formula

$$G(t,x;\tau) = E_x m(t, t_E^{\tau}T) + \ell(t_E) I(t_E^{<\tau}T) + \ell(t_E) I(t_E^{\tau}T) + g^{-1}(W_T) I(t_E^{\tau}T),$$

we get

$$G(t,x;\tau) \leq E_x [m(t, t_E^{\tau}T) + U(t_E^{\tau}T, W(t_E^{\tau}T))],$$

and applying (5.14) with $\xi = t_E \wedge \tau \wedge T$, we conclude that
\(G(t,x;t_E,\tau) \leq U(t,x)\), that is, (5.16) holds. The proof of (5.17) is similar.

We will need this lemma in preparation for the next theorem:

Lemma (5.18): The value of the payoff function \(G(t,x;\sigma,\tau)\) defined in (5.2) does not change if we replace \(A_{t,x} = \{\sigma: \sigma \geq t, W_\sigma > 0\}\) of its definition by the class \(A_{t,x}^* = \{\sigma^*: \sigma^* \geq t, W_\sigma^* \geq 0\}\).

Proof: Fix \(t \in [0,T]\). For each \(\sigma^* : \Omega \to [t,\infty)\) in \(A_{t,x}^*\), and \(\varepsilon > 0\), define \(\sigma_\varepsilon \in A_{t,x}\) by the relation

\[
\sigma_\varepsilon(\omega) = \sigma^*(\omega) \text{ if } W_{\sigma^*} > 0;
\]

\[
\sigma_\varepsilon(\omega) = \tau_\varepsilon(\omega) \text{ if } W_{\sigma^*} = 0,
\]

where \(\tau_\varepsilon(\omega) = \inf \{s: s > \sigma^*(\omega), W_s = \varepsilon\}\). We have

\[
|G(t,x;\sigma_\varepsilon,\tau) - G(t,x;\sigma^*,\tau)| \leq \mathbb{E}_X \left[ |m(t,x;\sigma_\varepsilon,\tau) - m(t,x;\sigma^*,\tau)| + |\mathcal{L}(\sigma_\varepsilon)I(\sigma_\varepsilon < \tau \wedge T) - \mathcal{L}(\sigma^*)I(\sigma^* < \tau \wedge T) + r(\tau)I(\tau < \sigma^* \wedge T) + I(\tau < \sigma^* \wedge T) + g^-(W_{T})|I(\sigma_\varepsilon \wedge T) - I(\sigma^* \wedge T)|I(\sigma^* < \sigma_\varepsilon)\right]
\]

\[
\leq \mathbb{E}_X \left[ \int_{\sigma^*}^{\sigma_\varepsilon} |h_{\lambda}(s,W_s)|ds + |\mathcal{L}(\sigma_\varepsilon) - \mathcal{L}(\sigma^*)|I(\sigma^* < \tau \wedge T) + C|I(\tau < \sigma^* \wedge T) - I(\tau < \sigma_\varepsilon \wedge T) + g^-(W_{T})|I(\sigma_\varepsilon \wedge T) + I(\sigma^* \wedge T)|I(\sigma^* < \sigma_\varepsilon)\right].
\]

Here \(C\) is the upper bound on \(r, \mathcal{L}\).
An immediate consequence of Lemma (5.18) is that the value of the game is not altered if we replace its rule by: \( P_1 \) has the right to stop if \( W \geq 0 \), \( P_2 \) if \( W < 0 \). An obvious extension of Lemma (5.18) permits us to say more: if \( A \) is a system of intervals on \([0,T]\), then the value of the game is not altered if \( P_1 \) has the right to stop if \( W_s \geq 0 \) for \( s \in A \) and if \( W_s > 0 \) for \( s \in A^c \), and \( P_2 \) has the right to stop if \( W_s < 0 \) for \( s \in A \) and if \( W_s \leq 0 \) for \( s \in A^c \).

(One could try to avoid all this "complication" by allowing \( P_1 \) to stop if \( W \geq 0 \) and \( P_2 \) if \( W \leq 0 \). There is nothing wrong with a non-empty intersection \( E \) of stopping regions, but, in order for the game to have a value function, we must require the penalty fee for termination to be identical on \( E \). In our context, we should require \( r(t) = c(t) \), but this would destroy the very motivation of our whole work, that is, to treat the case of non-identical cost functions.)

Theorem (5.19): Let (3.2), (3.3) and (5.6) hold. Let \((U,a,b)\) be a solution to (4.9). Then the game problem has value \( U \) and saddle point of sets \((E,F)\) given by (4.4), (4.5).

Proof: We apply Theorem (5.15). Measurability of \( U \) is immediate, by continuity. In view of Lemma (4.17), Lemma (5.18) and comments following it, we may assume \( t_e \in A_t,x \), \( t_f \in B_t,x \).
As $\varepsilon \to 0$, $\sigma_\varepsilon \to \sigma^*$ with $P_\mathcal{X}$-probability 1. It will be imposed later, but we already make use of it now, that we must allow at most polynomial growth on $|h_\mathcal{X}(t,\cdot)|$. With this restriction, and Gikman-Skorokhod inequality (see Fleming and Rishel (1975) p. 119), we see that the random variable \[ \int_0^T |h_\mathcal{X}(s, W_s(\omega))| \, ds \] possesses moments of all orders, so \[ E_{\sigma_\varepsilon} [ \int |h_\mathcal{X}(s, W_s)| \, ds ] \to 0 \] as $\varepsilon \to 0$. By continuity and boundedness of $\mathcal{L}$, we also have \[ E_{\sigma^*} [ |\mathcal{L}(\sigma_\varepsilon) - \mathcal{L}(\sigma^*)| I(\sigma^* \leq \tau \wedge T) ] \to 0 \] as $\varepsilon \to 0$. The other two factors in the last expression are also easily seen to converge to zero (recall that $g^\prime$ is bounded, c.f. (5.6)), proving the lemma.

We give an explanation about the necessity of the last lemma in our program. In the next theorem we want to identify the solution $(U, a, b)$ of Free boundary problem II with the value function of the game, with the boundaries $a$ and $b$ demarcating the optimal stopping regions for the players: $P_1$ should stop on $E = \{(t, x): 0 \leq t < T, x \geq b(t)\}$, $P_2$ on $F = \{(t, x): 0 \leq t < T, x \leq a(t)\}$. However, in Lemma (4.17), we were only able to establish that the boundaries satisfy $a(t) \leq 0 \leq b(t)$, $0 \leq t < T$, with at least one inequality strict for each $t$. Thus, given a solution $(U, a, b)$ to (4.9) we do not know, a priori, whether the boundaries "touch" the origin line $\{(t, 0), 0 \leq t < T\}$, in which case it would invalidate our game as originally proposed: $P_1$ has the right to stop if $W > 0$, $P_2$ if $W < 0$. 
If $U$ is a solution to (4.9), then (5.8) - (5.12) of Theorem (5.7) hold as a matter of definition. We proceed to show (5.13).

If $(t,x) \in F$ there is nothing to prove. Let $(t,x) \in F_{\varepsilon} \times [0,T) \times \mathbb{R}$ where $F_{\varepsilon}$ is a closed $\varepsilon$-neighborhood of $F$. Let $A_n = \{(t,x): 0 \leq t < T, |x| \geq n\}$, for $n \geq M_0$, where $(t,x) \in A_{M_0}^c$. Let $U_m$ be the mollifier $J_{1/m}$ of $U$, where $m > \frac{1}{\varepsilon}$. (For definition and properties of mollifiers, the reader is referred to Friedman (1964.).)

Since $U_m$ has derivatives of all orders, we can apply Ito's formula to $U_m(\lambda, W(\lambda))$ in the closure of $F_{\varepsilon} \times [0,T) \times \mathbb{R} - A_n$, getting

\begin{equation}
U_m(t,x) = E_x[U_m(\lambda, W(\lambda))] - E_x[\int_0^t L U_m(s, W(s))ds]
\end{equation}

where $\lambda = \lambda_{\varepsilon,n}$ is any $W$-stopping time with $t \leq \lambda \leq t_{\varepsilon} \wedge t_{A_n} \wedge T$.

We first study the behavior of

\begin{equation}
E_x[\int_0^\lambda L U_m(s, W(s))ds] = \int_0^T \int_{\mathbb{R}} L U_m(s,y) \Gamma(x, s-t, y) E_x[I_{\lambda}(s)|W(s) = y] dy ds,
\end{equation}

where $\Gamma(x, s, y)$ is the density of the transition probability function of the process $W$, $I_{\lambda}(s) = 1$ if $s < \lambda$ and 0 otherwise. From the estimate (see Friedman (1975) p. 141):

$$\Gamma(x, z, y) \leq K_1 \exp\left[-K_2 \frac{|x-y|^2}{|x|}\right],$$

we see that $E_x[I_{\lambda}|W] \in L^2([0,T) \times \mathbb{R})$. 

On the other hand, a quick comparison of $LU$ with the function $h_x$ through (4.10), (4.11) gives $LU \in L^2(A_n^C)$, for all $n$. (Here $\overline{B}$ is the closure of $B$.)

Recalling the properties of mollifiers, we have

$$LU_m \to LU \text{ in } L^2(A_n^C), \text{ for all } n.$$ Now we can let $m \to \infty$, $n \to \infty$ (with possibly $n$ at a smaller rate than $m$) and conclude that the right hand side of (5.21) converges to

$$E_x^\lambda [\int \int_T U(s,y)I(x,s-t,y)E_x[I(\lambda(s)|W(s)=y)] dy ds =$$

$$= E_x^\lambda [\int_T LU(s,W(s)) ds] \geq -E_x^\lambda [\int_T h_x(s,W(s)) ds].$$

In the last inequality, we applied (4.10), (4.11), and $\lambda$ is any stopping time such that $t \leq \lambda \leq t_f^\varepsilon \wedge T$. Noting again that

$$\int_T |h_x(s,W(s))| ds \text{ has moments of all orders, we can let } \varepsilon \downarrow 0 \text{ so}$$

(5.22) remains true for $t \leq \lambda \leq t_f^\varepsilon \wedge T$.

The other two factors in (5.20), $U_m(t,x)$ and $E_x[U_m(\lambda,W(\lambda))]$, are easily seen to converge respectively to $U(t,x)$ and $E_x[U(\lambda,W(\lambda))]$ on $[0,T] \times \mathbb{R} - F$ if we apply the properties that $U_m \to U$ uniformly in compact subsets and $U$ is bounded in $[0,T] \times \mathbb{R}$, by (4.13), (4.14), (4.15). With these remarks and (5.22), we see that (5.20) becomes, as $m \to \infty$, $n \to \infty$, $\varepsilon \downarrow 0$, 


\[ U(t,x) \leq E_x[U(\lambda, W(\lambda))] + E_x[\int_0^\lambda \frac{h_x(s, W(s))}{t} ds], \ t \leq \lambda \leq t_F \wedge T \]

that is, (5.13) holds. The proof of (5.14) is similar.

The existence of a unique solution to (4.9) is covered by Friedman (1973a) and Friedman (1976), although we must impose the additional restrictions on the cost functions:

(5.23i) the derivatives \( h_{tx}(t,x) \) and \( g''(x) \) exist and satisfy a polynomial growth condition on \([0,T] \times \mathbb{R}\) and \( \mathbb{R} \) respectively,

(5.23ii) the derivatives \( r''(t) \) and \( \ell''(t) \) exist,

and \( h_x(t,x) \) is bounded on \([0,T] \times \mathbb{R}\). Using some techniques as in Bensoussan and Lions (1982), one can replace boundedness of \( h_x \) by the less restrictive

(5.23iii) \( |h_x(t,x)| \) satisfies a polynomial growth condition on \([0,T] \times \mathbb{R}\).

Corollary (5.24): Let (3.2), (3.3), (5.6) and (5.23) hold. Then the unique solution \( U \) to (4.9) admits the stochastic representation

(5.25) \[ U(t,x) = \inf_{\sigma \in A_{t,x}} \sup_{\tau \in B_{t,x}} G(t,x;\sigma,\tau) \]

where \( G \) is given in (5.2).
6. CONSTRUCTION OF AN \( \epsilon \)-POLICY

We have seen in (4.8) that \( V \), the solution of the Free boundary problem I, is such that \( V(t,x) \leq J(t,x;S) \) for any policy \( S \). It will be shown next that, for any pre-assigned \( \epsilon > 0 \) and for any \( n \in \mathbb{N} \), it is possible to construct a policy \( S^\epsilon = (R^\epsilon, L^\epsilon) \) such that the inequality

\[
(6.1) \quad J(t,x;S^\epsilon) \leq V(t,x) + \epsilon
\]

holds uniformly on \([0,T] \times [-n,n]\).

First we have to look at some properties of the boundaries \( a(t) \), \( b(t) \), now identified as the limits of the continuation region for the game problem. From the solution of Free boundary problem II, we know that \( a \) and \( b \) are continuous functions. Now we show that we must have \( b(t) < \infty \), \( 0 \leq t < T \). Consider a point \((t_0, x_0)\), \( 0 \leq t_0 < T \), \( x_0 > 0 \) and large enough such that

\[
E_{x_0} \left[ \int_{t_0}^{\lambda \wedge T} h_X(s,W_s)ds \right] > 2C, \text{ where } C \text{ is the upper bound of the cost function } \ell \text{ for "premature termination" of player } P_1, \text{ and } \\
\lambda = \inf\{s, s \geq t: W_s = 0\}. \text{ Such a point } (t_0, x_0) \text{ always exist since } h_X(t, \cdot) \text{ is positive for } x > 0 \text{ and } h_X(t,x) \to \infty \text{ as } x \to \infty. \text{ If } W \text{ starts at } (t_0, x_0), \text{ it is clear that } P_1 \text{ should stop immediately and pay } \ell(t_0) \text{ to } P_2. \text{ But this means that } b(t_0) < x_0 < \infty.
On the other hand, consider a situation where the terminal cost function \( g^- (x) \) is identically zero and pick a starting point \( (t_1, x_1) \), where, no matter how large \( x_1 > 0 \) is, \( t_1 \) is so close to \( T \) that we have
\[
\mathbb{E}_{x_1} \left[ \int_{t_1}^T h_x(s, W_s) ds \right] < \frac{C}{2},
\]
where \( c \) is the lower bound of \( \ell \). Here \( (t_1, x_1) \) is in the continuation region, so \( b(t_1) > x_1 \). So we can have, for allowed choices of \( g^- \),
\[
\lim_{t \to T} b(t) = \infty.
\]

The same arguments show that \( a(t) > -\infty, 0 \leq t < T, \) with possibly \( \lim_{t \to T} a(t) = -\infty \). In all cases, we conclude that
\[
(6.2) \quad \text{the set } A_{\epsilon} = \{(t, x): 0 \leq t \leq T - \epsilon, a(t) \leq x \leq b(t)\} \text{ is a closed, bounded set, for any } \epsilon, 0 < \epsilon \leq T.
\]

A second comment is about the smoothness of the boundaries. We want to estimate the cost of control of the reflected BM inside \( A_{\epsilon} \), given in (6.2), but although it is known that for some cases (see Karatzas (1983)) the boundaries are continuously differentiable, we are not certain that in our situation they are. A simple device like the functions \( a_{\epsilon} \) and \( b_{\epsilon} \) to be presented next overcomes this difficulty.

Let \( \epsilon_1, \epsilon_2, \epsilon_3, \ldots \) be small positive constants, to be determined later. Let \((V, a, b)\) be a solution to (4.1). On the interval \([0, T - \epsilon_1]\) we are able to:
(i) approximate \( b(t) \) by a continuous differentiable function \( b_\varepsilon(t) \) such that \( 0 \leq b(t) - b_\varepsilon(t) \leq \varepsilon_2 \), and choose \( \varepsilon_2 \) in order to have \( V_X(t, b(t)) - V_X(t, b_\varepsilon(t)) \leq \varepsilon_3 \);

(ii) approximate \( a(t) \) by a continuous differentiable function \( a_\varepsilon(t) \) such that \( 0 \leq a_\varepsilon(t) - a(t) \leq \varepsilon_4 \), and choose \( \varepsilon_4 \) in order to have \( V_X(t, a_\varepsilon(t)) - V_X(t, a(t)) \leq \varepsilon_5 \).

(We recall that \( V_X(t, b(t)) = \ell(t) \) and \( V_X(t, a(t)) = -r(t) \).)

Let \( (t, x) \) be a point in \([0, T] \times [-n, n] \), and, to avoid trivialities, let \( n > \max\{\sup b(t), -\inf a(t); t \in [0, T-\varepsilon_1]\} \).

Our policy \( S^\varepsilon = (R^\varepsilon, L^\varepsilon) \) is described as:

(A) If \( (t, x) \) is in the closed region \( B_\varepsilon = \{(t, x): 0 \leq t \leq T-\varepsilon_1, a_\varepsilon(t) \leq x \leq b_\varepsilon(t)\} \), \( R^\varepsilon(L^\varepsilon) \) reflects \( W \) at the boundary \( a_\varepsilon(b_\varepsilon) \) so as to keep \( X^\varepsilon = x + W + R^\varepsilon - L^\varepsilon \) inside \( B_\varepsilon \). (For a construction of the reflected process, see Lions and Sznitman (1984)).

(B) If \( (t, x) \) is in \( B_\varepsilon \), \( x > b_\varepsilon(t) \), put \( L^\varepsilon_{0+} = x - b_\varepsilon(t), \ R^\varepsilon_{0+} = 0 \); if \( x < a_\varepsilon(t) \), put \( R^\varepsilon_{0+} = a_\varepsilon(t) - x, \ L^\varepsilon_{0+} = 0 \), and then proceed as in (A).

(C) On \([T-\varepsilon_1, T], R^\varepsilon = R^T_{T-\varepsilon_1}, L^\varepsilon = L^T_{T-\varepsilon_1} \).

Note that, for \( R^\varepsilon, L^\varepsilon \) so constructed we have

\[
\int_{[t, T-\varepsilon_1]} dR^\varepsilon_{s-t} < T + n; \quad \int_{[t, T-\varepsilon_1]} dL^\varepsilon_{s-t} < T + n.
\]
Now we evaluate the cost functional

\[(6.4) \quad J(t,x;S^e) = E_x \left[ \int_t^T h(s,X_s^e) ds + \int_{[t,T]} r(s) dR_{s-t}^e + \int_{[t,T]} \mathcal{L}(s) dL_{s-t}^e + g(X_T^e) \right] \]

through the function $V$. Applying generalized Ito's formula to $V(s,X_s^e)$ on the region $[0,T-\epsilon_1] \times [-n,n]$, we get

\[(6.5) \quad V(t,x) = E_x \left\{ \int_t^{T-\epsilon_1} h(s,X_s^e) ds + \int_{[t,T-\epsilon_1]} (-V_x(s,a_\epsilon(s))) dR_{s-t}^e + \int_{[t,T-\epsilon_1]} V_x(s,b_\epsilon(s)) dL_{s-t}^e - \sum_{t \leq s < T} [V(s,X_s^e) + V(s,X_{s+}^e) - V(s,X_s^e)\Delta X_s^e] + V(T-\epsilon_1,X_{T-\epsilon_1}) \right\}, \]

so we conclude that

\[(6.6) \quad J(t,x;S^e) - V(t,x) \leq E_x [I_1 + I_2 + I_3 + \Sigma_1 + |g(X_T^e) - V(t-\epsilon_1,X_{T-\epsilon_1}^e)|] \]

where $I_1 = \int_{T-\epsilon_1}^T h(s,X_s^e) ds,$

$I_2 = \int_{[t,T-\epsilon_1]} [r(s) + V_x(s,a_\epsilon(s))] dR_{s-t}^e,$

$I_3 = \int_{[t,T-\epsilon_1]} [\mathcal{L}(s) - V_x(s,b_\epsilon(s))] dL_{s-t}^e,$

$\Sigma_1 = \sum_{t \leq s < T} [V(s,X_s^e) - V(s,X_{s+}^e) - V_x(s,X_s^e)\Delta X_s^e].$
Now let $\varepsilon > 0$ be given, and consider $(t,x) \in [0,T] \times [-n,n]$. We recall that $X^\varepsilon$, on $[T-\varepsilon_1,T]$, is "free of control," i.e., it is a $(\mu, \sigma^2)$-BM starting at $X^\varepsilon_{T-\varepsilon_1}$, where $|X^\varepsilon_{T-\varepsilon_1}| < n$. Using Gilman-Skorokhod inequality plus condition 3.3 on $h$, we have

$$E_x I_1 \leq \varepsilon_1 \cdot C(1+n^m), \quad C \text{ constant;}$$

on the other hand, since $V(T,x) = g(x)$, $V$ is continuous on $[0,T] \times \mathbb{R}$, $X^\varepsilon$ is continuous on $[T-\varepsilon_1,T]$, it is possible to choose $\varepsilon_1$ such that

$$E_x I_1 \leq \frac{\varepsilon}{5}, \quad E_x |g(X^\varepsilon_T) - V(T-\varepsilon_1, X^\varepsilon_{T-\varepsilon_1})| \leq \frac{\varepsilon}{5},$$

hold simultaneously.

By construction, (see (ii), (6.3))

$$E_x I_2 \leq \varepsilon_1 \cdot \varepsilon_5 \cdot (T+n) \leq \frac{\varepsilon}{5},$$

and (see (i), (6.3))

$$E_x I_3 \leq \varepsilon_1 \cdot \varepsilon_3 \cdot (T+n) \leq \frac{\varepsilon}{5},$$

if we choose $\varepsilon_5, \varepsilon_3$ sufficiently small.

We omit here some little extra work to show that we can also have $E_x I_1 \leq \frac{\varepsilon}{5}$. Combining all the former inequalities into (6.6), we have proved the next result.
Theorem (6.7). Let (3.2), (3.3), (5.6), (5.23) hold, and let $V$ be a solution to (4.1). Then there exists an $\epsilon$-optimal policy for the control problem in the sense of (6.1).
7. COMMENTS

In (3.2.i) we imposed, for simplicity, that the running cost function $h$ should attain a minimum 0 at the origin: $h(t,0) = 0$, $0 \leq t \leq T$. Nothing changes in the arguments throughout this work if we allow the "optimum trajectory" to be a smooth line $m(t)$, with the minimum for $h$ through $m(t)$ to be a smooth function $c(t)$; that is

$$h(t,m(t)) = c(t), \quad h_x(t,m(t)) = 0; \quad 0 \leq t \leq T. \quad (7.1)$$

All other properties that we required for $h$ were used, and although we think some could be lifted, some proofs would have to be different.

In case we adopted (7.1), holding all other restrictions on $r, \ell, g$, we would have to redefine the game problem in a straightforward manner: $P_1$ has the right to stop the process if $W_s > m(s)$; $P_2$ if $W_s < m(s)$. With this modification, the value function $U$ is still interpreted as the solution of a Free boundary problem.

In (3.2.iv) we imposed the terminal cost function $g$ to have a minimum at the origin. This was never used in our work; we just put it there to make the system of cost functions seem compatible from the point of view of practical applications. All other restrictions on $g$ are essential for some proofs.
In (3.2.ii) and (3.2.iii) we imposed the cost of control functions, $r$ and $\ell$, to be monotonically non-increasing. This is, by far, the most artificial restriction on the cost functions, although we cover the important case in applications where $r$ and $\ell$ are positive constants. (Similar restrictions are not uncommon in the literature, as in Friedman (1973a)). In our work, this restriction was only used in Lemma (4.17), to establish a priori, quick and easy upper and lower bounds for the boundaries of Free boundary problem II: $a(t) \leq 0 \leq b(t), 0 \leq t \leq T$. This was used immediately in Lemma (4.19) to construct the solution to Free boundary problem I, and also, indirectly, in the game problem, to define the regions where $P_1$ and $P_2$ could stop the process. We can drop the restriction $r^-(t) \leq 0, \ell^-(t) \leq 0$, in (3.2) and add the more general (5.23ii), (which, in our context, is sufficient for the solution of Free boundary problem II) but a few modifications should be done:

A) Define $k(t)$ as the root of the function $U$, a solution to (4.9), in the $(t,x)$ plane, i.e., for $t \in [0,T]$,

$$
(7.2) \quad k(t) = \{x \in (a(t),b(t)) : U(t,x)=0\}.
$$

Since we have seen in Lemma (4.19) that $U_x > 0$ in $Dab$, $k(t)$ is uniquely defined. Since $U$ has continuous derivatives in $Dab$, $k$ is smooth.
B) Construct a solution $V$ of Free boundary problem I by

$$
V(t,x) = \int_0^T \left[ C_2 \frac{s^2}{2} - U_x(s,k(s)) + h(s,k(s)) \right] ds + \int_x^0 U(t,y) dy + g(k(T)).
$$

C) Redefine the rules of the game problem as: $P_1$ has the right to stop the process if $W_s > k(s)$; $P_2$ if $W_s < k(s)$. With the modifications (A), (B), (C), the equivalence between the control and game problems still holds as before.

Finally, we would like to mention that some results already established in the theory of stochastic games and variational inequalities look very promising to applications in more general control problems. For instance, it seems quite feasible to extend our control problem to the case where:

(i) the underlying random process $\xi$ is governed by equations of the type

$$
d\xi(t) = b(t,\xi(t)) dt + \sigma(t,\xi(t)) dB(t),
$$

where $B$ is a $(0,1)$BM;

(ii) the cost functions are allowed to be state-dependent:

$$
r = r(t,x), \ell = \ell(t,x),
$$

(iii) the state process is multi-dimensional.
BIBLIOGRAPHY


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We consider the finite horizon problem of tracking a Brownian Motion, with possibly non zero drift, by a process of bounded variation, in such a way as to minimize total expected cost of "action" and "deviation from a target state." The cost of "action" is given by two functions (of time), which represent price per unit of increase and decrease in the state process; the cost of "deviation" is incurred continuously at a rate given by a function convex in the state variable and a terminal cost function. We obtain the optimal cost function for this problem, as well an \( \varepsilon \)-optimal strategy, through the solution of a system of variational inequalities, which has a stochastic representation as the value function for an appropriate two-person-game.