Active Redundancy Allocation in Coherent Systems

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ABSTRACT

We introduce in this paper a new measure of component importance in coherent systems which is called redundancy importance. It is a measure of importance for the situation in which an active redundancy is to be made in a coherent system. This measure of component importance is compared with both the (Birnbaum) reliability importance and the structural importance of a component in a coherent system. Various models of component redundancy are studied, with particular reference to $k$ out of $n$ systems, parallel-series systems and series-parallel systems.
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1. Introduction.

In this paper we introduce and study a new measure of component importance in coherent systems which is termed redundancy importance. We use the notation and terminology for coherent systems and modules of coherent systems developed by Barlow and Proschan (1981). We shall assume that if \((C, \phi)\) represents a coherent system with components \(C = \{c_1, \ldots, c_n\}\) and structure function \(\phi\), then the components of the system act independently of each other. To indicate the state of component \(c_i\), the binary indicator variable \(x_i\) will take the value 1 \(\iff\) \(c_i\) is functioning (for \(i = 1, \ldots, n\)). The state of the coherent system is determined by the vector \(x = (x_1, \ldots, x_n)\) and the structure function \(\phi\). Let \(p = (p_1, \ldots, p_n)\) represent the vector of component reliabilities, and let \(h(p) = E(\phi(x))\) be the reliability function of the coherent system \((C, \phi)\). Dual to any vector \(p = (p_1, \ldots, p_n)\) of component reliabilities we define \(q = (q_1, \ldots, q_n) = (1 - p_1, \ldots, 1 - p_n) = 1 - p\). For any two numbers \(x, y\) we let \(x \uplus y = 1 - (1 - x)(1 - y)\) and \(x \cdot y = xy\), while for vectors \(x, y\) we let \(x \uplus y = (x_1 \uplus y_1, \ldots, x_n \uplus y_n)\) and \(x \cdot y = (x_1 y_1, \ldots, x_n y_n)\).

Given a coherent system \((C, \phi)\), let us suppose that for each \(i\) a spare component \(c_i^*\) with reliability \(p_i^*\) is available which may be placed in active (or parallel) redundancy with component \(c_i\). (Another type of redundancy which we do not investigate here is that of standby redundancy whereby the spare component is put in use only when the original fails.) Active redundancy of \(c_i^*\) with \(c_i\) would increase the reliability of 'position \(i\)' to \(p_i \uplus p_i^* = p_i + q_i p_i^*\) and the system reliability to \(h(p_1, \ldots, p_{i-1}, p_i \uplus p_i^*, p_{i+1}, \ldots, p_n)\). A well known engineering principle is that (active) redundancy at the component level is superior to redundancy at the system level. More precisely if \(p^* = (p_1^*, \ldots, p_n^*)\), then \(h(p \uplus p^*) \geq h(p) \uplus h(p^*)\). A dual result (with an interpretation that will be discussed later) is that \(h(p \cdot p^*) \leq h(p) \cdot h(p^*)\).

It is the purpose of this paper to investigate various models of active component redundancy. Generally we will be interested in maximizing system reliability subject to the allocation of one or more redundant components to the system. In section 2 we investigate component redundancy for \(k\) out of \(n\) systems while in section 3 we study component redundancy in the more general setting of modules of coherent systems.
There are various measures of component importance in coherent systems. The structural importance of component \( c_i \) is defined by

\[
I_\phi(i) = n_\phi(i)/2^{n-1} = (\# \text{ critical path vectors for } c_i)/2^{n-1} = \sum_{x,\bar{x} = 1} (\phi(1_i, x) - \phi(0_i, x))/2^{n-1}.
\]

It is clear that this measure of component importance does not depend on the vector \( p = (p_1, \ldots, p_n) \) of component reliabilities. In a \( k \) out of \( n \) system (the system functions iff at least \( k \) components function) the structural importance of any component is \( \binom{n-1}{k-1}/2^{n-1} \).

The (Birnbaum) reliability importance of component \( c_i \) is given by

\[
I_h(i) = \frac{\partial h}{\partial p_i} = h(1_i, p) - h(0_i, p).
\]

This measure of component importance (for \( c_i \)) depends on both \( \phi \) and \( p \) (but not \( p_i \) itself). Boland and Proschan (1983) show that in a \( k \) out of \( n \) system where \( p = (p_1, \ldots, p_n) \) is such that \( p_1 \leq \cdots \leq p_n \), then: (a) \( p_i \geq \frac{k-1}{n-1} \) for all \( i \Rightarrow I_h(1) \leq \cdots \leq I_h(n) \) and (b) \( p_i \leq \frac{k-1}{n-1} \) for all \( i \Rightarrow I_h(1) \geq \cdots \geq I_h(n) \).

We now define a new measure of component importance which we will label the redundancy importance of component \( i \).

**Definition 1.1.** Let \((C, \phi)\) be a coherent system of independent components where \( p = (p_1, \ldots, p_n) \) is the vector of component reliabilities. Assume that \( C^* = (c_1^*, \ldots, c_n^*) \) is a set of spares with respective reliabilities \( p^* = (p_1^*, \ldots, p_n^*) \) which are available for active redundancy (\( c_i^* \) may be put in active redundancy with \( c_i \)). The redundancy importance \( I_R(i) \) of component \( i \) is defined to be the improvement in reliability of the system which is achieved by putting \( c_i^* \) in active redundancy with \( c_i \).

Note that in a coherent system with independent components,

\[
(1.1) \quad h(p_1, \ldots, p_i + \Delta_i, \ldots, p_n) - h(p_1, \ldots, p_n) = \Delta_i I_h(i),
\]

and therefore

\[
I_R(i) = h(p_1, \ldots, p_i \cup p_i^*, \ldots, p_n) - h(p_1, \ldots, p_n) = (p_i \cup p_i^* - p_i) I_h(i) = p_i^* q_i I_h(i).
\]

Hence the redundancy importance of component \( c_i \) depends on its reliability \( p_i \), its reliability importance \( I_h(i) \) and the reliability \( p_i^* \) of the available redundant component \( c_i^* \).

An important problem is to determine the component in a coherent system which has the greatest redundancy importance. One situation when this problem might arise is where for a given coherent system \((C, \phi)\) a set of spare components \( C^* \) with respective reliabilities identical to those already in the system \((p_i^* = p_i \; \text{for all } i)\) is available for active redundancy. If we are limited to adding only one redundant component, then clearly the most important component \( c_i \) is that which maximizes \( p_i q_i I_h(i) \).
Another situation which might arise is where a single spare component \( c^* \) of reliability \( p^* \) is available for active redundancy with any of the \( n \) components in a coherent system (to be consistent with Definition 1.1, we may consider \( C^* = (c^*, c^*, \ldots, c^*) \), a set of \( n \) identical components \( c^* \)). This could happen for example in a \( k \) out of \( n \) system where at least structurally the components are all equivalent. In such a case in order to maximize the reliability of the system we should place the spare component in active redundancy with the component which maximizes \( q_i I_k(i) \) for \( i = 1, \ldots, n \). It should be observed that although the spare component \( c^* \) has the same reliability \( p^* \) no matter where it is placed in active redundancy, the improvement in reliability at position \( i \) is \( p_i \Pi p^* - p_i = q_i p^* \) which varies with \( i \). Were the improvement in reliability at position \( i \) constant over \( i \), then we note (from (1.1)) that the component with the greatest redundancy importance would coincide with the component with the greatest reliability (Birnbaum) importance. It should be clear therefore that in the model in which a spare component \( c^* \) may be placed anywhere, the component with the greatest redundancy importance may differ from that with the greatest reliability importance.

2. Redundancy in \( k \) out of \( n \) systems.

For a given vector \( p = (p_1, \ldots, p_n) \) for \( n \) component reliabilities, the reliability function for a \( k \) out of \( n \) system (of independent components) is given by

\[
h_{k|n}(p) = \sum_{\varepsilon, \epsilon \geq k} \prod_{i=1}^{n} p_i^{e_i} q_i^{1-e_i}
\]

where each \( \varepsilon = (\epsilon_1, \ldots, \epsilon_n) \) is a vector of zeros and ones and \( \varepsilon = \epsilon_1 + \cdot \cdot \cdot + \epsilon_n \). A \( k \) out of \( n \) system is a parallel system with reliability function \( h_{1|n}(p) = \Pi_{i=1}^{n} p_i \), and an \( n \) out of \( n \) system is a series system with reliability function \( h_{n|n}(p) = \Pi_{i=1}^{n} p_i \).

The following Lemma will be useful in assessing the redundancy importance of components in a \( k \) out of \( n \) system.

**Lemma 2.1.** Let \( p = (p_1, p_2, p_3, \ldots, p_n) \) where \( p_1 \leq p_2 \). Assume \( p_1^* \) and \( p_2^* \) are such that \( p_1^* \leq p_2^* \). Then

\( a) \quad h_{k|n}(p_1 \Pi p_1^*, p_2 \Pi p_2^*, p_3, \ldots, p_n) \leq h_{k|n}(p_1 \Pi p_2^*, p_2 \Pi p_1^*, p_3, \ldots, p_n) \) and

\( b) \quad h_{k|n}(p_1 \cdot p_1^*, p_2 \cdot p_2^*, p_3, \ldots, p_n) \geq h_{k|n}(p_1 \cdot p_2^*, p_2 \cdot p_1^*, p_3, \ldots, p_n). \)

**Proof:**

a) For a parallel system where \( k = 1 \),

\[
h_{1|n}(p_1 \Pi p_1^*, p_2 \Pi p_2^*, p_3, \ldots, p_n) = (p_1 \Pi p_1^*) \Pi (p_2 \Pi p_2^*) \Pi p_3 \Pi \ldots \Pi p_n
\]

\[
= (p_1 \Pi p_2^*) \Pi (p_2 \Pi p_1^*) \Pi p_3 \Pi \ldots \Pi p_n
\]

\[
= h_{1|n}(p_1 \Pi p_2^*, p_2 \Pi p_1^*, p_3, \ldots, p_n).
\]
Now let us assume \( k \geq 2 \). Then

\[
h_{k|n}(p_1 \cup p_1^*, p_2 \cup p_2^*, p_3, \ldots, p_n) = h_{k|n-2}(p_3, \ldots, p_n) + (1 - q_1 q_2 q_2^*) \sum_{\epsilon = (\epsilon_3, \ldots, \epsilon_n)} \prod_{i=3}^{n} p_i^{\epsilon_i} q_i^{1-\epsilon_i} + (p_1 \cup p_1^*) (p_2 \cup p_2^*) \sum_{\epsilon = (\epsilon_3, \ldots, \epsilon_n)} \prod_{i=3}^{n} p_i^{\epsilon_i} q_i^{1-\epsilon_i}.
\]

Hence

\[
h_{k|n}(p_1 \cup p_2^*, p_2 \cup p_1^*, p_3, \ldots, p_n) - h_{k|n}(p_1 \cup p_1^*, p_2 \cup p_2^*, p_3, \ldots, p_n)
= [(p_1 \cup p_2^*) (p_2 \cup p_1^*) - (p_1 \cup p_1^*) (p_2 \cup p_2^*)] \sum_{\epsilon = (\epsilon_3, \ldots, \epsilon_n)} \prod_{i=3}^{n} p_i^{\epsilon_i} q_i^{1-\epsilon_i}
= [(p_2 - p_1) (p_2^* - p_1^*)] \sum_{\epsilon = (\epsilon_3, \ldots, \epsilon_n)} \prod_{i=3}^{n} p_i^{\epsilon_i} q_i^{1-\epsilon_i}
\geq 0 \text{ since } p_1 \leq p_2 \text{ and } p_1^* \leq p_2^*.
\]

b) The dual of a \( k \) out of \( n \) system with component reliabilities \( p = (p_1, \ldots, p_n) \) is an \( n - k + 1 \) out of \( n \) system with component reliabilities \( q = 1 - p = (1 - p_1, \ldots, 1 - p_n) \). Therefore

\[
h_{k|n}(p_1 \cdot p_1^*, p_2 \cdot p_2^*, p_3, \ldots, p_n) = 1 - h_{n-k+1|n}(1 - p_1 \cdot p_1^*, 1 - p_2 \cdot p_2^*, 1 - p_3, \ldots, 1 - p_n)
= 1 - h_{n-k+1|n}(q_1 \cup q_1^*, q_2 \cup q_2^*, q_3, \ldots, q_n)
\]

and b) follows from a).

**Corollary 2.2.** Let \( p = (p_1, \ldots, p_n) \) be the vector of component reliabilities of a \( k \) out of \( n \) system where \( p_1 \leq p_2 \leq \cdots \leq p_n \). Then

\[
q_1 I_h(1) \geq q_2 I_h(2) \geq \cdots \geq q_n I_h(n)
\]

**Proof:** From Lemma 2.1 it follows that for a given \( p^* \),

\[
h_{k|n}(p_1 \cup 0, p_2 \cup p^*, p_3, \ldots, p_n) \leq h_{k|n}(p_1 \cup p^*, p_2 \cup 0, p_3, \ldots, p_n).
\]

Therefore from definition 1.1 it follows that

\[
p^* q_1 I_h(1) \geq p^* q_2 I_h(2)
\]

and more generally the conclusion of the corollary follows.
Remark 2.3. Consider the situation where a single spare component \( c^* \) with reliability \( p^* \) is available for active redundancy with any of the components in a \( k \) out of \( n \) system. It follows from the above corollary that the component \( c^* \) should be made redundant with the weakest component in the system - that is the weakest component has the greatest redundancy importance here. This is in contrast to the result of Boland and Proschan (1983) which states that in a \( k \) out of \( n \) system the component with the greatest reliability importance is the strongest when all \( p_i \geq \frac{k-1}{n-1} \), while it is the weakest when all \( p_i \leq \frac{k-1}{n-1} \).

Remark 2.4. For a given \( k \) out of \( n \) system let us suppose that a set of spare components \( c^* = (c_1^*, \ldots, c_n^*) \) with respective reliabilities identical to those in the system already (that is \( p_i^* = p_i \) for all \( i \)) is available for active redundancy. In this situation the component \( c_i \) with the greatest redundancy importance is that which maximizes \( p_i q_i I_k(i) \). If the system is a parallel system, then \( p_i q_i I_k(i) = p_i \prod_{j=1}^{n} q_j \) and the strongest component is the most important (has the greatest redundancy importance). It should be noted that the parallel case is very special since the same effect is achieved no matter where an active redundancy is made. In fact the result of an active redundancy in a 1 out of \( n \) system is a 1 out of \( n+1 \) system, and hence it is clear that the strongest component is the one which should be added. For a series system, the weakest is the most important in this sense. For a more general \( k \) out of \( n \) system there is no direct correspondence between component reliability and redundancy importance. For example one may construct 2 out of 3 systems such that the component with the greatest redundancy importance may be either the most reliable, the least reliable or neither of these.

We now investigate the problem of allocating several spares for active redundancy in a \( k \) out of \( n \) system. Initially we introduce the concept of arrangement increasing functions.

Definition 2.5. For a given vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we let \( x \downarrow = (x_{[1]}, \ldots, x_{[n]}) \) and \( x \uparrow = (x_{[n]}, \ldots, x_{[1]}) \) be respectively the vectors with the components of \( x \) arranged in decreasing (increasing) order. For any permutation \( \pi \) of \( \{1, \ldots, n\} \), we let \( x_{\pi} = (x_{\pi(1)}, \ldots, x_{\pi(n)}) \). For vectors \( x, y, u, v \in \mathbb{R}^n \), we write \( (x, y) \leq (u, v) \) if there exists a permutation \( \pi \) of \( \{1, \ldots, n\} \) such that \( x_{\pi} = u \) and \( y_{\pi} = v \). We define \( (x, y) \leq (u, v) \) if there exist a finite number of vectors \( z^1, \ldots, z^k \) such that (i) \( (x, y) \leq (x \uparrow, z^1) \) and \( (x \uparrow, z^k) \leq (u, v) \) and (ii) \( z^{i-1} \) can be obtained from \( z^i \) by an interchange of two components of \( z^i \), the first of which is less than the second. For example,

\[
((4,3,2,1),(0,2,3,.5)) \leq ((1,2,3,.4),(0,2,3,.5))
\]

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since
\[
((.4, .3, .2, .1), (0, .2, .3, .5)) \overset{a}{=} ((.1, .2, .3, .4), (.5, .3, .2, 0)) \\
\overset{a}{=} ((.1, .2, .3, .4), (0, .3, .2, .5)) \\
\overset{a}{=} ((.1, .2, .3, .4), (0, .2, .3, .5))
\]

**Definition 2.6.** A function \( g \) of two vector arguments for which \( g(x, y) \leq g(u, v) \) whenever \((x, y) \preceq (u, v)\) is said to be arrangement increasing (AI) by Marshall and Olkin (1979), and decreasing in transposition (DT) by Hollander, Proschan and Sethuraman (1977). We shall use the terminology arrangement increasing (AI) in order to emphasize that such a function \( g(x, y) \) increases in value as the arrangement of components in \( x \) becomes increasingly similar to the arrangement of components in \( y \). A function \( g \) is said to be arrangement decreasing or AD if \(-g\) is AI.

**Theorem 2.7.** Let \( p = (p_1, \ldots, p_n) \) and \( p^* = (p'_1, \ldots, p'_n) \) be two given vectors of component reliabilities. Then for any \( k = 1, \ldots, n \),

\[
\begin{align*}
(a) & \quad g_{\Pi}(p, p^*) = h_{k|n}(p \uplus p^*) \text{ is arrangement decreasing, and} \\
(b) & \quad g_{\Pi}(p, p^*) = h_{k|n}(p \cdot p^*) \text{ is arrangement increasing.}
\end{align*}
\]

**Proof:** This follows from Lemma 2.1 and definitions 2.5 and 2.6.

**Remark 2.8.** Let us suppose we have a \( k \) out of \( n \) system with components \( c_1, \ldots, c_n \) and respective reliabilities \( p = (p_1, \ldots, p_n) \) where \( p_1 \leq \cdots \leq p_n \). Assume that \( r (r \leq n) \) spare components \( c_1^*, c_2^*, \ldots, c_r^* \) with respective reliabilities \( p_1^* \leq \cdots \leq p_r^* \) are available for active redundancy with any of the \( r \) components in the given \( k \) out of \( n \) system. By considering the vector \((p_r^*, p_{r-1}^*, \ldots, p_1^*, 0, \ldots, 0)\), it follows from Theorem 2.7 (a) that the optimal allocation is to make \( c_r^* \) redundant with \( c_1, \ldots, c_{r-1}^* \) redundant with \( c_r \).

**Remark 2.9.** Boland and Proschan (1986) define a function \( g \) of \( s \) vector arguments to be arrangement increasing if the function increases in value as the components of the vectors become more similarly arranged. It may be shown therefore that if \( p^1, \ldots, p^s \) are given vectors of component reliabilities, then \( g_{\Pi}(p^1, \ldots, p^s) = h_{k|n}(\uplus_{i=1}^s p^i) \) is arrangement decreasing and \( g_{\Pi}(p^1, \ldots, p^s) = h_{k|n}(\prod_{i=1}^s p^i) \) is arrangement increasing for any \( k \geq 1 \) and \( n \).

The following examples help illustrate situations where redundancy in \( k \) out of \( n \) systems may be considered.
Example 2.10 - Parallel Redundancy.

a) $n$ locations are designated as desirable for planting a specific type of tree. It is desired that after a period of five years at least $k$ out of the $n$ locations have a healthy tree growing there. After the initial planting of $n$ small trees in the $n$ locations, it is decided to distribute the 'remaining' $r(\leq n)$ small trees, one to each of $r$ of the $n$ locations. The remaining trees are to be allocated in order maximize the probability that at least $k$ out of the $n$ locations contain at least one healthy tree after 5 years.

b) An intelligence organization feels that it is effectively operating in a country if it is actively receiving 'information' from at least $k$ out of $n$ key cities in the country. It has $r$ extra 'agents' with differing abilities which it desires to place in $r$ of the different cities in order to maximize the probability of continuing to obtain sufficient information.

c) In a battle or defense plan, a side has the naval advantage if it controls a majority of the $n$ key naval areas. It has $r$ extra destroyers to distribute, one each to $r$ of the key areas, in order to improve the chances of controlling power.

Example 2.11 - Series Redundancy.

If a component $c^*$ with reliability $p^*$ is put in series redundancy with component $c$ which has reliability $p$, then the resulting reliability (of the combination) is $p \cdot p^*$. Lemma 2.1 (b) and Theorem 2.7 (b) are results concerning series redundancy.

a) In a prison security system there are $n$ different entrance/exit locations where a prisoner might escape. At location $i$ there are various electronic and other devices connected in series through which a prisoner must pass in order to escape. Let $p_i =$ probability that a prisoner successfully passes through location (or exit) $i$. Hence the probability of a prisoner escaping is the reliability of a (1 out of $n$) parallel system. If we have $r$ extra security devices to distribute, one each in $r$ of the the various exits, we would be interested in allocating them to decrease as much as possible the chance of escape by a prisoner.

b) A safety system on a nuclear device gives a false alarm if $k$ or more of the $n$ independent safety devices in the system give a false alarm. We have $r(\leq n)$ extra sub-devices to attach in series to each of $r$ of the existing safety devices in order to lower the chance of a false alarm.

We conclude this section with a result concerning the optimal active redundancy allocation of $l$ identical spares of reliability $p^*$ to a $k$ out of $n$ system where the components each have reliability $p$. Let us suppose that we consider the allocation of $m_1$ spares to
location \(1, \ldots, m_n\) spares to location \(n\), where \(\ell = m_1 + \cdots + m_n\). The reliability of the allocation would then be

\[
R(m_1, \ldots, m_n) = h_{k \mid n}(p \oplus (U_{i=1}^{m_1} p^*), \ldots, (p \oplus (U_{i=1}^{m_n} p^*))).
\]

We show in fact that \(R(m_1, \ldots, m_n)\) is a Schur concave function of \((m_1, \ldots, m_n)\) for any \(k \geq 1\), that is the reliability of the allocation increases the more equally the spares are distributed among the \(n\) positions.

**Definition 2.12.** If \(x, y \in \mathbb{R}^n\), we say that \(x\) majorizes \(y\) \((x \succ y)\) if \(\sum_{i=1}^{j} x[i] \geq \sum_{i=1}^{j} y[i]\) holds for all \(j = 1, \ldots, n - 1\), and moreover \(\sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i]\). A real valued function \(f\) with the property that \(x \succ y \implies f(x) \leq f(y)\) (respectively \(f(x) \geq f(y)\)) is called Schur concave (Schur convex).

Basically a Schur concave function \(f\) of \(x = (x_1, \ldots, x_n)\) is one which decreases in value as the components of \(x\) (subject to the constraint that \(\Sigma x_i\) is fixed) become more dispersed. The treatise by Marshall and Olkin (1979) is an excellent source for examples and properties of such functions.

**Theorem 2.13.** \(R(m_1, \ldots, m_n) = h_{k \mid n}(p \oplus (U_{i=1}^{m_1} p^*), \ldots, (p \oplus (U_{i=1}^{m_n} p^*))\) is Schur concave.

**Proof:** If \(k = 1\), then \(R(m_1, \ldots, m_n)\) is constant. (Note that \(\Sigma m_i\) is fixed.) Now assume \(k \geq 2\). By the nature of majorization, it suffices to show that if \(m_1 < m_1 + 1 \leq m_2 - 1 < m_2\), then

\[
R(m_1, m_2, m_3, \ldots, m_n) \leq R(m_1 + 1, m_2 - 1, m_3, \ldots, m_n).
\]

Since

\[
R(m_1, m_2, m_3, \ldots, m_n) = \sum_{\epsilon, \epsilon \geq k \mid k = 3} \prod_{i=1}^{n} (1 - q(g^*)^{m_i})^{\epsilon_i}(q(g^*)^{m_i})^{1-\epsilon_i}
\]

\[
+ [1 - q^2(g^*)^{m_1 + m_2}] \sum_{\epsilon, \epsilon = k-1 \mid k = 3} \prod_{i=1}^{n} (1 - q(g^*)^{m_i})^{\epsilon_i}(q(g^*)^{m_i})^{1-\epsilon_i}
\]

\[
+ [1 - q(g^*)^{m_1}][1 - q(g^*)^{m_2}] \sum_{\epsilon, \epsilon = k-2 \mid k = 3} \prod_{i=1}^{n} (1 - q(g^*)^{m_i})^{\epsilon_i}(q(g^*)^{m_i})^{1-\epsilon_i}
\]

(where \(\epsilon\) is any vector of \(n - 2\) coordinates equal to 1 or 0), it suffices to show that

\[
(1 - q(g^*)^{m_1 + 1})(1 - q(g^*)^{m_2 - 1}) \geq (1 - q(g^*)^{m_1})(1 - q(g^*)^{m_2}).
\]

However this is true since

\[
q(g^*)^{m_1}(1 - q^*) - q(g^*)^{m_2 - 1}(1 - q^*) = qp^*((g^*)^{m_1} - (g^*)^{m_2 - 1}) > 0.
\]
3. Redundancy in Modules of Coherent Systems.

In this section we will assume that the coherent system \((C, \phi)\) has a modular decomposition. Hence there exist a set of disjoint modules \(\{ (A_1, x_1), \ldots, (A_r, x_r) \}\) with an organizing structure function \(\psi\) such that

(a) \(C = \cup_{i=1}^{r} A_i, \quad A_i \cap A_j = \emptyset \) for all \(i \neq j\), and
(b) \(\phi(x) = \psi[x_1(x^{A_1}), \ldots, x_r(x^{A_r})]\).

Series-parallel and parallel-series systems are basic examples of systems with a modular decomposition. The reliability function \(h_\phi\) has the form \(h_\phi(p) = h_\psi(h_{x_1}(p^{A_1}), \ldots, h_{x_r}(p^{A_r}))\) from which it follows that the reliability importance of component \(c_{ij}\) (the \(j^{th}\) component in the \(i^{th}\) module) is given by

\[
I_{h_\phi}(c_{ij}) = I_{h_\psi}(x_i)I_{h_{x_i}}(j).
\]

Therefore the reliability of component \(c_{ij}\) is the product of its importance within the \(i^{th}\) module multiplied by the importance of the \(i^{th}\) module within the system.

Let \(C^*\) be a set of spare components with respective reliabilities \(p^{c_{ij}}\) available for redundancy in the system \((C, \phi)\). Then the redundancy importance of component \(c_{ij}\) is given by

\[
I_U(i, j) = p^{c_{ij}} q_{ij} I_{h_\psi}(x_i)I_{h_{x_i}}(j).
\]

Example 3.1 - Series-parallel systems.

Suppose the coherent system \((C, \phi)\) is composed of \(r\) parallel systems connected in series. Let \(n_i\) be the number of components in the \(i^{th}\) parallel system. Assume \(C^* = (c^{\ast}_{ij})\) represents a set of spares available for redundancy. Then

\[
I_U(i, j) = p^{c_{ij}} q_{ij} I_{h_\psi}(x_i)I_{h_{x_i}}(j)
\]

\[
= p^{c_{ij}} q_{ij} \left( \prod_{l \neq i}^{n_i} \prod_{k=1}^{n_l} p^{\ell k} \right) \prod_{k \neq j} q_{ik} = p^{c_{ij}} \frac{\prod_{k=1}^{n_i} q_{ik}}{1 - \prod_{k=1}^{n_i} q_{ik}} h_\phi(p)
\]

Hence finding the component with the maximum redundancy importance is equivalent to maximizing \(p^{c_{ij}} \frac{1 - h_{x_i}(p^{A_i})}{h_{x_i}(p^{A_i})}\).

Let us consider the situation where one component \(c^*\) with reliability \(p^*\) is available for redundancy anywhere in the series-parallel system. Then, by replacing each \(p^{c_{ij}}\) above by \(p^*\), it follows that redundancy should be made with any component (in a parallel system attaching an active redundancy to any component has the same effect) in the weakest parallel subsystem. If all components in \((C, \phi)\) are equally reliable \((p_{ij} = p\) for all \((i, j)\)), then clearly this would be the subsystem with the fewest components.
Another possible scenario for a redundancy allocation in a series–parallel system is where all components within any parallel subsystem are equally reliable \( (p_{i1} = \cdots = p_{in_i} = p_i \text{ for all } i) \), and the one redundancy which is to be made is of reliability equal to those in the allocated subsystem. For example in a series parallel system where \( r = 2, n_1 = 3, n_2 = 2, p_1 = .5 \) and \( p_2 = .7 \), the following table illustrates that although the components in the 2nd parallel subsystem have greater reliability importance, those in the 1st have greater redundancy importance.

![Figure 3.1](image)

<table>
<thead>
<tr>
<th>Components in Subsystem</th>
<th>Reliability</th>
<th>Reliability Importance</th>
<th>Redundancy Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.5</td>
<td>.2275</td>
<td>.056875</td>
</tr>
<tr>
<td>2</td>
<td>.7</td>
<td>.2625</td>
<td>.055125</td>
</tr>
</tbody>
</table>

Table 3.1

In this general scenario, if all components are equally reliable in the system \( (C, \phi) \) (that is \( p_1 = \cdots = p_r \)), then the optimal redundancy allocation is made anywhere in the smallest parallel subsystem. If all subsystems are of the same size \( (n_1 = \cdots = n_r = n) \) but where the reliabilities \( p_1, \ldots, p_r \) may vary, then we are interested in maximizing \( q^n_i/(1 + q_i + \cdots + q_i^{n-1}) \). Therefore the allocation should be made to the subsystem with the minimum \( p_i \).

**Example 3.2 - Parallel–Series Systems.**

In a parallel–series system \( (C, \phi) \) there are \( r \) series systems connected in parallel. Using the notation of Example 3.1, it follows that

\[
I_{11}(i, j) = p_{ij}^r q_{ij}^r h_{x_i}(x_i) h_{x_i}(j)
\]

\[
= p_{ij}^r q_{ij} \left[ \prod_{\ell \neq i} (1 - \prod_{j=1}^{n_i} p_{\ell j}) \right] \prod_{k \neq j} p_{ik}
\]

\[
= p_{ij}^r \frac{q_{ij}}{p_{ij}} \frac{h_{x_i}(p^{A_i})}{1 - h_{x_i}(p^{A_i})} [1 - h_\phi(p)].
\]

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Hence we are interested in maximizing $\frac{p_{ij} \cdot h_{x_i}(p^{a_i})}{p_{ij} \cdot h_{x_i}(p^{a_i})}$.

In the situation where $p_{ij} = p_{ij}$ for all $(i, j)$ — that is, the redundant spare matches the reliability of the component where it will be placed — the problem reduces to maximizing $q_{ij} \cdot h_{x_i}(p^{a_i})$.

Assume all components in any series subsystem are equally reliable (that is for any $i$, $p_{i1} = \cdots = p_{in_i} = p_i$). If all components are equally reliable ($p_1 = \cdots = p_r$), then the optimal redundancy allocation should be made in the smallest series subsystem. If all subsystems are of the same size ($n_1 = \cdots = n_r = n$) — but where the reliabilities $p_1, \ldots, p_r$ may vary — then the optimal redundancy allocation should be made in the subsystem with the maximum $p_i$. 
References


Active Redundancy Allocation in Coherent Systems

Philip J. Boland, Emad El-Neweihi and Frank Proschan

We introduce in this paper a new measure of component importance in coherent systems which is called redundancy importance. It is a measure of importance for the situation in which an active redundancy is to be made in a coherent system. This measure of component importance is compared with both the (Birnbaum) reliability importance and the structural importance of a component in a coherent system. Various models of component redundancy are studied, with particular reference to k out of n systems, parallel-series systems and series-parallel systems.