LAPLACE ORDERING AND ITS RELIABILITY APPLICATIONS

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ABSTRACT

Two arbitrary life distributions $F$ and $G$ can be ordered with respect to their Laplace transforms. We say $\bar{F}$ is Laplace-smaller than $\bar{G}$ if $\int_{0}^{\infty} e^{-st}\bar{F}(t)dt \leq \int_{0}^{\infty} e^{-st}\bar{G}(t)dt$ for all $s > 0$. Interpretations of this ordering concept in reliability, operations research, and economics are described. General preservation properties are presented. Using these preservation results we derive useful inequalities and discuss their applications to $M/G/1$ queues, time series, coherent systems, shock models and cumulative damage models.
1. **INTRODUCTION.**

The main purpose of this paper is to compare survival functions and via their Laplace transforms. The Laplace transform is a powerful tool useful in many areas of probability and statistics. See Feller (1971), Chapters 8 and 9. In studying nonnegative random variables, statisticians sometimes find it more convenient to use Laplace transforms rather than characteristic functions.

Suppose $X$ is a nonnegative random variable having absolutely continuous distribution $F(x)$ with density function $f(x)$. Then the ordinary Laplace transform of the density function $f$ is defined by

$$f^*(s) = E(e^{-sX}) = \int_0^\infty e^{-st} f(t) \, dt$$

provided the integral exists for $s \geq 0$. Since differentiation under the integral sign is permissible (the resulting integral is bounded and continuous), we get

$$(-1)^n f^{(n)}(s) = \int_0^\infty e^{-st} t^n f(t) \, dt, \quad \text{where } f^{(n)}(s) = \frac{d^n}{ds^n} f^*(s).$$

It follows that $F$ possesses a finite $n^{th}$ moment if and only if a finite limit $f^{(n)}(0)$ exists. For $n = 1$, $E(X) = -f''(0)$ and for $n = 2$, $E(X^2) = f^{(2)}(0)$. As in the case of characteristic functions, the Laplace transform uniquely determines the distribution: Let $f(t)$ be continuous at $t$ and bounded on $[0, \infty)$, then

$$\lim_{n \to \infty} \left( \frac{(-1)^n}{n!} s^{n+1} f^*(s) \right)_{s = (n+1)/t} = f(t)$$

(Widder, 1946).

Convergence is uniform in every finite closed interval throughout which $f(t)$ is continuous. The above inversion formula has also been used by Feller (1971, Chapter 7) to prove the law of large numbers.
Klefsjö (1983) suggested an ordering concept in terms of a class of life distributions which is strictly larger than the harmonic new better than used in expectation (HNBUE) class. He investigated those distributions for which

$$
\int_0^\infty e^{-st} F(t) dt \geq \frac{\mu}{1 + s\mu} \quad \text{for } s \geq 0,
$$

(1.2)

where $F$ is a life distribution (i.e., $F(0^-) = 0$) with survival function $\bar{F} = 1 - F$ and finite mean $\mu = \int_0^\infty \bar{F}(t) dt$. The right-hand side of (1.2) can be written as $\int_0^\infty e^{-st} \bar{G}(t) dt$ where $\bar{G}(t) = e^{-t/\mu}$.

In this paper, we compare survival functions $\bar{F}$ and $\bar{G}$ where $\bar{G}$ is not constrained to be exponential.

The concept of Laplace ordering is formally defined in Section 2 and an equivalent definition is given. Also, the discrete analogue of Laplace ordering is defined. Various interpretations of the ordering based on this equivalent definition are described in Section 2. Section 3 presents general preservation properties of the Laplace ordering. Applications of the preservation results to $M/G/1$ queues and first order autoregressive time series are presented in Section 4. In Section 5, we investigate applications of Laplace ordering in reliability such as formation of coherent systems, shock models, and cumulative damage models.

2. DEFINITIONS AND INTERPRETATIONS.

2.1. Definitions.

Stoyan (1983, p. 22) considered the following ordering concept among probability distributions based on comparison of their Laplace-Stieltjes transforms.
Definition 2.1.1. For distribution functions $F$ and $G$ of nonnegative random variables $X$ and $Y$, $F$ is smaller than $G$ (or $X$ is smaller than $Y$) with respect to the Laplace – Stieltjes transform if

$$E(e^{-sX}) = \int_0^\infty e^{-st}dF(t) \geq \int_0^\infty e^{-st}dG(t) = E(e^{-sY}), s \geq 0. \quad (2.1)$$

We write $F \leq^L G$.

Let $\bar{F} = 1 - F$ be the survival function of $F$. Then the Laplace transform of $\bar{F}$ with respect to Lebesgue measure is defined by

$$F^*(s) = \int_0^\infty e^{-st} \bar{F}(t) dt \text{ for } s \geq 0. \quad (2.2)$$

It is easy to check that if $F$ and $G$ are absolutely continuous with respect to Lebesgue measure and have finite means, then (2.1) is equivalent to

$$\int_0^\infty e^{-st} \bar{F}(t) dt \leq \int_0^\infty e^{-st} \bar{G}(t) dt. \quad (2.3)$$

If (2.3) holds, we say $\bar{F}$ is Laplace – smaller than $\bar{G}$ and write $\bar{F} <^L \bar{G}$. Since (2.1) and (2.3) are equivalent, the notation $\bar{F} <^L \bar{G}, F <^L G$, and $X <^L Y$ will be used interchangeably. From the definition it is clear that $\bar{F}$ has a smaller Laplace transform than does $\bar{G}$ uniformly in $s$.

Let $X$ be a discrete nonnegative random variable such that $P(X = k) = p_k$, $k \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\bar{P}_k = P(X > k)$, $k = 0, 1, 2, \ldots$ denote the corresponding survival function with $1 = \bar{P}_0 \geq \bar{P}_1 \geq \bar{P}_2 \geq \ldots$ and $\sum_{k=0}^\infty \bar{P}_k = \mu$. Then the discrete version of $F^*(s)$ is

$$F^*(s) = \int_0^\infty e^{-st} \bar{F}(t) dt$$

$$= \sum_{k=0}^\infty \bar{P}_k e^{-sk}. \quad (2.4)$$

Let

$$P(p) = \sum_{k=0}^\infty \bar{P}_k p^k.$$
Since the Laplace transform $F^*(s)$ differs from the probability generating function $P(p)$ only by the change of variable $p = e^{-s}$, the following definition is natural.

**Definition 2.1.2.** Let $\bar{P}_k$ and $\bar{Q}_k$ be survival functions with support on the integers $k = 0, 1, 2, \ldots$. Let

$$\sum_{k=0}^{\infty} \bar{P}_k p^k \leq \sum_{k=0}^{\infty} \bar{Q}_k p^k$$

for all $p, 0 \leq p \leq 1$. We say $\bar{P}_k$ is generating function smaller than $\bar{Q}_k$, and write $\bar{P}_k \prec \bar{Q}_k$.

One motivation for studying comparison (2.3) is that in many problems the solution is obtained not in terms of a distribution function, but rather in terms of its Laplace transform. This is true in many queueing problems and some reliability problems.

**Definition 2.1.3.** A nonnegative function $\phi$ on $[0, \infty)$ is completely monotone (c.m.) if it possesses derivatives $\phi^{(n)}$ of all orders and

$$(-1)^n \phi^{(n)}(t) \geq 0 \text{ for } t \geq 0 \text{ and } n = 1, 2, \ldots.$$ 

Equivalently, $\phi$ is c.m. if and only if it is of the form

$$\phi(t) = \int_0^{\infty} e^{-tx} d\lambda(x) \text{ for } t > 0,$$

where $\lambda$ is a measure, not necessarily finite, on $[0, \infty)$ (cf. Feller, 1971, p. 439).

**Definition 2.1.4.** A sequence $\{a_n\}$ of nonnegative real numbers is called completely monotone if

$$(-1)^r \Delta^r a_k \geq 0 \text{ for } r = 1, 2, \ldots,$$

where $\Delta a_k = a_{k+1} - a_k$ and $\Delta^r a_k = \Delta^{r-1}(\Delta a_k)$.
Equivalently \((a_k)\) is a c.m. sequence with \(a_0 = 1\) if and only if

\[
a_k = \int_0^1 x^k d\lambda(x),
\]

where \(\lambda\) is a probability measure on \([0, 1]\) (cf. Feller, 1971, p. 225).

2.2. Interpretations.

The expression (2.2) and the inequality (2.3) have a variety of interpretations corresponding to models in reliability, inventory, and maintenance.

(a). One simple interpretation of (2.2) is the mean life of a series system of two independent components, one having exponential survival function and the other having survival function \(\bar{F}\).

(b). A machine has survival function \(\bar{F}\) and produces one unit of output per hour when functioning. The present value of one unit produced at time \(t\) is \(1 \cdot e^{-st}\), where \(s\) is the discount rate. Then the expected present value of total output produced during the life of the machine is \(\int_0^\infty e^{-st} \bar{F}(t)dt\). Thus \(\bar{F}_1 < L \bar{F}_2\) imples that a machine governed by survival function \(\bar{F}\) produces a smaller expected total present value than does a machine governed by survival function \(\bar{F}_2\).

(c). Suppose a man buys an insurance policy, and his remaining life has survival function \(\bar{F}(t)\), where \(t\) is the time measure from the moment of purchase. If he pays one dollar per unit of time, then the total expected payment during his remaining lifetime has present value \(\int_0^\infty e^{-st} \bar{F}(t)dt\), where \(s\) is the discount rate.

(d). A man buys an annuity which pays one dollar per unit of time during his remaining lifetime. The present value of total payment received has expectation \(\int_0^\infty e^{-st} \bar{F}(t)dt\).

(e). Consider a population of \(n\) producers at time \(t = 0\). The expected proportion of producers
still functioning at time \( t \) is \( \bar{F}(t) \). Thus the output during \( (t, t + dt) \) is approximately \( n \bar{F}(t)dt \), with present value \( n \bar{F}(t)e^{-st}dt \), where \( s \) is the discount factor. Hence \( n \int_0^\infty e^{-st} \bar{F}(t)dt \) is the present value of the total output of the original population.

(f). A system consists of statistically independent units 1 and 2 arranged in series, with respective random lifetimes \( X \) and \( Y \) governed by survival functions \( e^{-st} \) and \( \bar{F}(t) \), respectively. Then the probability of system failure due to unit 1 is \( s \int_0^\infty e^{-st} \bar{F}(t)dt \).

(g). Let \( e^{-st} \) be the present value of the maintenance cost: say, one dollar paid at time \( t > 0 \) with the discount rate \( s \). Then \( E(e^{-sT}) \) is the expected present value of the maintenance cost at a random time \( T \) having distribution \( F \) and density \( f \). From (2.2)

\[
F^*(s) = \frac{1}{s} [1 - \int_0^\infty e^{-st}dF(t)]
\]

\[
= E(\int_0^T e^{-st}dt),
\]

where \( \int_0^T e^{-st}dt \) may be viewed as the present value of the maintenance cost during \([0, T]\). Thus the Laplace transform \( \int_0^\infty e^{-st} \bar{F}(t)dt \) can be interpreted as the expected present value of the total maintenance cost paid during the lifetime of a system.

(h). Let \( E_1(t) \) be the profit rate of a firm at time \( t \) when there are no rivals and \( E_2(t) \) be the profit rate after rival entry. Let \( F(t) \) be the probability that rival entry will occur by time \( t \). Then \( \int_0^\infty e^{-st}(E_1(t) \bar{F}(t) + E_2(t)F(t))dt \) represents the expected present value of the total profit during the firm's lifetime when future profits are discounted at rate \( s \).

Kamien and Schwartz (1981) present a more general case involving a price policy. For other interpretations interesting from a reliability standpoint, see Klefsjö (1983).
(i). Expressions (2.4) and (2.5) can be interpreted as follows:

Suppose that a device is exposed to random shocks which may cause its failure. Let \( \bar{P}_k(t) \) be the probability that the device survives \( k \) shocks during \([0, t]\). Each time a shock occurs, a maintenance action is performed with probability \( 1 - p \). Then \( \bar{P}_k(t)p^k \) is the probability that no maintenance is performed and the device survives \( k \) shocks in the time interval \([0, t]\). Thus the probability generating function \( \sum_{k=0}^{\infty} \bar{P}_k(t)p^k \) is the probability that the device survives past time \( t > 0 \) without a maintenance being performed.

Some ingenious probabilistic interpretations of generating functions with respect to random point processes can be found in Råde (1972).

3. GENERAL PRESERVATION RESULTS.

The following theorem gives a characterization of the Laplace ordering in terms of functions with completely monotone derivatives.

**Theorem 3.1.** \( \bar{F}_1 \leq^L \bar{F}_2 \) if and only if

\[
E(\phi(X_1)) \leq E(\phi(X_2))
\]

for each nonnegative function \( \phi \) having a c.m. derivative and such that the expectations exist.

**Proof.** Suppose \( \bar{F}_1 \leq^L \bar{F}_2 \) and nonnegative \( \phi \) has a c.m. derivative. Then there exists a nonnegative measure \( \lambda \) such that

\[
\phi'(t) = \int_0^\infty e^{-tx} d\lambda(x).
\]

Consequently,

\[
\phi(t) = \int_0^\infty \frac{1 - e^{-tx}}{x} d\lambda(x).
\]
Using the Fubini theorem and $\bar{F}_1 \ll L \bar{F}_2$, we get
\[
\int_0^\infty \phi(t) dF_1(t) = \int_0^\infty \int_0^\infty \frac{1 - e^{-tx}}{x} d\lambda(x) \ dF_1(t) \\
= \int_0^\infty \int_0^\infty \frac{1 - e^{-tx}}{x} dF_1(t) \ d\lambda(x) \\
\leq \int_0^\infty \int_0^\infty \frac{1 - e^{tx}}{x} dF_2(t) \ d\lambda(x) \\
= \int_0^\infty \int_0^\infty \frac{1 - e^{tx}}{x} d\lambda(x) \ dF_2(t) \\
= \int_0^\infty \phi(t) dF_2(t).
\]

Conversely, suppose that (3.1) holds for each nonnegative $\phi$ having a c.m. derivative. Then (3.1) holds for $\phi(x) = 1 - e^{-tx}$, which implies $\bar{F}_1 \ll L \bar{F}_2$. ||

Corollary 3.2. For every nonnegative function $\psi$ with a c.m. derivative, $X_1 \ll L X_2$ implies $\psi(X_1) \ll L \psi(X_2)$.

**Proof.** Let $\phi$ be a nonnegative function with a c.m. derivative. Using criteria 1 and 2 of Feller (1971, p. 441) and the fact that $\frac{d}{dx} \phi(\psi(x)) = \psi'(x) \cdot \phi'(\psi(x))$, we see that $\phi(\psi(x))$ is a nonnegative function with a c.m. derivative. Now the conclusion follows as a consequence of Theorem 3.1. ||

The following therom is very useful in establishing preservation results.

**Theorem 3.3.** Let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ each be a random vector with mutually independent components. Suppose that $X_i \ll L Y_i, i = 1, 2, \ldots, n$. Then
\[
\psi(X_1, \ldots, X_n) \ll L \psi(Y_1, \ldots, Y_n)
\]
for all nonnegative functions $\psi$ on $\mathbb{R}_+^n$ such that $\frac{\partial}{\partial x_i} \psi(x_1, \ldots, x_n)$ is c.m. in $x_i, i = 1, 2, \ldots, n$.

**Proof.** Since the left side of the inequality depends only on the distribution of $X$, while the right side depends only on the distribution of $Y$, without loss of generality we assume $X$ and $Y$
mutually independent. We shall prove the result by induction. By Corollary 3.2, the result holds for \( n = 1 \). Assume that the result holds for vectors of size \( n - 1 \). Let \( \phi \) be a nonnegative function with a c.m. derivative. Then

\[
E(\phi(\psi(X_1, \ldots, X_n))) = E(E(\phi(\psi(X_1, \ldots, X_n)|X_1))) \\
\leq E(E(\phi(\psi(X_1, Y_2, \ldots, Y_n)|X_1)) \\
= E(\phi(\psi(X_1, Y_2, \ldots, Y_n))) \\
= E(E(\phi(\psi(X_1, Y_2, \ldots, Y_n)|Y_2, \ldots, Y_n)) \\
\leq E(E(\phi(\psi(Y_1, Y_2, \ldots, Y_n)|Y_2, \ldots, Y_n)) \\
= E(\phi(\psi(Y_1, \ldots, Y_n))).
\]

The first inequality above follows from the induction assumption and the fact that \( X_1 \) is independent of \( \{Y_2, \ldots, Y_n\} \) and \( \{X_2, \ldots, X_n\} \), whereas the second inequality follows from the fact that the result holds for \( n = 1 \) and \( \{Y_2, \ldots, Y_n\} \) is independent of \( X_1 \) and \( Y_1 \). This completes the proof. ||

**Corollary 3.4.** Let \( \{X_1, \ldots, X_n\} \) and \( \{Y_1, \ldots, Y_n\} \) be as in Theorem 3.3. Then for all \( a_i \geq 0, i = 1, 2, \ldots, n, \sum_{i=1}^{n} a_i X_i <^L \sum_{i=1}^{n} a_i Y_i. \)

**Corollary 3.5.** Let \( \{X_1, \ldots, X_n\} \) and \( \{Y_1, \ldots, Y_n\} \) be as in Theorem 3.3. Then \( \prod_{i=1}^{n} X_i <^L \prod_{i=1}^{n} Y_i. \)

Next we present the following preservation result under mixing.

**Theorem 3.6.** Let \( \{F_{i\alpha}: \alpha \in \Lambda\}, i = 1, 2, \) be two sets of distribution functions such that \( \bar{F}_{1\alpha} <^L \bar{F}_{2\alpha} \) for all \( \alpha \in \Lambda \). Then for any distribution function \( G \) on \( \Lambda, \int_{\Lambda} \bar{F}_{1\alpha}(x)dG(\alpha) <^L \int_{\Lambda} \bar{F}_{2\alpha}(x)dG(\alpha). \)

The following preservation theorem is very useful in comparing compound distributions and
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