Schur Structure Functions

by

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FSU Technical Report No. M-784
AFOSR Technical Report No. 88-225

February, 1988

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⁴Research supported under AFOSR Grant 85-0230.
⁵Research supported under AFOSR Grant 88-0040.

Key Words and Phrases: Schur functions, majorization, multistate structures, Schur structure functions, coherent systems, system performance.

AMS Subject Classification: 62N05
Abstract

We define two new classes of multistate coherent systems by requiring, among other conditions, that their structure functions be Schur-concave (Schur-convex). The M+1 performance levels of both the systems and their components are represented by the set \{0,1,\ldots,M\}. We present basic structural properties of the new classes. In particular we study in some detail the number and form of the critical upper (lower) vectors to the various performance levels. We also present some probabilistic aspects of the new classes.
1. Introduction.

The theory of multistate coherent systems has been developed by reliability researchers to describe a system and its components which can operate at more than two levels (states) of performance. See, e.g., Barlow and Wu (1978), El-Neweihi et al. (1978), Ross (1979), Griffith (1980), Block and Savits (1982), Natvig (1982), and Block, Griffith and Savits (1987).

A basic notion in this theory is the structure function which relates the level of performance of the system to the levels of performance of its components. In this paper we study structure functions that are Schur-concave (Schur-convex). These mathematical properties model the corresponding engineering properties that many systems perform better (worse) when the levels of performance of their components are more homogeneous. These Schur properties, successfully utilized in many other contexts (see Marshall and Olkin, 1979), prove to be quite useful in the multistate model as well. We prove that monotone structure functions possessing the Schur property enjoy a number of basic and interesting structural and probabilistic properties.

In Section 2, we present definitions, notation, and terminology. Section 3 contains the definition and the basic structural properties of two new classes of multistate coherent systems with discrete state space. The main defining property for those new classes is the Schur property. Lemma 3.3 gives a simple but important bound on the level of performance of a system in one of these classes. In Theorem 3.6 we prove that the only Schur-convex structure function for which redundancy at the system level is as good as redundancy at the component level is the parallel system. The structure of the critical upper (lower) vectors to various levels of system performance is examined in some detail in Lemma 3.8. These vectors play the same basic role that min path (cut) vectors play in the binary models. A sharp upper bound on the number of critical upper (lower) vectors to various levels of system performance is obtained in Proposition 3.10 for the case of two components.

Finally, in Section 4, we derive some probabilistic bounds and use a well known preservation theorem due to Proschan and Sethuraman (1977) to show that the Schur property is inherited under specified conditions by the expected performance function.
2. Notation, terminology, and preliminaries

The vector $x = (x_1, ..., x_n)$ denotes the vector of states of components $1, ..., n$.

$(j, x) \equiv (x_1, ..., x_{i-1}, j, x_{i+1}, ..., x_n)$, where $j=0,1, ..., M$.

$(i, x) \equiv (x_1, ..., x_{i-1}, ..., x_n)$.

$j \equiv (j, ..., j)$, where $j=0,1, ..., M$.

$x \vee y \equiv \max (x, y)$.

$x \lor y = \max (x, y)$.

$\vee \lor y = (x_1, ..., y)$, where $j=0,1, ..., M$.

$x \land y \equiv \min (x, y)$.

$x \land y \equiv (x_1, ..., y_n)$.

$x \geq y$ means $x_i \geq y_i$ for $i=1, ..., n$.

$x > y$ means $x_i \geq y_i$ for $i=1, ..., n$ and $x_i > y_i$ for some $i$.

When we say $\sigma(x_1, ..., x_n)$ is nondecreasing, we mean $\sigma$ is nondecreasing in each argument.

Given a set $S$, $S^n$ denotes its nth Cartesian product.

We now give definitions of majorization and Schur-functions and state some well known results.

Given a vector $x = (x_1, ..., x_n)$, let $x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$ denote a nonincreasing rearrangement of $x_1, ..., x_n$.

A vector $x$ is said to majorize a vector $y$ if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]}, \quad j=1, ..., n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

in symbols, $x \geq^m y$. If a function $f$ satisfies the property that $f(x) \leq (\geq) f(y)$ whenever $x \geq^m y$, then $f$ is called a Schur-concave (Schur-convex) function. Functions which are either Schur-concave or Schur-convex are called Schur functions (or alternatively, are said to have the Schur property). Clearly a Schur function is necessarily permutation invariant.
Examples of Schur-concave (Schur-convex) functions are \( \sum_{i=1}^{n} f_i(x_i) \) and \( \prod_{i=1}^{n} f_i(x_i) \), where \( f_1: R \to R \) is concave (convex) and \( f_2: R \to R \) is log-concave (log-convex).

We now introduce the definition of total positivity of order 2 and of the semigroup property which will be used in Section 4. A function \( \sigma(\lambda, x) \) on \( R \) is said to be totally positive of order 2 (TP2) if (a) \( \sigma(\lambda, x) \geq 0 \) and (b) \( \lambda_1 \leq \lambda_2, x_1 \leq x_2 \) imply that \( \sigma(\lambda_1, x_1) \sigma(\lambda_2, x_2) \geq \sigma(\lambda_1, x_2) \sigma(\lambda_2, x_1) \). A function \( \sigma(\lambda, x) \), defined on \((0, \infty) \times [0, \infty)\), is said to satisfy the semigroup property in \( \lambda \) if

\[
\sigma(\lambda_1 + \lambda_2, x) = \int_{0}^{\infty} \sigma(\lambda_1, x-y) \sigma(\lambda_2, y) \, d\mu(y),
\]

where \( \mu \) denotes either Lebesgue measure on \([0, \infty)\) or counting measure on the nonnegative integers.

3. Two new classes of multistate coherent systems and their structural properties.

A basic notion in the theory of binary coherent systems is the structure function \( \sigma: \{0, 1\}^n \to \{0, 1\} \) which determines the state of the system in terms of the states of its components. The following two conditions are required for a binary system to be a coherent structure [Barlow and Proschan 1981, Def. 2.1, p.6] :

(i) The function \( \sigma(x) \) is nondecreasing.

(ii) For each \( i \), there exists a vector \((1, \ldots, 1)_i \) such that \( \sigma(0,1, \ldots, 1)_i < \sigma(1, \ldots, 1)_i \).

Recently researchers felt the need to develop the theory of multistate coherent systems to describe more adequately the performance of components and systems which have more than two levels of performance.

Again a basic notion in such a theory is the structure function \( \sigma: S^n \to S \), where \( S = \{0,1,\ldots,M\} \) is the set representing levels of performance ranging from perfect functioning \( M \) to total failure \( 0 \). (We concentrate in this paper on the case of finite \( S \).) Condition (i) is extended in a straightforward fashion by requiring \( \sigma \) to be nondecreasing. However the relevance condition (Condition (ii)) has been extended by researchers in many different ways, each leading to a distinct class of multistate "coherent" systems.

In this section we introduce two new classes of multistate coherent systems. The basic feature of these new classes is that their structure functions are required to be Schur
functions. This Schur property embodies the realistic fact that many systems perform better (worse) when the performance levels of their components are more homogeneous.

Specifically, we stipulate three conditions:

3.1 Definition. A system of $n$ components is said to be a Schur-concave (Schur-convex) multistate coherent system if its structure function $\sigma = [\sigma^*_{S^n}] = [\sigma^*_{S^n}]$, where $[a] \triangleq \langle a \rangle$ is the largest (smallest) integer less (greater) than or equal to $a$ and $\sigma^*_{S^n}$ is the restriction to $S^n$ of a function $\sigma^* : [0, M]^n \rightarrow [0, M]$ which satisfies:

(i) $\sigma^*$ is nondecreasing.
(ii) $\sigma^*$ is Schur-concave (Schur-convex).
(iii) $\sigma^*(j) = j$ for $0 \leq j \leq M$.

In the remainder of this paper we use the acronym SCV (SCX) for Schur-concave (Schur-convex). A member of the SCV (SCX) class is often identified with its structure function.

Recall that the dual structure function $\sigma^D$ of a binary structure function $\sigma$ is defined by $\sigma^D(x) = 1 - \sigma(1-x)$. Similarly the dual structure function $\sigma^D$ of a multistate structure function $\sigma$ is defined by $\sigma^D(x) = M - \sigma(M-x)$.

The following Lemma shows that the SCV class and SCX class are dual to one another.

3.2 Lemma. Let $\sigma : \{0,1,...,M\}^n \rightarrow \{0,1,...,M\}$. Then $\sigma \in$ SCV class if and only if $\sigma^D \in$ SCX class.

Proof. Note that $\sigma = [\sigma^*_{S^n}]$ iff $\sigma^D = <\sigma^*_{S^n}>$, where $\sigma^D(x) = M - \sigma^*(M-x)$ for all $x \in [0,M]^n$. Also $x \preceq y$ implies $M-x \succeq M-y$. Consequently $\sigma^*$ is Schur-concave iff $\sigma^D$ is Schur-convex. □

This duality between the two classes will be exploited quite often in this paper to show that the structural properties enjoyed by one of the two classes have their natural counterparts for the other class. □

Some examples of SCV structure functions are $\min_{1 \leq i \leq n} x_i$, $\left[ \sum_{i=1}^{n} \frac{x_i}{n} \right]$, and $\left[ \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \right]$. Duals of the above structure functions constitute examples of SCX structure functions. Note that the structure function $\min_{1 \leq i \leq n} x_i$ and its dual $\max_{1 \leq i \leq n} x_i$ are the well-known series and parallel (multistate) structure functions respectively.
The following lemma contains a simple but important upper (lower) bound on the performance level of an SCV (SCX) structure function.

3.3 Lemma. Let \( \varnothing \) be an SCV (SCX) structure function. Then \( \varnothing(\mathbf{x}) \leq (\geq) \bar{x} \), where \( \bar{x} = \sum_{i=1}^{n} \frac{x_i}{n} \).

Proof. The vector \( \mathbf{x} \) majorizes the vector \((\bar{x}, \ldots, \bar{x})\). The conclusion now follows from conditions (ii) and (iii) of Definition 3.1. \( \Box \)

The following lemma shows that the components of an SCV (SCX) structure enjoy a certain degree of relevance to system performance. This justifies the use of the term "coherent" in Definition 3.1.

3.4 Lemma. Let \( \varnothing \) be and SCV (SCX) system of \( n \) components. Then for component \( i, \ 1 \leq i \leq n \), at level \( j, j \geq 1 \) \((j \leq M-1)\), there exists a vector \((\mathbf{x}_i, \mathbf{x})\) such that \( \varnothing(i, \mathbf{x}) \geq (\leq) j \) and \( \varnothing((j-1), \mathbf{x}) < j \) \((\varnothing((j+1), \mathbf{x}) > j) \).

Proof. Note that \( \varnothing(i, j) = j \). The average of the coordinates of the vector \((i-1, j) \) \((i+1, j)\) is \( < (>) j \). The result is now a direct consequence of Lemma 3.3. \( \Box \)

3.5 Remark. The relevance property enjoyed by SCV structure functions is due to Natvig (1982). The relevance property enjoyed by SCX structure functions is the natural dual of Natvig's relevance.

It should be noted that the worst (best) structure in the SCV (SCX) class is the series (parallel) structure. The following theorem characterizes these two well-known systems within these two new classes.

3.6 Theorem. Let \( \varnothing: \{0,1,\ldots,M\}^n \rightarrow \{0,1,\ldots,M\} \). Then

(a) \( \varnothing \) is SCV and \( \varnothing(\mathbf{x} \land \mathbf{y}) = \varnothing(\mathbf{x}) \land \varnothing(\mathbf{y}) \) for all \( \mathbf{x} \) and \( \mathbf{y} \) if and only if \( \varnothing(\mathbf{x}) = \min_{1 \leq i \leq n} x_i \) for all \( \mathbf{x} \).

(b) \( \varnothing \) is SCX and \( \varnothing(\mathbf{x} \lor \mathbf{y}) = \varnothing(\mathbf{x}) \lor \varnothing(\mathbf{y}) \) for all \( \mathbf{x} \) and \( \mathbf{y} \) if and only if \( \varnothing(\mathbf{x}) = \max_{1 \leq i \leq n} x_i \) for all \( \mathbf{x} \).

Proof. It suffices to prove (a) since (b) follows immediately by considering the dual structure.

Assume that \( \varnothing(\mathbf{x}) = \min_{1 \leq i \leq n} x_i \). Then trivially, \( \varnothing \) is SCV and \( \varnothing(\mathbf{x} \land \mathbf{y}) = \varnothing(\mathbf{x}) \land \varnothing(\mathbf{y}) \) for all \( \mathbf{x} \) and \( \mathbf{y} \).

To prove the reverse implication, suppose that \( \varnothing \) is SCV for which \( \varnothing(\mathbf{x} \land \mathbf{y}) = \varnothing(\mathbf{x}) \land \varnothing(\mathbf{y}) \) for all \( \mathbf{x} \) and \( \mathbf{y} \). The vector \( \mathbf{j} = (j_1, M) \land (j_2, M) \land \ldots \land (j_n, M) \). Thus \( j = \varnothing(\mathbf{j}) = \varnothing(\mathbf{j}_i) \),
$M \land \sigma(j_2, M) \land \ldots \land \sigma(j_n, M)$. Since $\sigma$ is permutation invariant, we must have $\sigma(j, M) = j$ for all $1 \leq i \leq n$. Since $j$ is arbitrarily chosen, we must have $\sigma(x) = \min_{1 \leq i \leq n} x_i$.

3.7 Remark. Note that the proof does not depend on any relevance property of the components forming the system. This contrasts with the proofs presented by El. Neweihi et al. (1978) and Griffith (1980) for results similar to Theorem 3.6.

In the theory of multistate coherent structures, the critical upper (lower) vectors to various performance levels play the same basic role that the min path (cut) vectors play in the binary case. For completeness we review their definitions: A vector $x$ is said to be a critical upper (lower) vector to level $j$ if $\sigma(x) \geq (\leq) j$ and $y < (>) x$ implies that $\sigma(y) < (>) j$, where $j \geq 1$ ($j \leq M-1$). Note that the vector $j$ is a critical upper (lower) vector to level $j$ for the SCV (SCX) structure, where $j \geq 1$ ($j \leq M-1$). Indeed any vector $x$ for which $\sigma(x) = \bar{x} \equiv \sum_{i=1}^{n} \frac{x_i}{n}$ is a critical upper (lower) vector to level $\bar{x}$ in the SCV (SCX) class. For a critical upper vector $x$ to level $j$ for SCV structure function $\sigma$, define the following subset of $\{1, \ldots, n\}: P(x) = \{i : x_i \geq j\}$, where $j \geq 1$. Similarly, given a critical lower vector $y$ to level $j$ for SCX structure function $\sigma$, we define $K(x) = \{i : x_i \leq j\}$, where $j \leq M-1$. The set $P(x)$ ($K(x)$) is analogous to the path set (cut set) in the binary case.

For SCV (SCX) structure function $\sigma$, let $\{x_{1}^{(j)}, \ldots, x_{n_j}^{(j)}\}$ be the set of all critical upper (lower) vectors to level $j$, where $j \geq 1$ ($j \leq M-1$). Let $P_{1}^{(j)}, \ldots, P_{n_j}^{(j)} (K_{1}^{(j)}, \ldots, K_{n_j}^{(j)})$ be defined by $P_{\ell}^{(j)} = P(x_{\ell}^{(j)}) (K_{\ell}^{(j)} = K(x_{\ell}^{(j)}))$, where $1 \leq \ell \leq n_j$ and $j \geq 1$ ($j \leq M-1$). The following lemma presents an interesting property for these "path" ("cut") sets.

3.8 Lemma. Let $A^{(j)} (B^{(j)})$ be the set of cardinalities of $P_{\ell}^{(j)} (K_{\ell}^{(j)})$, $1 \leq \ell \leq n_j$, described above, where $j \geq 1$ ($j \leq M-1$). Then $m \in A^{(j)} (B^{(j)})$ implies $(m+1) \in A^{(j)} (B^{(j)})$, where $1 \leq m \leq n$ and $j \geq 1$ ($j \leq M-1$).

Proof. It suffices to prove the lemma for $A^{(j)}$, since by considering duals we observe that $B^{(j)} = A^{(M-j)}$, where $j \leq M-1$.

Let $j \geq 1$ be given and assume $m \in A^{(j)}$, where $1 \leq m \leq n$. We must have $m = \|P(x_{\ell}^{(j)})\|$ for some $1 \leq \ell \leq n$, where $\|E\|$ denotes the cardinality of the set $E$. Without loss of generality assume that $x_{\ell}^{(j)} = (j+k_1, \ldots, j+k_m, j-k_{m+1}, \ldots, j-k_n)$, where $k_1 \geq k_2 \geq \ldots \geq k_m \geq 0$ and $1 \leq k_{m+1} \leq \ldots \leq k_n$. The average of the coordinates of the vector $x_{\ell}^{(j)}$ must be at least $j$. 

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Therefore we have \( k_1 + \ldots + k_m - k_{m+1} > 0 \). Let \( s \geq 0, 0 \leq r \leq m \) be integers such that \( k_1 + \ldots + k_m - k_{m+1} = s(m+1)+r \). Consider the vector \( \mathbf{y} = \left( \frac{j+s+1}{\text{r-times}}, \ldots, \frac{j+s+1}{\text{r-times}} \right) \), \( j-k_{m+2}, \ldots, j-k_n \). Clearly \( \mathbf{x}^{(j)} \leq \mathbf{y} \) and so \( \sigma(\mathbf{y}) \geq j \). A vector \( \mathbf{y}^* \) can be constructed with the following properties: \( \mathbf{y}^* \leq \mathbf{y} \), \( \sigma(\mathbf{y}^*) \geq j \), and if \( y_i > j \) or \( y_i < j \), then any reduction in the \( i \)th coordinate results in deterioration of system performance to a level below \( j \), \( 1 \leq i \leq n \).

Without loss of generality let \( \mathbf{y}^* = (\hat{y}_1, \ldots, \hat{y}_i, j, \ldots, j, y_{m+2}, \ldots, y_n) \), where \( \hat{y}_1 \geq \ldots \geq \hat{y}_i > y_{m+2} \geq \ldots \geq y_n \). Consider the vectors \( \mathbf{y}^{**} = (\hat{y}_1, \ldots, \hat{y}_i, j-1, j, \ldots, y_{m+2}, \ldots, y_n) \) and \( \mathbf{y}^{***} = (\hat{y}_1, \ldots, \hat{y}_i-1, j, \ldots, y_{m+2}, \ldots, y_n) \). Clearly \( \mathbf{y}^{**} \geq \mathbf{y}^{***} \) and consequently \( \sigma(\mathbf{y}^{**}) \leq \sigma(\mathbf{y}^{***}) < j \). This establishes that if \( x < y^* \), then \( \sigma(x) < j \). The vector \( \mathbf{y}^* \) is thus a critical upper vector to level \( j \) and \( \sigma(\mathbf{y}^*) = m+1 \).

This lemma yields the following lower bound on \( n_j \), the number of critical upper vectors to level \( j \) for an SCV structure \( \sigma \), where \( j \geq 1 \). A similar result holds for the number of critical lower vectors to level \( j \leq M-1 \) for an SCX structure \( \sigma \) since \( B^{(j)} = A^{(M-j)} \), where \( A^{(j)}, B^{(j)} \) are described above and \( j \leq M-1 \).

**3.9 Corollary.** Let \( A^{(j)} \) be the set of cardinalities described above, \( m^{(j)} \) be the first element in \( A^{(j)} \), and \( k \) be the number of critical vectors to level \( j \). Then \( k \geq \sum_{a=m^{(j)}}^{n} \binom{n}{a} \).

**Proof.** The proof is immediate by Lemma 3.8 and the fact that an SCV structure is permutation invariant.

Next we derive two upper bounds for the number of critical upper (lower) vectors to level \( j \) corresponding to an SCV (SCX) structure function \( \sigma \) of two components, where \( j \geq 1 \) (\( j \leq M-1 \)). The first of these two upper bounds is simple but crude in the sense that it is not based on the Schur property of the structure function. When this property is utilized, a better upper bound is obtained.

**3.10 Proposition.** Let \( \sigma \) be an SCV structure function of two components. Let \( k \) be the number of critical upper vectors to level \( j \), \( j \geq 1 \). Then the following are successively better upper bounds for \( n_j \):

(a) \( 1 + 2j \ (M-j+1) \),

(b) \( 1 + (M-j) \ (M-j+1) - ((M-2j) \lor 0) \ (M-2j+1) \lor 0) \).
Proof. (a) Let $E = \{ x : x \leq (j - 1) \}$ and $F = \{ x : x > j \}$. In view of conditions (i) and (iii) of Definition 3.1, a critical upper vector to level $j$ must be an element of $(0, 1, \ldots, M)^2 \setminus (E \cup F)$. Hence $\eta_j \leq (M+1)^2 - j^2 - (M-j+1)^2 + 1 = 1 + 2j(M-j+1), j \geq 1$.

(b) Let $E$ and $F$ be as defined in part (a) and let $S_\ell = \{ x \in \{0, 1, \ldots, M\}^2 : x_1 + x_2 = \ell \}, 0 \leq \ell \leq 2M$. Let $C$ be the set of critical upper vectors to level $j, j \geq 1$. By Lemma 3.3 and conditions (i) and (iii) of Definition 3.1, we have $C \subseteq \left( \bigcup_{\ell = 2j}^{M+j-1} S_\ell \right) \setminus F$. It is not difficult to show that the number of elements in $S_{2j+r} \setminus F$ is $1 + 2((M-j) - (M-2j) \lor 0)$ for $r = 0$ and is $2((M-j-r) - (M-2j-r) \lor 0)$ for $r = 1, \ldots, M-j-1, j \geq 1$. The conclusion now follows by summing over $r$.

3.11. Corollary. Let $\sigma$ be an SCX structure function of two components. Let $n_j$ be the number of critical lower vectors to level $j, j \leq M-1$. The following are successively better upper bounds for $n_j$.

(a) $1 + 2(M-j)(j+1)$,
(b) $1 + j(j+1) - (2j-M) \lor 0) (2j-M+1) \lor 0)$.

Proof. The proof follows immediately from Proposition 3.10 by considering duals. The details are omitted.

The following example attests to the sharpness of the upper bound in (b) of Proposition 3.10.

3.12. Example. Let $M = 2$. Define $\sigma$ by $\sigma(0,0) = \sigma(1,0) = \sigma(0,1) = 0, \sigma(2,0) = \sigma(0,2) = \sigma(1,1) = \sigma(1,2) = \sigma(2,1) = 1, \sigma(2,2) = 2$. Then the vectors $(2,0), (1,1)$ and $(0,2)$ are the critical upper vectors to level 1. Of course $(2,2)$ is the only critical upper vector to level 2.

Finally, we show that the series (parallel) structure $\sigma$ is the only structure of the Barlow-Wu type (i.e., $\sigma$ has the representation $\sigma(x) = \max_{i \in P_j} \min_{s \in S_r} x_{p_i}$, where $P_1, \ldots, P_r$ are the minimal path sets of some binary coherent system) that is SCV (SCX).

3.13. Proposition. Let $\sigma : \{0, 1, \ldots, M\}^n \rightarrow \{0, 1, \ldots, M\}$ be a structure function with the representation $\sigma(x) = \max_{i \in P_j} \min_{s \in S_r} x_{p_i}$, where $P_1, \ldots, P_r$ are the minimal path sets of some binary coherent system $\sigma'$. Then $\sigma$ is SCV (SCX) if and only if $\sigma'$ is a series (parallel) structure.

Proof. Obviously the series (parallel) structure is of the Barlow-Wu type and also is SCV (SCX).

Next let $\sigma$ be SCV (SCX) having the above representation. The binary structure $\sigma'$
must be permutation invariant and consequently $\varnothing$ is a $k$-out-of-$n$ system for some $k$. It follows that $\varnothing(x) = x_{(n-k+1)}$ where $x_{(i)}$ is the $i$th value in the increasing ordering of $x_1,...,x_n$. For $\varnothing$ to be also Schur-concave (Schur-convex), $k$ must be $n$ (1). \[ \]

**Stochastic properties.**

In Section 3 we have studied deterministic aspects of SCV (SCX) structures. In this section, we consider some of their stochastic properties. In particular, we obtain bounds which can be used when exact system performance is difficult to evaluate. We also use the expected performance function to characterize the series (parallel) structures. Finally we illustrate by an example the use of a well known preservation theorem due to Proschan and Sethuraman (1977) to show that the Schur property is inherited by the expected performance function.

Let $X_i$ denote the random state of component $i$, with $P[X_i = j] = p_{ij}$. $P[X_i \leq j] = P_i(j)$ and $\overline{P}_i(j) = 1 - P_i(j)$, where $j = 0,1,...,M$ and $i = 1,...,n$. $P_i$ represents the performance distribution of component $i$.

Let $X = (X_1,...,X_n)$ be the random vector representing the states of components $1,...,n$, where the $X_1,...,X_n$ are assumed to be mutually independent. Then $\varnothing(X)$ is the random variable representing the state of an SCV (SCX) structure, with $P[\varnothing(X) = j] = p_j$, $P[\varnothing(X) \leq j] = P_\varnothing(j)$, and $\overline{P}_\varnothing(j) = 1 - P_\varnothing(j)$, $j = 0,1,...,M$. $P_\varnothing$ represents the performance distribution of the system and $P_\varnothing = P_\varnothing(\overline{P}_1,...,\overline{P}_n)$. Let the expected performance function of the system be defined by $h_\varnothing = h_\varnothing(\overline{P}_1,...,\overline{P}_n) = E\varnothing(X)$.

The following proposition gives a lower (upper) bound on the performance distribution of an SCV (SCX) structure.

**4.1. Proposition.** Let $P$ be the performance distribution of an SCV (SCX) structure function $\varnothing$. Then

$$ P_\varnothing(j) \geq (\leq) \sum_{i=1}^{j_1} \cdot \sum_{i=1}^{j_n} \left( \prod_{i=1}^{n} P_{ij_i} \right) $$

where $p_{ij_i} = P[X_i = j_i], \ i = 1,...,n$, and $j_1,...,j_n$ in the summation satisfy $\sum_{i=1}^{n} j_i \leq nj$.

**Proof.** The proof follows immediately from Lemma 3.3. The details are therefore omitted. \[ \]

Let $\varnothing$ be an SCV (SCX) structure function. Let $M$ be as defined in Corollary 3.9 where $j \geq 1$ ($j \leq M-1$). The following proposition gives another lower (upper) bound for the performance distribution of the system $\varnothing$ when its components are identically distributed.
4.2 Proposition. Let $P_\varnothing$ be the performance distribution of an SCX (SCV) structure function $\varnothing$ whose components are identically distributed with common performance distribution $P$. Then

$$P_\varnothing(j) \geq \sum_{k=1}^{M(j)-1} \binom{n}{k} (P(j))^k (P(j))^{n-k} \left( \leq \sum_{k=m(j)}^{n} \binom{n}{k} (P(j))^k (P(j))^{n-k} \right)$$

where $j \geq 1$ ($j \leq M - 1$).

Proof. Let $\varnothing$ be an SCV structure. In view of Corollary 3.9 the random vector $\varnothing(X)$ is at state $j$ or better implies that at least $m(j)$ of its components are at state $j$ or better where $j \geq 1$. Using the fact that the components are mutually independent and identically distributed, we get the required lower bound. The upper bound for an SCX structure is obtained by using a duality argument.

In what follows we denote by $P_\sim$ the vector $(P_1, \ldots, P_n)$, where $P_i = 1 - P_i$ and $P_i$ is the performance distribution for component $i$, $i = 1, \ldots, n$. Given two such vectors $P_\sim, P'_\sim$ let

$$P_\sim \cdot P'_\sim = (P_1 \cdot P'_1, \ldots, P_n \cdot P'_n)$$

and $P_\sim \lor P'_\sim = (P_1 \lor P'_1, \ldots, P_n \lor P'_n)$ where $P_i \lor P'_i = 1 - P_i \cdot P'_i$. We also denote the expected performance function of a series (parallel) system of two components by $h_\land(h_\lor)$.

The following proposition characterizes the series (parallel) structure within the SCV (SCX) class in terms of the expected performance function.

4.3 Proposition. Let $h_\varnothing(P_\sim)$ be the expected performance function of a Schur structure $\varnothing$. Then

(a) $\varnothing$ is an SCV structure and $h_\varnothing(P_\sim \cdot P'_\sim) = h_\land(P_\varnothing(P_\sim), P_\varnothing(P'_\sim))$ for all $P_\sim, P'_\sim$ iff $\varnothing$ is series.

(b) $\varnothing$ is an SCX and $h_\varnothing(P_\sim \lor P'_\sim) = h_\lor(P_\varnothing(P_\sim), P_\varnothing(P'_\sim))$ for all $P_\sim, P'_\sim$ iff $\varnothing$ is parallel.

Proof. (a) We only prove the "only if" part since the other implication is straightforward. Assume that $\varnothing$ is an SCV structure. Let $X_i(X'_i)$ be the random variable representing the state of component $i$ with performance distribution $P_i(P_i')$. Assume that $X_1, \ldots, X_n, X'_1, \ldots, X'_n$ are mutually independent. Clearly $h_\varnothing(P_\sim \cdot P'_\sim) = E\varnothing(X_\land X'_\land)$ and
\( h(\tilde{P}, \tilde{P}') = E(\sigma(\tilde{X}) \land \sigma(\tilde{X}')). \) So we must have \( E[\sigma(\tilde{X} \land \tilde{X}') - \sigma(\tilde{X}) \land \sigma(\tilde{X}')] = 0 \) for all \( \tilde{P} \) and \( \tilde{P}' \). This implies that \( \sigma(\tilde{X} \land \tilde{X}') = \sigma(\tilde{X}) \land \sigma(\tilde{X}') \) for all \( \tilde{X} \) and \( \tilde{X}' \). In view of Theorem 3.6, \( \sigma \) must be the series structure. (b) follows easily from (a) by using a duality argument.

Finally we illustrate the use of a well known preservation theorem due to Proschlan and Sethuraman (1977) to show that the Schur property can be inherited by the expected performance function.

Let \( \sigma \) be an SCV (SCX) structure and \( h(\sigma) \) be its expected performance function. Assume that \( X_i \), the random state of component \( i \), has a binomial distribution with parameters \( M_i, p \), \( i=1, \ldots, n \). Observe that \( E\sigma(\tilde{X}) = \sum_{\tilde{X}} \sigma(\tilde{X}) \left( \prod_{i=1}^{n} (\frac{M}{X_i})^p (1-p)^{M-x} \right) \) may now be viewed as a function of \( M_1, \ldots, M_n \). Thus we may write \( h(\sigma) = h(\sigma(M_1, \ldots, M_n)) \). It can be shown that \( (\frac{M}{X})^p (1-p)^{M-x} \) is a TP2 function in \( M \), \( x \) which satisfies the semigroup property in \( M \) (see Section 2). By Theorem 1.1 of Proschlan and Sethuraman (1977) \( h(\sigma(M_1, \ldots, M_n)) \) is an SCV (SCX) function.

The following is an example illustrating the preceding situation.

**4.4. Example.** Consider a binary parallel system \( S_i \) consisting of \( M_i \) binary components whose lifetimes are represented by \( T_{i1}, \ldots, T_{iM_i} \), \( i=1, \ldots, n \). Assume that all the random variables are mutually independent with common distribution \( F \). At any fixed point in time \( t_0 \) the number of working components in \( S_i \) is a binomial random variable with parameters \( M_i \) and \( p \), where \( p = \overline{F}(t_0) \) and \( i=1, \ldots, n \). The system \( S_i \) may now be viewed as a multistate component whose state \( X_i \) is the number of working components in \( S_i \), \( i=1, \ldots, n \). Let \( M = \max(M_1, \ldots, M_n) \) and note that \( p_{ij} = P(X_i=j) = 0 \) for \( j=M_i+1, \ldots, M \). The \( n \) multistate components now form a multistate system whose structure function \( \sigma \) is a Schur function.
References


