REDUNDANCY IMPORTANCE AND ALLOCATION OF SPARES IN COHERENT SYSTEMS

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Abstract

We study the model in which a set of spares is available for redundancy in a coherent system. In some circumstances, parallel (or active) redundancy is used to improve the reliability of the system, while in others series redundancy is used to improve a different measure of utility. Hence we define the two concepts of parallel and series redundancy importance for components in a coherent system relative to an available set of redundant spares. These measures of importance are compared with the structural importance and reliability importance of a component. Various results for the optimal allocation of redundant spares are given, with particular reference to $k$ out of $n$ systems and modules of coherent systems.
Redundancy Importance and Allocation of Spares in Coherent Systems

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1. Introduction.

In this paper we discuss the concept of 'redundancy importance' for components in a coherent system, and compare it to other well known measures of component importance. Generally speaking, we will be using this concept to optimize the allocation of one or more redundant spare components to the coherent system. In some situations, we will be implementing parallel (or active) redundancies to the system to improve system reliability. In other instances, series redundancy will be appropriate in order to decrease the probability of occurrence of an undesirable outcome. Hence we define the two concepts of parallel and series redundancy importance for components in a coherent system relative to an available set of redundant spares. These measures of importance are compared with the structural importance of a component, and the reliability (or Birnbaum) importance of a component. In Section 2, particular optimization results are given with respect to \( k \) out of \( n \) systems, while in Section 3 we study component redundancy in the more general setting of modules of coherent systems. The results in this paper summarize and extend those in Boland, El-NEWEIH and ProSchan (1988).

For the most part, we use the notation and terminology for coherent systems and modules of coherent systems developed by Barlow and Proschan (1981). We let \( (C, \phi) \) denote a coherent system with components \( C = \{c_1, \ldots, c_n\} \) and structure function \( \phi \). We assume that the components of the system act independently of each other. To indicate the state of component \( c_i \), the binary indicator variable \( x_i \) takes the value 1 if \( c_i \) is functioning and 0 if failed for \( i = 1, \ldots, n \). The state of the coherent system is determined by the vector \( \mathbf{x} = (x_1, \ldots, x_n) \) and the structure function \( \phi \). Let \( \mathbf{p} = (p_1, \ldots, p_n) \) represent the vector of component reliabilities, and let \( h(\mathbf{p}) = E(\phi(\mathbf{x})) \) be the reliability function of the coherent system \( (C, \phi) \). For any two numbers \( x, y \) we let \( x \ll y = 1 - (1 - x)(1 - y) \) and \( x \ll y = xy \), while for vectors \( \mathbf{x}, \mathbf{y} \) we let \( \mathbf{x} \ll \mathbf{y} = (x_1 \ll y_1, \ldots, x_n \ll y_n) \) and \( \mathbf{x} \ll \mathbf{y} = (x_1 y_1, \ldots, x_n y_n) \).

Various measures have been proposed for component importance in coherent systems. The structural importance of component \( c_i \) is defined by

\[
I_{\phi}(i) = n_{\phi}(i)/2^{n-1} = \left( \# \text{ critical path vectors for } c_i \right)/2^{n-1}
= \sum_{\mathbf{x}, x_i = 1} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})]/2^{n-1}.
\]
It is clear that this measure of component importance does not depend on the vector \( p = (p_1, \ldots, p_n) \) of component reliabilities. In a \( k \) out of \( n \) system (the system functions iff at least \( k \) components function) the structural importance of each component is \( \binom{n-1}{k-1}/2^{n-1} \). The (Birnbaum) reliability importance of component \( c_i \) is given by \( I_h(i) = \frac{\partial h}{\partial p_i} = h(1,i,p) - h(0,i,p) \). This measure of component importance for \( c_i \) depends on both \( \phi \) and \( p \) (but not \( p_i \) itself). Boland and Proschan (1983) show for a \( k \) out of \( n \) system in which \( p_1 \leq \cdots \leq p_n \) that: (a) \( p_i \geq \frac{k-1}{n-1} \) for all \( i \mapsto I_h(1) \leq \cdots \leq I_h(n) \) and (b) \( p_i \leq \frac{k-1}{n-1} \) for all \( i \mapsto I_h(1) \geq \cdots \geq I_h(n) \).

In a coherent system of independent components, when the reliability of component \( i \) is increased (decreased) from \( p_i \) to \( p_i + \Delta(p_i - \Delta) \), the reliability of the entire system is increased (decreased) by the amount \( \Delta \frac{\partial h}{\partial p_i} \). Hence if components are ordered with respect to this measure, they are ordered with respect to Birnbaum reliability importance. However there are reliability operations on components (such as the addition of a parallel or series redundancy) which do not produce a common increase (or decrease) \( \Delta \) in component performance. For example, if we connect in parallel (series) to component \( c_i \) an independent component \( c_i^* \) of reliability \( p_i^* \), then the increase (decrease) in reliability at ‘position \( i \)’ is \( \Delta_i = p_i \Pi p_i^* - p_i = p_i^* q_i(\Delta_i = p_i - p_i \Pi p_i^* = q_i^* p_i) \). We will see that even in the situation where the available spares \( \{c_1^*, \ldots, c_n^*\} \) are each of reliability \( p^* \), the component which is most important in Birnbaum’s sense is not necessarily the one which would result in the greatest increase (decrease) in system reliability if a parallel (series) redundancy were to be performed on it. In order to study the importance of components when performing parallel (series) redundancies on a coherent system, we give the following definitions of redundancy importance:

**Definition 1.1.** Let \( (C, \phi) \) be a coherent system of independent components where \( p = (p_1, \ldots, p_n) \) is the vector of component reliabilities. Let \( C^* = \{c_1^*, \ldots, c_n^*\} \) denote a set of spares with respective reliabilities \( p = (p_1^*, \ldots, p_n^*) \) available for redundancy in the coherent system. Normally we will be interested in either parallel or series redundancy, where \( c_i^* \) may be put in redundancy with \( c_i \). The parallel (series) redundancy importance \( I_{\Pi}(i)(I_{\Pi}(i)) \) of component \( i \) is defined to be the increase (decrease) in reliability of the system which results from putting \( c_i^* \) in parallel (series) redundancy with \( c_i \). Hence

\[
I_{\Pi}(i) = h(p_1, \ldots, p_i \Pi p_i^*, \ldots, p_n) - h(p_1, \ldots, p_n) = (p_i \Pi p_i^* - p_i)I_h(i) = p_i^* q_i I_h(i),
\]

and

\[
I_{\Pi}(i) = h(p_1, \ldots, p_n) - h(p_1, \ldots, p_i \Pi p_i^*, \ldots, p_n) = (p_i - p_i \Pi p_i^*)I_h(i) = q_i^* p_i I_h(i).
\]

Note that the (parallel or series) redundancy importance of a component \( c_i \) depends on its reliability \( p_i \), the reliability \( p_i^* \) of the available redundant component \( c_i^* \), and the (Birnbaum) reliability importance \( I_h(i) \).
Generally speaking, we will usually be interested in maximizing (minimizing) system reliability subject to the allocation of one or more parallel (series) redundant components to the system. Possible models when considering parallel redundancy are:

(a) a single spare component $c^*$ of reliability $p^*$ is available for parallel redundancy with any of the components in a coherent system (e.g., in a $k$ out of $n$ system where components are structurally equivalent). In this situation we are interested in maximizing

$$I_{II}(i) = p^* q_i I_h(i).$$

(b) a set of spare components $C^*$ with respective reliabilities identical to those already in the system ($p_i^* = p_i$ for all $i$) is available for parallel redundancy. If one allocation is to be made we are interested in maximizing

$$I_{II}(i) = p_i q_i I_h(i).$$

Example 1.2.

Consider the relatively simple coherent system (see Figure 1 above) with reliability function $h(p) = h(p_1, p_2, p_3) = p_1 \Pi(p_2 \Pi p_3) = p_1 (1 - (1 - p_2)(1 - p_3))$. Suppose that $p_1 = .9, p_2 = .7$ and $p_3 = .6$. Then it is clear that component 1 has the greatest structural and (Birnbaum) reliability importance. Suppose however that we wish to make one parallel redundancy allocation at one of the component positions so as to maximize system reliability. Where should this allocation be made? Which component is the most important in this sense? Relative to the two models previously discussed, we have

(a) When one component $c^*$ with reliability $p^*$ is available (here $c_1^* = c_2^* = c_3^*$), then the component $c^*$ should be made redundant with $c_2$. 

Figure 1
(b) When for each position a component with corresponding reliability is available for redundancy \((p_1^* = .9, p_2^* = .7, p_3^* = .6)\), then component \(c_1^*\) should be made redundant with \(c_1\).

2. Component Redundancy in \(k\) out of \(n\) systems.

For a given vector \(p = (p_1, \ldots, p_n)\) of \(n\) component reliabilities, the reliability function of a \(k\) out of \(n\) system (of independent components) is given by

\[
h_{k|n}(p) = \sum_{\epsilon, \epsilon \geq k} \Pi_{i=1}^{n} p_i^{\epsilon_i} q_i^{1-\epsilon_i}
\]

where each \(\epsilon = (\epsilon_1, \ldots, \epsilon_n)\) is a vector of zeros and ones and \(\epsilon = \epsilon_1 + \cdots + \epsilon_n\). A \(1\) out of \(n\) system is a parallel system with reliability function \(h_{1|n}(p) = \Pi_{i=1}^{n} p_i\), and an \(n\) out of \(n\) system is a series system with reliability function \(h_{n|n}(p) = \Pi_{i=1}^{n} p_i\).

Note generally that \(h_{k|n}(p) = 1 - h_{n-k+1|n}(q)\), and hence we have the following duality relationship between parallel and series redundancy in \(k\) out of \(n\) systems:

\[
h_{k|n}(p \parallel p^*) = 1 - h_{n-k+1|n}(q \parallel q^*).
\]

The concepts of arrangement increasing (AI) and arrangement decreasing (AD) functions are useful in determining the optimal allocation of spare components in \(k\) out of \(n\) systems.

**Definition 2.1.** For a given vector \(x = (x_1, \ldots, x_n) \in R^n\), we let \(x \downarrow = (x_{[1]}, \ldots, x_{[n]})\) and \(x \uparrow = (x_{[n]}, \ldots, x_{[1]})\) be respectively the vectors with the components of \(x\) arranged in decreasing (increasing) order. For any permutation \(\pi\) of \(\{1, \ldots, n\}\), we let \(x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})\). For vectors \(x, y, u, v\) in \(R^n\), we write \((x, y)^{\downarrow}_{\downarrow}(u, v)\) if there exists a permutation \(\pi\) of \(\{1, \ldots, n\}\) such that \(x_\pi = u\) and \(y_\pi = v\). We define \((x, y)^{\uparrow}_{\downarrow}(u, v)\) if there exists a finite number of vectors \(z^1, \ldots, z^k\) such that (i) \((x, y)^{\downarrow}_{\downarrow}(x \uparrow, z^1)\) and \((x \uparrow, z^k)^{\downarrow}_{\downarrow}(u, v)\) and (ii) \(z^{i-1}\) can be obtained from \(z^i\) by an interchange of two components of \(z^i\), the first of which is less than the second. For example,

\[
((4, .3, 2, 1), (0, 2, 3, 5))^{\downarrow}_{\downarrow} ((1, 2, 3, 4), (0, 2, 3, 5))
\]

since

\[
((4, .3, 2, 1), (0, 2, 3, 5))^{\downarrow}_{\downarrow} ((1, 2, 3, 4), (5, 3, 2, 0))
\]

\[
^{\downarrow}_{\downarrow} ((1, .2, 3, 4), (0, 3, 2, 5))
\]

\[
^{\downarrow}_{\downarrow} ((1, .2, 3, 4), (0, 2, 3, 5)).
\]

**Definition 2.2.** A function \(g\) of two vector arguments for which \(g(x, y) \leq g(u, v)\) whenever \((x, y)^{\downarrow}_{\downarrow}(u, v)\) is said to be arrangement increasing (AI) by Marshall and Olkin (1979), and
decreasing in transposition (DT) by Hollander, Proschan and Sethuraman (1977). We shall use the terminology arrangement increasing (AI) in order to emphasize that such a function \( g(x, y) \) increases in value as the arrangement of components in \( y \) becomes increasingly similar to the arrangement of components in \( x \). A function \( g \) is said to be arrangement decreasing or AD if \( -g \) is AI.

The following theorem is proved in Boland, El-Neweihi and Proschan (1988):

**Theorem 2.3.** Let \( p = (p_1, \ldots, p_n) \) and \( p^* = (p_1^*, \ldots, p_n^*) \) be two given vectors of component reliabilities. Then for any \( k = 1, \ldots, n, \)

\[
(a) \quad g_n(p, p^*) = h_{k|n}(p \sqcup p^*) \text{ is arrangement decreasing, and}
\]

\[
(b) \quad g_n(p, p^*) = h_{k|n}(p \Pi p^*) \text{ is arrangement increasing.}
\]

**Corollary 2.4.** Let \( p = (p_1, \ldots, p_n) \) be the vector of component reliabilities of a \( k \) out of \( n \) system where \( p_1 \leq p_2 \leq \ldots \leq p_n \). Then

\[
q_1 I_h(1) \geq q_2 I_h(2) \geq \ldots \geq q_n I_h(n)
\]

and

\[
p_1 I_h(1) \leq p_2 I_h(2) \leq \ldots \leq p_n I_h(n).
\]

**Proof:** Let \( \mathbf{1} = (1, 1, \ldots, 1) \) and \( \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 in the \( i \)th position for \( i = 1, \ldots, n \). For a given \( p^* \in (0, 1) \), it follows from Theorem 2.3 that

\[
h_{k|n}(p \sqcup p^* \mathbf{e}_i) - h_{k|n}(p) = p^* q_i I_h(i) \geq p^* q_j I_h(j) = h_{k|n}(p \sqcup p^* \mathbf{e}_j) - h_{k|n}(p)
\]

and

\[
h_{k|n}(p) - h_{k|n}(p \Pi (1 - q^* \mathbf{e}_i)) = q^* p_i I_h(i) \leq q^* p_j I_h(j) = h_{k|n}(p) - h_{k|n}(p \Pi (1 - q^* \mathbf{e}_j))
\]

whenever \( i < j \).

**Remark 2.5.** Corollary 2.4 implies that whenever a single component \( c^* \) with reliability \( p^* \) is available for parallel redundancy with any one component in a \( k \) out of \( n \) system, then the maximum improvement in system reliability is obtained by making it redundant with the weakest component. This result is somewhat surprising when compared with the result of Boland and Proschan (1983) which states that in a \( k \) out of \( n \) system the strongest component is the most important (in the sense of Birnbaum) whenever all \( p_i \geq \frac{k-1}{n-1} \), while the weakest is most important (in the sense of Birnbaum) whenever all \( p_i \leq \frac{k-1}{n-1} \).
Remark 2.6. If we have available for active redundancy in a \( k \) out of \( n \) system a set of \( n \) spares \( C^* = (c_1^*, \ldots, c_n^*) \) with reliabilities identical to those already in the system (\( p_i^* = p_i \) for \( i = 1, \ldots, n \)), then the component with the maximum parallel redundancy importance is that \( c_i^* \) maximizing \( p_i q_i I_h(i) \). Although the sequences \( \{q_i I_h(i)\} \) and \( \{p_i I_h(i)\} \) are both monotone in \( i \) whenever \( p_1 \leq p_2 \leq \ldots \leq p_n \), the same cannot be said in general about \( \{p_i q_i I_h(i)\} \). One may, for example, construct 2 out of 3 systems where the component with the greatest parallel redundancy importance may be either the most reliable, the least reliable, or neither of these.

We now prove a corollary to Theorem 2.3 which may be helpful when deciding between expanding a \( k \) out of \( n \) system and improving the already existing system by means of a redundancy.

**Corollary 2.7.** Let \( p = (p_1, \ldots, p_n) \) and \( p^* = (p_1^*, \ldots, p_n^*) \) be probability vectors of dimensions \( n \) and \( r (r \leq n) \) respectively. Then

\[
(a) \quad h_{k+n+r}(p_1, \ldots, p_n, p_1^*, \ldots, p_r^*) \geq h_{k+n}(p_1 \Pi p_1^*, \ldots, p_r \Pi p_r^*, p_{r+1}, \ldots, p_n) \quad \text{and}
\]

\[
(b) \quad h_{k+r+n+r}(p_1, \ldots, p_n, p_1^*, \ldots, p_r^*) \leq h_{k+n}(p_1 \Pi p_1^*, \ldots, p_r \Pi p_r^*, p_{r+1}, \ldots, p_n)
\]

for \( k = 1, \ldots, n \).

**Proof:**

(a) \[ h_{k+n+r}(p_1, \ldots, p_n, p_1^*, \ldots, p_r^*) = h_{k+n+r}((p_1, \ldots, p_n, 0, \ldots, 0) \Pi (0, \ldots, 0, p_1^*, \ldots, p_r^*)) \]

\[ \geq h_{k+n+r}((p_1, \ldots, p_n, 0, \ldots, 0) \Pi (p_1^*, \ldots, p_r^*, 0, \ldots, 0)) \]

(by Theorem 2.3a)

\[ = h_{k+n}(p_1 \Pi p_1^*, \ldots, p_r \Pi p_r^*, p_{r+1}, \ldots, p_n). \]

(b) \[ h_{k+r+n+r}(p_1, \ldots, p_n, p_1^*, \ldots, p_r^*) = h_{k+r+n+r}((p_1, \ldots, p_n, 1, \ldots, 1) \Pi (1, \ldots, 1, p_1^*, \ldots, p_r^*)) \]

\[ \leq h_{k+r+n+r}((p_1, \ldots, p_n, 1, \ldots, 1) \Pi (p_1^*, \ldots, p_r^*, 1, \ldots, 1)) \]

(by Theorem 2.3b)

\[ = h_{k+n}(p_1 \Pi p_1^*, \ldots, p_r \Pi p_r^*, p_{r+1}, \ldots, p_n). \]

Alternatively, (b) may be proved by using part (a) together with the duality relationship (2.2).

Given a \( k \) out of \( n \) system with component reliabilities \( p_1, \ldots, p_n \), let us suppose we have an additional component with reliability \( p^* \) which may be used to improve the system in two different ways. On the one hand, it may be put in active (parallel) redundancy with one of the components already in the system. On the other hand, it may be added to the system in such a way that the result is a \( k \) out of \( n + 1 \) system with (independent) component reliabilities \( p_1, \ldots, p_n, p^* \).
Corollary 2.7 states that the later option is always preferable. For example, suppose a politician has campaigned in \( n \) (of \( N \)) areas in an electoral district. Success will result if the politician wins \( k \leq n \) or more of the areas. Faced with the option of either revisiting an area (an analogue of active redundancy) or visiting another area for the first time, the above corollary suggests that the later action is preferable.

**Remark 2.8.** Suppose now we have a \( k \) out of \( n \) system with components \( c_1, \ldots, c_n \) with respective reliabilities \( p = (p_1, \ldots, p_n) \) where \( p_1 \leq \ldots \leq p_n \). Assume that \( r(r \leq n) \) spare components \( c_1^*, \ldots, c_r^* \) with respective reliabilities \( p_1^* \leq \ldots \leq p_r^* \) are available for redundancy with any of the \( r \) components in the system. It follows from Theorem 2.3 that for parallel redundancy, the optimal allocation (giving the greatest improvement in system reliability) results from making \( c_r^* \) redundant with \( c_1, \ldots, c_{r-1}^* \) redundant with \( c_r \). On the other hand for series redundancy, the greatest reduction in reliability is obtained by making \( c_r^* \) redundant with \( c_{n-r+1}, \ldots, c_1^* \) redundant with \( c_n \).

Many examples are given in Boland, El-Neweihi and Proschan (1988) of practical situations where redundancy in \( k \) out of \( n \) systems may be considered. The following examples are typical:

**Example 2.9.**

(a) (Parallel Redundancy). Weather instruments are located in \( n \) areas of a small country, and signals are periodically sent to a central weather station. An accurate picture of the current weather may be obtained if at least \( k \) of the instruments are accurately relaying information. \( r(\leq n) \) new devices are obtained and are to be distributed to \( r \) of the locations to improve local weather detection. The allocation is to be made in order to maximize the probability that the central station obtains a true picture of the weather.

(b) (Series Redundancy) \( n \) patrol boats are policing a coastal area to stop drug trafficking. Let \( p_i \) = probability that a drug transporter can pass undetected by patrol boat \( i \). The probability of a drug trafficker's mission being successful is then the reliability of a 1 out of \( n \) (parallel) system. We have \( r \) extra detection devices to distribute to \( r \) of the boats, and wish to do so in such a way that we minimize the chances of a drug trafficker slipping through.

Now consider the problem of allocating \( \ell \) identical spares of reliability \( p^* \) to a \( k \) out of \( n \) system where the components each have reliability \( p \). If we are implementing parallel redundancy where \( m_1 \) spares are allocated to location 1, \ldots, \( m_n \) are allocated to location \( n \), where \( \ell = m_1 + \cdots + m_n \), then the reliability of the resulting system would be

\[
R(m_1, \ldots, m_n) = h_{k|n}(p \prod_{i=1}^{m_1}(p^*), \ldots, p \prod_{i=1}^{m_n}(p^*)).
\]

For series redundancy, the reliability would be

\[
S(m_1, \ldots, m_n) = h_{k|n}(p^{\ell m_1}, \ldots, p^{\ell m_n}).
\]
Boland, El-Neweihi and Proschan (1988) show that $R(m_1,\ldots,m_n)$ is a Schur concave function of $(m_1,\ldots,m_n)$ for any $k \geq 1$; that is the reliability of the allocation increases more the more equally the spares are distributed among the $n$ positions. In a similar manner, one may show that $S(m_1,\ldots,m_n)$ is a Schur convex function (that is, increases in value as the spares become more unequally distributed).

We now proceed to introduce an added degree of randomness to some of our results. Suppose for example that $P^* = (P_1^*,\ldots,P_n^*)$ is a random vector of component reliabilities with density $f(\lambda,p^*)$. Here $\lambda$ is a parameter vector. If $f(\lambda,p^*)$ is arrangement increasing in $\lambda$ and $p$, then by the composition theorem for arrangement increasing functions (see Hollander, Proschan and Sethuraman [1977] for a proof as well as examples of AI densities) it follows that:

**Theorem 2.11.** Let $P^*$ be a random vector with arrangement increasing density $f(\lambda,p^*)$, and let $p = (p_1,\ldots,p_n)$ be a vector of component reliabilities. Then

(a) \[ g_n(\lambda,p) = \int h_{k|n}(p \Pi p^*)f(\lambda,p^*)dp^* \text{ is arrangement decreasing} \]

and

(b) \[ g_n(\lambda,p) = \int h_{k|n}(p\Pi p^*)f(\lambda,p^*)dp^* \text{ is arrangement increasing.} \]

As an application of this result, suppose manufacturer $i$ produces component $i$ with random reliability $P_i^*$ where $P_i^* \leq \cdots \leq P_n^*$. Then, when parallel redundancy is to be employed, the optimal strategy is to match standby spare $i$ with component $n+1-i$, for $i = 1,\ldots,n$ (where $p_1 \leq \cdots \leq p_n$). This will yield the largest expected system reliability for $k$ out of $n$ systems. The smallest expected system reliability will be achieved by matching standby spare $i$ with component $i$.

We next present a dynamic version of Theorem 2.3. Let $T_1 \leq \cdots \leq T_n$ be the independent random variables representing the lifetimes of $n$ components forming a $k$ out of $n$ system. Similarly let $T_1^* \leq \cdots \leq T_n^*$ be the random variables representing the lifetimes of $n$ spare components available for parallel (respectively series) redundancy with the $n$ components in the system. We are interested in allocating one spare to each position. Let $\pi$ and $\pi^*$ be two permutations of $\{1,2,\ldots,n\}$ and let $v_{k|n}$ represent the lifetime function for a $k$ out of $n$ system. Let $T_\pi \vee T_\pi^* = (T_{\pi(1)} \vee T_{\pi^*(1)},\ldots,T_{\pi(n)} \vee T_{\pi^*(n)})$ and $T_\pi \wedge T_\pi^*$ be defined in a similar fashion, where $\vee$ denotes maximum and $\wedge$ denotes minimum.

**Theorem 2.12.** Let $f$ be any increasing function on $R$. Then

(a) \[ g_\vee(\pi,\pi^*) = E(f(v_{k|n}(T_\pi \vee T_\pi^*))) \text{ is arrangement decreasing,} \]

and

(b) \[ g_\wedge(\pi,\pi^*) = E(f(v_{k|n}(T_\pi \wedge T_\pi^*))) \text{ is arrangement increasing.} \]
PROOF: (a) Suppose \( (\pi, \pi^*) \leq_\pi (\pi_1, \pi^*_1) \). Then for any \( t \geq 0 \),

\[
\begin{align*}
\text{Prob}(\nu_k|\pi(T_\pi \vee T_{\pi^*}) > t) &= h_{k|n}[\bar{F}_{\pi(1)}(t), \ldots, \bar{F}_{\pi(n)}(t)] \Pi (\bar{F}_{\pi^*(1)}(t), \ldots, \bar{F}_{\pi^*(n)}(t)) \\
&\geq h_{k|n}[\bar{F}_{\pi_1(1)}(t), \ldots, \bar{F}_{\pi_1(n)}(t)] \Pi (\bar{F}_{\pi^*_1(1)}(t), \ldots, \bar{F}_{\pi^*_1(n)}(t)) \quad \text{by Theorem 2.3},
\end{align*}
\]

where \( T_i \sim F_i \) and \( T_{i^*} \sim F_{i^*}^* \). Hence

\[
v_{k|n}(T_\pi \vee T_{\pi^*}) \leq v_{k|n}(T_{\pi_1} \vee T_{\pi^*_1}),
\]

proving (a). The proof of (b) is similar.

As a consequence, results that previously were proved for a fixed point in time are true stochastically. For instance if \( T_1 \leq_{st} \cdots \leq_{st} T_n \) and we are allowed to make one parallel (respectively series) redundancy with lifetime \( T^* \), then

\[
v_{k|n}(T_1 \vee T^*, T_2, \ldots, T_n) \geq v_{k|n}(T_1, \ldots, T_{n-1}, T_n \vee T^*)
\]

(respectively \( v_{k|n}(T_1 \wedge T^*, T_2, \ldots, T_n) \leq_{st} v_{k|n}(T_1, \ldots, T_{n-1}, T_n \wedge T^*) \)). For example we might have \( T_i \sim \text{Exp}(\lambda_i), i = 1, \ldots, n \) where \( \lambda_1 \geq \cdots \geq \lambda_n \) and \( T^* \sim \text{Exp}(\lambda^*) \).

We conclude this section with an extension of our results to multistate systems. We now assume that each of the \( n \) components as well as the system itself can be in any one of \( M + 1 \) states denoted by \( \{0, 1, \ldots, M\} \), where 0 represents total failure and \( M \) perfect functioning. Suppose \( X_1, \ldots, X_n \) represent the random states of the \( n \) components and that \( X_1 \leq_{st} \cdots \leq_{st} X_n \). We let \( Y_{ij} = 1 \iff X_i \geq j \) be a binary random variable indicating whether or not component \( i \) is functioning at level \( j \) or better. Suppose \( 1 \leq k_1 \leq \cdots \leq k_M \leq n \) are \( M \) given integers and let \( \phi_j \) be a binary \( k_j \) out of \( n \) system, \( j = 1, \ldots, M \). We define \( \phi(X) = \sum_{j=1}^{M} \phi_j(Y_j) \) to be the structure function representing the state of our system where \( Y_j = (Y_{1j}, \ldots, Y_{nj}) \). The structure function \( \phi \) defined here belongs to a class of multistate systems defined by Natvig (1982). We now let \( X_1^* \leq_{st} \cdots \leq_{st} X_n^* \) be the random states of \( n \) components available for parallel (series) redundancy with the \( n \) components forming the system \( \phi \) (one spare to each position). We prove the following Corollary to Theorem 2.12.

**Corollary 2.13.** Let \( \pi \) and \( \pi^* \) be two permutations of \( \{1, 2, \ldots, n\} \), and \( f \) be an increasing function on \( R \). Then

(a) \( g_{\vee}(\pi, \pi^*) = E(f(\phi(X_\pi \vee X_{\pi^*}^*))) \) is arrangement decreasing, and

(b) \( g_{\wedge}(\pi, \pi^*) = E(f(\phi(X_\pi \wedge X_{\pi^*}^*))) \) is arrangement increasing.
PROOF: (a) Let $p^j = (p^j_1, \ldots, p^j_n)$, where $p^j_i = P[X_i \geq j], j = 1, \ldots, M$, and let $p^{*j}$ be similarly defined. Now

$$P[\phi(X \cup X^*_{\pi^*}) \geq j] = h_{k_j | \pi}(p^{*j}_\pi \parallel p^{*j}_{\pi^*}).$$

Hence if $(\pi, \pi^*) \leq (\pi_1, \pi^*_1)$, then $\phi(X \cup X^*_{\pi^*}) \leq \phi(X \cup X^*_{\pi_1})$ by Theorem 2.12. This proves (a).

(b) is proved similarly.

It follows therefore that if we have only a single spare component to add as a parallel redundancy to one of our components, then the optimal decision in terms of stochastically maximizing system performance is to allocate the spare to the (stochastically) weakest position. Suppose however we are considering a series redundancy to one of our components. Allocating the spare to the (stochastically) weakest position would stochastically decrease the system reliability the least, while an allocation to the (stochastically) strongest position would yield the greatest stochastic decrease in system reliability. In our example 2.9b of series redundancy, we were interested in allocating spares so as to decrease the probability of a 'bad event'. However there are situations where we are interested in high reliability (probability of a 'good event'), and series redundancy is forced upon us. In such a case, we would be interested in allocating the redundancies in order to minimize the decrease in system reliability.

3. Component Redundancy in Modules of Coherent Systems.

In this section we will assume that the coherent system $(C, \phi)$ has a modular decomposition. Hence there exists a set of disjoint modules $\{(A_1, \chi_1), \ldots, (A_r, \chi_r)\}$ with an organizing structure function $\psi$ such that

(a) $C = \cup_{i=1}^r A_i$, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and
(b) $\phi(x) \equiv \psi(\chi_1(x^{A_1}), \ldots, \chi_r(x^{A_r})).$

Series–parallel and parallel–series systems are basic examples of systems with a modular decomposition. The reliability function $h_\phi$ has the form $h_\phi(p) = h_\psi(h_{\chi_1}(p^{A_1}), \ldots, h_{\chi_r}(p^{A_r}))$ from which it follows that the reliability importance of component $c_{ij}$ (the $j$th component in the $i$th module) is given by

$$I_{h_\phi}(c_{ij}) = I_{h_{\chi_i}}(j)I_{h_\psi}(\chi_i).$$

That is, the reliability importance of component $c_{ij}$ is the product of its importance within the $i$th module multiplied by the importance of the $i$th module within the system.

Let $C^*$ be a set of spare components with respective reliabilities $(p^*_i)$ available for redundancy in the system $(C, \phi)$. Then the parallel redundancy importance of component $c_{ij}$ is given by

$$I_{\Pi}(i, j) = p^*_i q_{ij}I_{h_{\chi_i}}(j)I_{h_\psi}(\chi_i),$$
while the series redundancy importance is given by

\[ I^i_{\Pi}(i,j) = q^i_{ij}p_{ij}I_{h_{\chi_i}}(j)I_{h_{\psi}}(\chi_i). \]

Note that the parallel (series) redundancy importance of component \( c_{ij} \) is the product of its parallel (series) redundancy importance within the \( i^{th} \) module multiplied by the (Birnbaum) importance of the \( i^{th} \) module within the system.

We now consider examples of parallel redundancy in \( k \) out of \( r \) – parallel systems, and series redundancy in \( k \) out of \( r \) – series systems. Examples of optimal parallel redundancy in series–parallel and parallel–series systems are given in Boland, El-Neweihi and Proschan (1988).

**Example 3.1.** \( k \) out of \( r \) – parallel systems

Here \( \chi_i \) is a \( 1 \) out of \( n_i \) parallel structure for \( i = 1, \ldots, r \), and \( \psi \) is a \( k \) out of \( r \) structure function. When \( k = r \), we have a series parallel system.

By viewing a parallel redundancy to component \( j \) within module \( i \) as a parallel redundancy to the whole module acting as a ‘supercomponent’, we can arrive at some conclusions about optimal allocation of spares.

(a) Suppose that \( p^i_{ij} = p^* \) for all \( i \) and \( j \), and one spare component is to be allocated. Then the optimal (parallel) redundancy should be made ‘anywhere’ within the weakest subsystem, i.e., the one with the smallest value among \( h_{\chi_i}(p_{i1}, \ldots, p_{in_i}) = 1 - \prod_{j=1}^{n_i} q_{ij}, \quad i = 1, \ldots, r. \) If in particular components within a parallel subsystem are homogeneous, i.e. \( p_{ij} = p_i \) for all \( i \) and \( j \), the weakest subsystem is that \( i \) which maximizes \( q^i_{ni}. \) If furthermore \( p_1 = \ldots = p_r \), then the weakest subsystem corresponds to the \( i \) with the smallest \( n_i. \)

(b) Suppose now that \( \ell \leq r \) spare components are available for parallel redundancy, one to each of \( \ell \) subsystems, then it is optimal to allocate the strongest spare to the weakest subsystem, the second strongest spare to the second weakest subsystem, etc.

**Example 3.2.** \( k \) out of \( r \) – series systems.

Here \( \chi_i \) is an \( n_i \) out of \( n_i \) series system for \( i = 1, \ldots, r \), and \( \psi \) is again a \( k \) out of \( r \) structure function. When \( k = 1 \), we have a parallel series system. We consider the problem of allocating spares for series redundancy with components in the system. Of course a series redundancy to component \( j \) within module \( i \) is equivalent to a series redundancy to the \( i^{th} \) module.

(a) Suppose that \( p^i_{ij} = p^* \) for all \( i \) and \( j \) and one series redundancy is to be made. The greatest reduction in system reliability would be made by allocating it to the strongest subsystem,
(i.e., selecting \( i \) to maximize \( \prod_{j=1}^{n_i} p_{ij} \)). If in particular \( p_{ij} = p_i \), for all \( i \) and \( j = 1, \ldots, n_i \), then we select the \( i \) maximizing \( p_i^{n_i} \). If all \( p_i \) are equal, we select the subsystem \( i \) with the minimal \( n_i \). If on the other hand the \( p_i \) vary but \( n_1 = \cdots = n_r \), then we select the subsystem with the maximum \( p_i \).

(b) Suppose now that \( \ell \leq r \) spare components are available for series redundancy, one to each of \( \ell \) subsystems. Then the optimal method (for maximizing reduction in system reliability) is to allocate the weakest component to the strongest series subsystem, the second weakest to the second strongest subsystem, etc.
References


We study the model in which a set of spares is available for redundancy in a coherent system. In some circumstances, parallel (or active) redundancy is used to improve the reliability of the system, while in others series redundancy is used to improve a different measure of utility. Hence we define the two concepts of parallel and series redundancy importance for components in a coherent system relative to an available set of redundant spares. These measures of importance are compared with the structural importance and reliability importance of a component. Various results for the optimal allocation of redundant spares are given, with particular reference to $k$ out of $n$ systems, arrangement increasing, Schur concave.