ON THE ESTIMATION OF COEFFICIENTS OF SIMULTANEOUS
LINEAR EXPLOSIVE MODEL OF HIGHER ORDERS WITH MOVING
AVERAGE ERRORS GENERATING A PAIR OF TIME SERIES

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Abstract: In this paper, estimation of coefficients of a simultaneous linear partially explosive model of higher orders with moving average errors is considered. It has been shown that the above model can be decomposed into a purely explosive model and an autoregressive model. A two stage estimation procedure is carried out towards proposing estimators for the partially explosive model. The asymptotic properties of these estimators are also studied.

Key words and phrases: partially explosive model, autoregressive model, moving average errors, Yule-Walker type estimators.
1. INTRODUCTION

Consider the model, for $t \geq 1$

$$
X(t + 2) - c_1 X(t + 1) - c_2 X(t) - c_3 Y(t + 1) - c_4 Y(t) - c_5 = \varepsilon_1(t + 2)
$$

$$
Y(t + 2) - d_1 Y(t + 1) - d_2 Y(t) - d_3 X(t + 1) - d_4 X(t) - d_5 = \varepsilon_2(t + 2)
$$

(1.1)

with the following explanatory assumptions.

ASSUMPTION 1.1: $(\varepsilon_i(t); t \geq 1)$ $i = 1, 2$ are independent families of random variables consisting of independent and identically distributed random variables with $E\varepsilon_i(t) = 0$ and $E\varepsilon_i^2(t) = \sigma_i^2$ (positive and finite) $i = 1, 2$.

ASSUMPTION 1.2: On setting $\varepsilon_i(t) = 0$ for $i = 1, 2$ and for $t \geq 0$; let for $t \geq 1$

$$
\overline{\varepsilon}_1(t) = \sum_{i=0}^{L} a_i \varepsilon_i(t - i); \quad \overline{\varepsilon}_2(t) = \sum_{j=0}^{K} b_j \varepsilon_j(t - j); a_0 = b_0 = 1.
$$

ASSUMPTION 1.3: Denoting by

$$
P(z) = z^4 - (c_1 + d_1)z^3 - (c_2 - c_1 d_1 + d_2 + c_3 d_3)z^2 - (c_3 d_4 + c_4 d_3 - c_1 d_2 - d_1 c_2)z - (c_4 d_4 - c_2 d_2)
$$

i) $P(z)$ has the factorization $P(z) = P_1(z)P_2(z)$

ii) $P_1(z) = z^2 - (\alpha_1 + \beta_1)z + (\alpha_1 \beta_1 - \alpha_2 \beta_2)$ has roots $\phi_1$ and $\phi_2$ having the placements either

\begin{align*}
\text{a) } & \quad |\phi_1| > |\phi_2| > 1 \\
\text{b) } & \quad \phi_1 = \phi_2 = \phi_0; |\phi_0| > 1, \quad \phi_i(\text{real}) \quad i = 1, 2
\end{align*}

iii) $P_2(z) = z^2 - (\gamma_1 + \delta_1)z + (\gamma_1 \delta_1 - \gamma_2 \delta_2)$ has roots $\rho_1$ and $\rho_2$ with $|\rho_1|, |\rho_2| < 1$.

Models such as (1.1) give rise to a fourth order stochastic difference equations in $X(t)$ and $Y(t)$ and the dynamic stability of the model depends on the placements of the four roots of the common characteristic polynomial associated with the stochastic difference equations. If all the roots lie within the unit circle the model is said to be autoregressive in nature. If all the roots are real and lie outside the unit circle the model is said to be explosive. When some of them lie inside the unit circle and some of them outside we call such models as partially explosive. Venkataraman (1974) has made a passing reference to the model of the type (1.1). However, he has not made a detailed
study on such models. In case the model is autoregressive with moving average errors, we can 
generalize the results of Suresh Chandra and Gopal (1987). In the presence of explosive roots one 
can visualize the generalization of Theorems 2, 3, and 5 of Venkataraman (1974).

In this paper, we have considered (1.1) as a partially explosive model (i.e. two roots of the 
associated characteristic polynomial lie within the unit circle and the other two lie outside the 
unit circle); proposed estimators for the coefficients \((c, d) = (c_i \quad i = 1, \ldots, 5; d_j \quad j = 1, \ldots, 5)\) 
and derived its asymptotic properties (in Section 5). It has been shown in Section 2, that the 
Assumption 1.3 is necessary and sufficient to have a re-representation of the model (1.1) as

\begin{align}
X(t + 1) - \alpha_1 X(t) - \alpha_2 Y(t) - \alpha_3 &= G(t + 1) \\
Y(t + 1) - \beta_1 Y(t) - \beta_2 X(t) - \beta_3 &= H(t + 1); \quad t \geq 1
\end{align}

(1.2)

where

\begin{align}
G(t + 1) - \gamma_1 G(t) - \gamma_2 H(t) &= \tilde{\epsilon}_1(t + 1) \\
H(t + 1) - \delta_1 H(t) - \delta_2 G(t) &= \tilde{\epsilon}_2(t + 1); \quad t \geq 1
\end{align}

(1.3)

with \(P_1(z)\) and \(P_2(z)\) as the characteristic polynomials associated with (1.2) and (1.3) respectively. 
Treating (1.2) as purely explosive model generating a pair of related time series, in the 
first stage we have established limit distribution properties of the least-squares estimators of 
\((\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)\) occurring in (1.2) in Section 3. As a second stage estimation we propose 
a Yule-Walker type estimator for \((\gamma, \delta) = (\gamma_1, \gamma_2; \delta_1, \delta_2)\) occurring in (1.3) using the estimated 
residuals and the Basic limit theorem of Suresh Chandra and Gopal (1987), in Section 4. These 
two estimators are suitably combined to propose a CAN estimator for \((c, d)\) occurring in (1.1). 
These details are given in Section 5.

2. A BASIC THEOREM FOR REWRITING THE MODEL (1.1)

THEOREM 2.1: The model (1.1) with the associated characteristic polynomial \(P(z)\) allows for 
a unique decomposition as in (1.2) and (1.3) with the associated characteristic polynomials \(P_1(z)\) 
and \(P_2(z)\) respectively if and only if Assumption 1.3 holds.

PROOF: A combined reading of (1.2) and (1.3) results in the re-representation (1.1) with

\begin{align}
\text{(2.1)} & \quad \text{i)} \quad c_1 = \alpha_1 + \gamma_1 \quad \text{vi)} \quad d_1 = \beta_1 + \delta_1 \\
\text{ii)} \quad c_2 = -(\alpha_1 \gamma_1 + \beta_2 \gamma_2) \quad \text{vii)} \quad d_2 = -(\beta_1 \delta_1 + \delta_2 \alpha_2)
\end{align}
iii) \( c_3 = \alpha_2 + \gamma_2 \)  
iv) \( c_4 = -(\alpha_2 \gamma_1 + \beta_1 \gamma_2) \) 
v) \( c_5 = \alpha_3 - (\alpha_3 \gamma_1 + \gamma_2 \beta_3) \)

and with \((P_1(z) \cdot P_2(z))\) serving as the characteristic polynomial associated with the model (1.1).

Conversely, it is possible to show that the model (1.1) together with the Assumption 1.3 induces the relationships (1.2) and (1.3) along with the conditions on \( P_1(z) \) and \( P_2(z) \) thereof.

In order to justify this claim, we recall from Assumption 1.3 that

\[
\begin{align*}
\text{i)} & \quad \phi_1 + \phi_2 = \alpha_1 + \beta_1 \\
\text{ii)} & \quad \rho_1 + \rho_2 = \gamma_1 + \delta_1 \\
\text{iii)} & \quad \phi_1 \phi_2 = \alpha_1 \beta_1 - \alpha_2 \beta_2 \\
\text{iv)} & \quad \rho_1 \rho_2 = \gamma_1 \delta_1 - \gamma_2 \delta_2.
\end{align*}
\]

It is therefore sufficient if we can show that the 14 relations in (2.1) and (2.2) allow for a unique solution for \( \alpha, \beta, \gamma, \) and \( \delta \). Further, since equations (v) and (x) in (2.1) are linear in \( \alpha_3 \) and \( \beta_3 \), given \( \gamma_1, \gamma_2, \delta_1, \) and \( \delta_2 \), it is enough if the remaining twelve equations provide a unique solution for \( \alpha, \beta, \gamma \) and \( \delta \) (excluding \( \alpha_3 \) and \( \beta_3 \)).

We note that equations (i) and (iv) in (2.1) and (i) and (ii) in (2.2) are linearly dependent. Hence for given \( \alpha_1 \), it can be seen that, \( \beta_1, \gamma_1 \) and \( \delta_1 \), can be uniquely determined by the equations.

\[
\begin{align*}
\beta_1 &= (\phi_1 + \phi_2) - \alpha_1 \\
\gamma_1 &= (c_1 - \alpha_1) \\
\delta_1 &= d_1 - (\phi_1 + \phi_2) + \alpha_1
\end{align*}
\]

Consequently given \( \alpha_1 \) one can solve for \( \alpha_2 \) and \( \gamma_2 \) from (iiii) and (iv) of (2.1) utilizing the information on \( \beta_1 \) and \( \gamma_1 \) from (2.3). In parallel, one can solve for \( \beta_2 \) and \( \delta_2 \) from (viiii) and (ixii) of (2.1) by virtue of (2.3). In fact it can be checked that these solutions are linear in \( \alpha_1 \) on remembering
that $(\gamma_1 - \beta_1) = c_1 - (\phi_1 + \phi_2)$ and $(\delta_1 - \alpha_1) = d_1 - (\phi_1 + \phi_2)$. To be precise these solutions are

\[
\begin{align*}
\gamma_2 &= \frac{(c_3 c_3 + c_4)}{(c_1 - (\phi_1 + \phi_2))} \cdot \alpha_1 \\
\alpha_2 &= -\frac{c_3 (\phi_1 + \phi_2) + c_4}{(c_1 - (\phi_1 + \phi_2))} + \frac{c_3}{(c_1 - (\phi_1 + \phi_2))} \cdot \alpha_1 \\
\delta_2 &= \frac{d_3 (d_1 - (\phi_1 + \phi_2)) + d_4}{(d_1 - (\phi_1 + \phi_2))} + \frac{d_3}{(d_1 - (\phi_1 + \phi_2))} \cdot \alpha_1 \\
\beta_2 &= -\frac{d_4}{(d_1 - (\phi_1 + \phi_2))} - \frac{d_3}{(d_1 - (\phi_1 + \phi_2))} \cdot \alpha_1
\end{align*}
\]

Thus if $\alpha_1$ can be uniquely determined from (2.1) and (2.2), the rest of the constants can be uniquely determined from (2.3) and (2.4). By virtue of (2.3) and (2.4) it can be checked that

\[
\begin{align*}
\phi_1 \phi_2 + c_2 &= m_1 \alpha_1 + m_2 \\
\rho_1 \rho_2 + d_2 &= -m_1 \alpha_1 + m_3
\end{align*}
\]

where

\[
\begin{align*}
m_1 &= -\frac{c_3 d_3}{d_1 - (\phi_1 + \phi_2)} \\
m_2 &= \frac{c_3 d_4}{d_1 - (\phi_1 + \phi_2)} \\
m_3 &= c_1 - (\phi_1 + \phi_2)
\end{align*}
\]

Since,

\[
\phi_1 \phi_2 + c_2 - m_2 = m_3 - \rho_1 \rho_2 - d_2
\]

there exists a unique solution for $\alpha_1$. Hence the theorem.

3. FIRST STAGE ESTIMATION OF $(\alpha, \beta)$

Theorem 2.1 facilitates the re-presentation of the model (1.1) in terms of models (1.2) and (1.3). Let us now focus our attention on the model (1.2). Assumption 1.3 together with (1.3) enables us to rewrite

\[
\begin{align*}
G(t) &= \sum_{r=0}^{\infty} h(r)[\bar{\varepsilon}_1(t-r) - \partial_1 \bar{\varepsilon}_1(t-r-1) + \gamma_2 \bar{\varepsilon}_2(t-r-1)] \\
H(t) &= \sum_{r=0}^{\infty} h(r)[\bar{\varepsilon}_2(t-r) - \gamma_1 \bar{\varepsilon}_2(t-r-1) + \partial_2 \bar{\varepsilon}_1(t-r-1)]
\end{align*}
\]

where

\[
h(r) = \begin{cases} 
0 & \text{for } r < 0 \\
1 & \text{for } r = 0
\end{cases}
\]
and for \( r \geq 1 \),

\[
\begin{align*}
\text{i)} & \quad h(r) - (\gamma_1 + \partial_1)h(r - 1) + (\gamma_1 \partial_1 - \gamma_2 \partial_2)h(r - 2) = 0 \\
\text{ii)} & \quad h(r) = (\rho_1 - \rho_2)^{-1}\rho_1^{r+1} + (\rho_2 - \rho_1)^{-1}\rho_2^{r+1}
\end{align*}
\]

(3.2)

Assumption 1.3 renders \( \sum h(r) \) absolutely convergent. These observations enable us to conclude that the process \((G(t); t \geq 1)\) and \((H(t); t \geq 1)\) are asymptotically wide sense stationary processes.

It follows from (1.2) that for \( t \geq 1 \)

\[
\begin{align*}
X(t + 2) - (\alpha_1 + \beta_1)X(t + 1) + (\alpha_1 \beta_1 - \alpha_2 \beta_2)X(t) - \tilde{\alpha}_3 &= \bar{G}(t + 2) \\
Y(t + 2) - (\alpha_1 + \beta_1)Y(t + 1) + (\alpha_1 \beta_1 - \alpha_2 \beta_2)Y(t) - \tilde{\beta}_3 &= \bar{H}(t + 2)
\end{align*}
\]

(3.3)

where

\[
\begin{align*}
\text{i)} & \quad \bar{\alpha}_3 = (1 - \beta_1)\alpha_3 + \alpha_2 \beta_3 \\
\text{ii)} & \quad \bar{\beta}_3 = (1 - \alpha_1)\beta_3 + \beta_2 \alpha_3 \\
\text{iii)} & \quad \bar{G}(t + 2) = G(t + 2) - \beta_1 G(t + 1) + \alpha_2 H(t + 1) \\
\text{iv)} & \quad \bar{H}(t + 2) = H(t + 2) - \alpha_1 H(t + 1) + \beta_2 G(t + 1)
\end{align*}
\]

(3.4)

We now introduce,

\[
\begin{align*}
G_1 &= \sum_{r=1}^{\infty} \phi_1^{-r}\bar{G}(r); & G_2 &= \sum_{r=1}^{\infty} \phi_2^{-r}\bar{H}(r) \\
H_1 &= \sum_{r=1}^{\infty} \phi_2^{-r}\bar{G}(r); & H_2 &= \sum_{r=1}^{\infty} \phi_2^{-r}\bar{H}(r)
\end{align*}
\]

(3.5)

Note: In this paper all random variables represented by infinite series are assumed to be convergent in mean square sense.

We prove the following lemmas useful in the sequel.

**LEMMA 3.1**: Under the basic assumptions on (1.2), the following representations hold:

a) when \( |\phi_1| > |\phi_2| > 1 \),

\[
\begin{align*}
X(t) &= P_1 \phi_1^t G_1 + P_2 \phi_2^t H_1 + W_1(t) \\
Y(t) &= P_1 \phi_1^t G_2 + P_2 \phi_2^t H_2 + W_2(t)
\end{align*}
\]
(3.16) \[ \phi_1^{-2N} \phi_2^{-2N} G_2 \det \tilde{M}_1(1) \]
\[ = \phi_1^{-2N} \phi_2^{-2N} G_2 \det \left[ R_{21} \left( \frac{-G_1}{G_2} \right) \tilde{M}_1(1) \right] \]
\[ = \phi_1^{-2N} \phi_2^{-2N} \left[ \left( \sum \phi(t) G(t + 1) \right) \left( \sum Y^2(t) \right) - \left( \sum \phi(t) Y(t) \right) \left( \sum G'(t + 1) Y(t) \right) \right] \]
\[ = \phi_2^{-N} P_1^2 P_2^2 G_2^2 K_0 \left[ (\phi_1^2 - 1)^{-1} \sum \phi_2^t G'(t + 1) - (\phi_1^2 - 1)^{-1} \sum \phi_1^t G'(t + 1) \right] + o_p(1) \]

i) \[ \phi_1^{-2N} \phi_2^{-2N} G_2 \det \tilde{M}_1(2) \]
\[ = \phi_1^{-2N} \phi_2^{-2N} \left[ \left( \sum \phi(t) X(t) \right) \left( \sum G'(t + 1) Y(t) \right) - \left( \sum G'(t + 1) \phi(t) \right) \left( \sum X(t) Y(t) \right) \right] \]
\[ = \phi_2^{-N} P_1^2 P_2 G_1 G_2 K_0 \left[ (\phi_1 \phi_2 - 1)^{-1} \sum \phi_1^t H'(t + 1) - (\phi_2^2 - 1)^{-1} \sum \phi_2^t H'(t + 1) \right] + o_p(1) \]

iii) \[ \phi_1^{-2N} \phi_2^{-2N} G_2 \det \tilde{M}_2(1) \]
\[ = \phi_1^{-2N} \phi_2^{-2N} \left[ \left( \sum \phi(t) X(t) \right) \left( \sum H'(t + 1) X(t) \right) - \left( \sum H'(t + 1) \phi(t) \right) \left( \sum X(t) Y(t) \right) \right] \]
\[ = \phi_2^{-N} P_1^2 P_2 G_1 G_2 K_0 \left[ (\phi_1^2 - 1)^{-1} \sum \phi_1^t H'(t + 1) - (\phi_2^2 - 1)^{-1} \sum \phi_2^t H'(t + 1) \right] + o_p(1) \]

iv) \[ \phi_1^{-2N} \phi_2^{-2N} G_2 \det \tilde{M}_2(2) \]
\[ = \phi_1^{-2N} \phi_2^{-2N} \left[ \left( \sum \phi(t) Y^2(t) \right) \left( \sum H'(t + 1) \phi(t) \right) - \left( \sum \phi(t) Y(t) \right) \left( \sum H'(t + 1) Y(t) \right) \right] \]
\[ = \phi_2^{-N} P_1^2 P_2^2 G_2^2 K_0 \left[ (\phi_1^2 - 1)^{-1} \sum \phi_2^t H'(t + 1) - (\phi_1 \phi_2 - 1)^{-1} \sum \phi_1^t H'(t + 1) \right] + o_p(1) \]

The statements (a), (b) and (c) follow from (3.11), (3.15) and (3.16) on closely adopting the arguments in the proof of Theorem 4 of Venkataraman (1974) through invoking Lemma 5 of Venkataraman (1974). Towards proving (d), we note that

(3.17) \[ \tilde{\theta}(1) = M_0 (\tilde{\alpha}_1 - \alpha_1) + (\tilde{\alpha}_2 - \alpha_2) \]
\[ = (\det \tilde{M}_1)^{-1} \det \begin{pmatrix} 0 & -M_0 & 1 \\ \sum G'(t + 1) X(t) & \sum X(t) Y(t) & \sum Y^2(t) \\ \sum G'(t + 1) Y(t) & \sum Y(t) \phi(t) & \sum \phi(t) Y(t) \end{pmatrix} \]
\[ = (\det \tilde{M}_1)^{-1} G_2^{-2} \det \begin{pmatrix} 0 & 0 & -1 \\ \sum G'(t + 1) \phi(t) & \sum \phi^2(t) & \sum \phi(t) Y(t) \\ \sum G'(t + 1) Y(t) & \sum Y(t) \phi(t) & \sum Y^2(t) \end{pmatrix} \]
where

\begin{align*}
\text{i)} & \quad W_1(t) = (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \alpha_3 - P_1 \sum_{r=1}^{\infty} \phi_1^{-r} \tilde{G}(t + r) - P_2 \sum_{r=1}^{\infty} \phi_2^{-r} \tilde{H}(t + r) \\
\text{ii)} & \quad W_2(t) = (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \beta_3 - P_1 \sum_{r=1}^{\infty} \phi_1^{-r} \tilde{G}(t + r) - P_2 \sum_{r=1}^{\infty} \phi_2^{-r} \tilde{H}(t + r) \\
\text{iii)} & \quad P_1 = \phi_1 (\phi_1 - \phi_2)^{-1}; \quad P_2 = \phi_2 (\phi_2 - \phi_1)^{-1}
\end{align*}

b) when $\phi_1 = \phi_2 = \phi_0; |\phi_0| > 1$

\begin{align*}
X(t) &= (t + 1) \phi_0^t G_1 - \phi_0^t K_1 + \tilde{W}_1(t) \\
Y(t) &= (t + 1) \phi_0^t G_2 - \phi_0^t K_2 + \tilde{W}_2(t)
\end{align*}

where

\begin{align*}
\text{i)} & \quad K_i = \sum_{r=1}^{\infty} r \phi_0^{-r} \tilde{G}_i(r), \quad i = 1, 2 \\
\text{ii)} & \quad \tilde{W}_1(t) = (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \alpha_3 + \sum_{r=1}^{\infty} r \phi_0^{-r} \tilde{G}(t + r) - \sum_{r=1}^{\infty} \phi_0^{-r} \tilde{G}(t + r) \\
\text{iii)} & \quad \tilde{W}_2(t) = (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \beta_3 + \sum_{r=1}^{\infty} r \phi_0^{-r} \tilde{H}(t + r) - \sum_{r=1}^{\infty} \phi_0^{-r} \tilde{H}(t + r)
\end{align*}

PROOF: Equations (3.3) have explicit solutions that

\begin{align*}
X(t) &= \sum_{r=0}^{t-1} h(r) \tilde{G}(t + r) + (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \alpha_3 \\
Y(t) &= \sum_{r=0}^{t-1} h(r) \tilde{H}(t + r) + (1 - \alpha_1 - \beta_1 + \alpha_1 \beta_1 - \alpha_2 \beta_2)^{-1} \beta_3
\end{align*}

(3.6)

on remembering that $\epsilon_i(t) = 0$ for $t \leq 0$. A substitutional evaluation of (3.6) based on the fact that

\begin{equation}
\begin{aligned}
    h(r) &= P_1 \phi_1^{-r} + P_2 \phi_2^{-r} \quad \text{if} \quad |\phi_1| > |\phi_2| > 1 \\
    &\quad = (r + 1) \phi_0 \quad \text{if} \quad \phi_1 = \phi_2 = \phi_0; |\phi_0| > 1
\end{aligned}
\end{equation}

(3.7)
yields the lemma.
LEMMA 3.2: Let $\phi^*$ represent a real root of the equation
\[ z^2 - (\alpha_1 + \beta_1)z + (\alpha_1 \beta_1 - \alpha_2 \beta_2) = 0. \]
such that $|\phi^*| > 1$ and let
\[
J_1 = \sum_{r=1}^{\infty} \phi^{*-r} \hat{G}(r)
\]
\[
J_2 = \sum_{r=1}^{\infty} \phi^{*-r} \hat{H}(r)
\]
then under the basic assumptions,
\[
(\phi^* - \alpha_1)J_1 - \alpha_2 J_2 = (\phi^* - \beta_1)J_2 - \beta_2 J_1 = 0.
\]
This lemma is similar to Lemma 2 of Venkataraman (1974) and hence the proof is omitted. As a direct consequence of Lemma 3.2, we have
\[
(3.8) \quad (\phi_1 - \alpha_1)G_1 - \alpha_2 G_2 = (\phi_1 - \beta_1)G_2 - \beta_2 G_1 = 0
\]
\[
(\phi_2 - \alpha_1)H_1 - \alpha_2 H_2 = (\phi_2 - \beta_1)H_2 - \beta_2 H_1 = 0
\]
Consequently it is interesting to note that
\[
(3.9) \quad \frac{G_1}{G_2} = \frac{\alpha_2}{(\phi_1 - \alpha_1)} = \frac{(\phi_1 - \beta_1)}{\beta_2} \quad \text{(a.s.)}
\]
\[
\frac{H_1}{H_2} = \frac{\alpha_2}{(\phi_2 - \alpha_1)} = \frac{(\phi_2 - \beta_1)}{\beta_2} \quad \text{(a.s.)}
\]
In this section, we propose two sets of consistent estimators $(\tilde{\alpha}, \tilde{\beta})$ and $(\bar{\alpha}, \bar{\beta})$ for $(\alpha, \beta) = (\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3)$ possessing some useful asymptotic properties. To be specific, they are obtained as follows.

\[
(3.10) \quad \text{a)} \quad (\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\beta}_1, \tilde{\beta}_2) \text{are obtained by minimizing the sum of squares}
\]
\[
\sum_{t=1}^{N-1} [X(t+1) - \tilde{\alpha}_1 X(t) - \tilde{\alpha}_2 Y(t)]^2 + \sum_{t=1}^{N-1} [Y(t+1) - \beta_1 Y(t) - \beta_2 X(t)]^2
\]
with respect to $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and
\[
\bar{\alpha}_3 = (N-1)^{-1} \sum_{t=1}^{N-1} [X(t+1) - \bar{\alpha}_1 X(t) - \bar{\alpha}_2 Y(t)]
\]
\[
\bar{\beta}_3 = (N-1)^{-1} \sum_{t=1}^{N-1} [Y(t+1) - \bar{\beta}_1 Y(t) - \bar{\beta}_2 X(t)]
\]
b) \((\hat{\alpha}, \hat{\beta})\) is obtained by minimizing the sum of squares

\[
\sum_{t=1}^{N-1} [X(t+1) - \alpha_1 X(t) - \alpha_2 Y(t) - \alpha_3]^2 + \sum_{t=1}^{N-1} [Y(t+1) - \beta_1 Y(t) - \beta_2 X(t) - \beta_3]^2
\]

with respect to \((\alpha, \beta)\).

First we will derive the asymptotic properties of \((\hat{\alpha}, \hat{\beta})\) under the case: \(|\phi_1| > |\phi_2| > 1\). The following theorem, which incidentally establishes the consistency of \((\hat{\alpha}, \hat{\beta})\), summarizes some of the asymptotic properties of \((\hat{\alpha}, \hat{\beta})\) which are needed in the sequel.

**THEOREM 3.1:** Let \(|\phi_1| > |\phi_2| > 1\) and \(P[G_i = 0] = 0\) for \(i = 1, 2\). Then under the basic assumptions on (1.2), the following statements hold.

a) \((\phi_2^N (\hat{\alpha}_i - \alpha_i) i = 1, 2; \phi_2^N (\hat{\beta}_j - \beta_j) j = 1, 2)\) converges in law, as \(N \to \infty\), to a random vector \((\xi_2(1), \xi_2(2), \xi_2(3), \xi_2(4))\), say, with mean zero.

b) If \(M_0 = \frac{\alpha_2}{(\phi_1 - \alpha_1)} = \frac{\alpha_2}{\beta_2}\) then

\[
M_0 \xi_2(1) + \xi_2(2) = 0
\]

\[
\xi_2(3) + M_0 \xi_2(4) = 0
\]

c) \((\xi_2(1), \xi_2(3))\) is distributed like \((U_1, V_1)\), say, where \(U_1\) and \(V_1\) are ratios of two independent random variables which are themselves distributed like certain linear combinations of \(\epsilon_i(t); t \geq 1\).

d) If \(\tilde{\theta}(1) = M_0 (\hat{\alpha}_1 - \alpha_1) + (\hat{\alpha}_2 - \alpha_2)\) and \(\tilde{\theta}(2) = (\hat{\beta}_1 - \beta_1) + M_0 (\hat{\beta}_2 - \beta_2)\) then each of \(\phi_1^N \tilde{\theta}(i), i = 1, 2\) is bounded in probability.

e) \(N^{\frac{1}{2}} (\hat{\alpha}_3 - \alpha_3)\) and \(N^{\frac{1}{2}} (\hat{\beta}_3 - \beta_3)\) are bounded in probability.

**PROOF:** By definition,

\[
(\hat{\alpha}_i - \alpha_i) = (\text{det } \tilde{M}_1)^{-1} \text{det } \tilde{M}_1(i); i = 1, 2
\]

\[
(\hat{\beta}_j - \beta_j) = (\text{det } \tilde{M}_2)^{-1} \text{det } \tilde{M}_2(j); j = 1, 2
\]
where

i) \( \tilde{M}_1 = \left[ \begin{array}{cc} \sum X^2(t) & \sum Y(t)X(t) \\ \sum Y(t)X(t) & \sum Y^2(t) \end{array} \right] \)

ii) \( \tilde{M}_1(i) \) is obtained from \( \tilde{M}_1 \) on replacing its i-th column by the column vector 
\[
(\sum X(t)G'_i(t+1), \sum Y(t)G'_i(t+1))' \quad \text{for } i = 1, 2
\]

iii) \( \tilde{M}_2 \) is obtained from \( \tilde{M}_1 \) on interchanging its rows and columns.

iv) \( \tilde{M}_2(j) \) is obtained from \( \tilde{M}_2 \) on replacing its j-th column by the column vector
\[
(\sum Y(t)H'_j(t+1), \sum X(t)H'_j(t+1))' \quad \text{for } j = 1, 2.
\]

v) \( G'(t) = G(t) + \alpha_3; \quad H'(t) = H(t) + \beta_3. \)

iv) The range of summation sign \( \sum \), when it is not specified is assumed to be over \( t \) from 1 to \( N - 1 \), henceforth.

An evaluation based on Lemma 3.1 (a), (3.6) and (3.8) will yield that, if \( \phi(t) = G_2X(t) - G_1Y(t) \)
then
\[
\phi(t) = P_2\phi^t_2K_0 + \left[ G_1\sum_{r=1}^{\infty}\phi^{-r}_2G(t+r) - G_2\sum_{r=1}^{\infty}\phi^{-r}_2H(t+r) \right]
\]
where \( K_0 = G_2H_1 - G_1H_2. \)

It may be checked that the expression within the square brackets on the right hand side of (3.12) has absolute expectation bounded by \( A_0. \) It, therefore, follows from Lemma 3.1 on closely adopting the arguments in the proof of Theorem 2 of Venkataraman (1974), we have,

\[
\phi_1^{-2N}\sum X(t)Y(t) = (\phi_1^2 - 1)^{-1}P_1^2G_1G_2 + o_p(1)
\]
\[
\phi_1^{-2N}\sum X^2(t) = (\phi_1^2 - 1)^{-1}P_1^2G_1^2 + o_p(1)
\]
\[
\phi_1^{-2N}\sum Y^2(t) = (\phi_1^2 - 1)^{-1}P_1^2G_2^2 + o_p(1)
\]
\[
\phi_1^{-N}\phi_2^{-N}\sum \phi(t)X(t) = (\phi_1\phi_2 - 1)^{-1}P_1P_2G_1K_0 + o_p(1)
\]
\[
\phi_1^{-N}\phi_2^{-N}\sum \phi(t)Y(t) = (\phi_1\phi_2 - 1)^{-1}P_1P_2G_2K_0 + o_p(1)
\]

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\[
\begin{align*}
\phi_1^{-N} \sum X(t) &= (\phi_1 - 1)^{-1} P_1 G_1 + o_p(1) \\
\phi_1^{-N} \sum Y(t) &= (\phi_1 - 1)^{-1} P_1 G_2 + o_p(1) \\
\phi_2^{-2N} \sum \phi^2(t) &= (\phi_2 - 1)^{-1} P_2^2 K_0^2 + o_p(1) \\
\phi_2^{-N} \sum \phi(t) &= (\phi_2 - 1)^{-1} P_2 K_0 + o_p(1) \\
\phi_1^{-N} \sum G(t + 1)X(t) &= P_1 (\sum \phi_1^{-N} G(t + 1))G_1 + o_p(1) \\
\phi_1^{-N} \sum G(t + 1)Y(t) &= P_1 (\sum \phi_1^{-N} H(t + 1))G_2 + o_p(1) \\
\phi_1^{-N} \sum H(t + 1)X(t) &= P_1 (\sum \phi_1^{-N} H(t + 1))G_1 + o_p(1) \\
\phi_1^{-N} \sum H(t + 1)Y(t) &= P_1 (\sum \phi_1^{-N} H(t + 1))G_2 + o_p(1) \\
\phi_2^{-N} \sum G(t + 1)\phi(t) &= P_2 (\sum \phi_2^{-N} G(t + 1))K_0 + o_p(1) \\
\phi_2^{-N} \sum H(t + 1)\phi(t) &= P_2 (\sum \phi_2^{-N} H(t + 1))K_0 + o_p(1)
\end{align*}
\]

We also note that by virtue of our basic assumptions that

\[\sum \phi_i^{-N} G(t + 1), \quad \sum \phi_i^{-N} H(t + 1) \quad i = 1, 2\]

are bounded in probability.

Towards evaluating the random determinants in (3.11), we use some elementary matrix operations on them to simplify the computation. Let \( R_{ij}(h) \) denote the elementary operation of adding to the \( j \)-th row of a matrix, \( h \) times \( i \)-th row and \( C_{ij}(h) \) denote such operations with respect to its columns. The successive operations are indicated in a left to right order.

Thus settled, a substituitional evaluation based on (3.13) will yield that

\[
\begin{align*}
(3.15) \quad \phi_1^{-2N} \phi_2^{-2N} G_2^2 \det \hat{M}_1 &= \phi_1^{-2N} \phi_2^{-2N} G_2^2 \det [R_{21}\left(-\frac{G_1}{G_2}\right) \cdot C_{21}\left(-\frac{G_1}{G_2}\right) \hat{M}_1] \\
&= \phi_1^{-2N} \phi_2^{-2N} G_2^2 \det [R_{12}\left(-\frac{G_1}{G_2}\right) \cdot C_{12}\left(-\frac{G_1}{G_2}\right) \hat{M}_1] \\
&= \phi_1^{-2N} \phi_2^{-2N} \left[ \sum \phi^2(t) \sum Y^2(t) - (\sum \phi(t)Y(t))^2 \right] \\
&= P_1^2 P_2^2 G_2^2 K_0^2 [(\phi_2^2 - 1)^{-1}(\phi_1^2 - 1)^{-1} - (\phi_1 \phi_2 - 1)^2] + o_p(1)
\end{align*}
\]

and consequently the right hand side of (3.15) converges in probability, as \( N \to \infty \), to a non-zero value, by virtue of our basic assumptions. Exactly similar manipulations yield that
on effecting two elementary transformations of (i) adding $-M_0$ times third column to the second column and (ii) adding $-M_0$ times third row to the second row. Thus

$$
\tilde{\theta}(1) = -(det \tilde{M}_1)\frac{1}{2}G_2^{-1}[\sum G'(t + 1)\phi(t)\sum Y(t)\phi(t) - \sum G'(t + 1)Y(t)] = -\phi_1^{-N}(\phi_2^{-2N}\phi_2^{-2N}G_2^{-1}det \tilde{M}_1)^{-1}[(\phi_2^{-N}\sum G'(t + 1)\phi(t))(\phi_1^{-N}\phi_2^{-N}\sum Y(t)\phi(t)) - (\phi_2^{-2N}\sum \phi^2(t))(\phi_1^{-N}\sum G'(t + 1)Y(t))]
$$

A combined reading of (3.14), (3.15) and relevant expressions in (3.13) will enable us to infer from (3.18) that $\phi_1^N\tilde{\theta}(1)$ is bounded in probability. Exactly similar arguments yield the proof of the statement relating to $\tilde{\theta}(2)$.

Towards proving (e) we note that

$$
(N - 1)\tilde{\alpha}_3 = -(\tilde{\alpha}_1 - \alpha_1)\sum X(t) - (\tilde{\alpha}_2 - \alpha_2)\sum Y(t) - \sum G(t + 1) + (N - 1)\alpha_3
$$

Thus substituting for $\tilde{\alpha}_2 - \alpha_2$ from the definition of $\tilde{\theta}(1)$,

$$
(N - 1)(\tilde{\alpha}_3 - \alpha_3) = -(\tilde{\alpha}_1 - \alpha_1)G_2^{-1}\sum \phi(t) - \sum G(t + 1) - \tilde{\theta}(1)\sum Y(t)
$$

on recalling the definition of $\phi(t)$. Hence we can rewrite (3.20) as

$$
(N - 1)^\frac{1}{2}(\tilde{\alpha}_3 - \alpha_3) = -(N - 1)^{-\frac{1}{2}}[\phi_2^N(\tilde{\alpha}_1 - \alpha_1)]G_2^{-1}[\phi_2^{-N}\sum \phi(t)] - [(N - 1)^{-\frac{1}{2}}\sum G(t + 1)] - (N - 1)^{-\frac{1}{2}}[\phi_1^N\tilde{\theta}(1)][\phi_1^{-N}\sum Y(t)]
$$

The expressions within the five square brackets on the right hand side of (3.21) are bounded in probability by virtue of (a), (3.13), Lemma 2.1 of Suresh Chandra and Gopal (1987), (d) and (3.13) respectively. Consequently $(N - 1)^\frac{1}{2}(\tilde{\alpha}_3 - \alpha_3)$ and hence $N^\frac{1}{2}(\tilde{\alpha}_3 - \alpha_3)$ is bounded in probability. Similar proof holds for $N^\frac{1}{2}(\tilde{\beta}_3 - \beta_3)$. Hence the theorem.

The following theorem summarizes the asymptotic properties of the other estimator $(\hat{\alpha}, \hat{\beta})$ which may be proved following the same lines of the proof of the previous theorem and hence the proof is omitted.

**THEOREM 3.2:** Let $|\phi_1| > |\phi_2| > 1$ and $P[G_i = 0] = 0$ for $i = 1, 2$. Then under the basic assumptions, the following statements hold.
a) \( \phi_2^N(\hat{\alpha}_i - \alpha_i) \ i = 1, 2; \ \phi_2^N(\hat{\beta}_j - \beta_j) \ j = 1, 2 \) converges in law, as \( N \to \infty \), to a random vector \((\xi_2(5), \xi_2(6), \xi_2(7), \xi_2(8))\), say, with mean zero.

b) If \( M_0 = \frac{\alpha_2}{(\phi_1 - \alpha_1)} = \frac{(\phi_1 - \beta_1)}{\beta_2} \), then

\[
M_0 \xi_2(5) + \xi_2(6) = 0
\]
\[
\xi_2(7) + M_0 \xi_2(8) = 0
\]

c) \((\xi_2(5), \xi_2(7))\) is distributed like \((U_2, V_2)\), say, where \( U_2 \) and \( V_2 \) are ratios of two independent random variables which are themselves distributed like certain linear combinations of \( \epsilon_i(t) \), \( t \geq 1, i = 1, 2 \).

d) If \( \hat{\theta}(1) = M_0(\hat{\alpha}_1 - \alpha_1) + (\hat{\alpha}_2 - \alpha_2) \) and \( \hat{\theta}(2) = (\hat{\beta}_1 - \beta_1) + M_0(\hat{\beta}_2 - \beta_2) \), then each of \( \phi_1^N(\hat{\theta}(i), i = 1, 2 \) is bounded in probability.

e) \( N^{\frac{1}{2}}(\hat{\alpha}_3 - \alpha_3) \) and \( N^{\frac{1}{2}}(\hat{\beta}_3 - \beta_3) \) are bounded in probability.

Next we study the asymptotic properties of these two estimators under the case \( \phi_1 = \phi_2 = \phi_0; |\phi_0| > 1 \).

**THEOREM 3.3:** Let \( \phi_1 = \phi_2 = \phi_0; |\phi_0| > 1 \) and \( P[G_i = 0] = 0 \) for \( i = 1, 2 \). Then under the basic assumptions on \((1.2)\) the following statements hold.

a) \( N^{-1}\phi_0^N(\hat{\alpha}_i - \alpha_i) \ i = 1, 2; N^{-1}\phi_0^N(\hat{\beta}_j - \beta_j) \ j = 1, 2 \) converges in law, as \( N \to \infty \), to a random vector \((\xi_3(1), \xi_3(2), \xi_3(3), \xi_3(4))\), say, with mean zero.

b) If \( M_0 = \frac{\alpha_2}{(\phi_0 - \alpha_1)} = \frac{(\phi_0 - \beta_1)}{\beta_2} \) then

\[
M_0 \xi_3(1) + \xi_3(2) = 0
\]
\[
\xi_3(3) + M_0 \xi_3(4) = 0
\]

c) \((\xi_3(1), \xi_3(2))\) is distributed like \((U_3, V_3)\), say where \( U_3 \) and \( V_3 \) are ratios of two independent random variables which are themselves distributed like certain linear combinations of \( \epsilon_i(t) \), \( t \geq 1, i = 1, 2 \).

d) If \( \hat{\theta}(3) = M_0(\hat{\alpha}_1 - \alpha_1) + (\hat{\alpha}_2 - \alpha_2) \) and \( \hat{\theta}(4) = (\hat{\beta}_1 - \beta_1) + M_0(\hat{\beta}_2 - \beta_2) \), then each of \( \phi_0^N(\hat{\theta}(i), i = 3, 4 \) is bounded in probability.
PROOF: By definition

\[(\bar{a}_i - a_i) = (\det \bar{M}_1)^{-1} \det \bar{M}_1(i) \quad i = 1, 2\]

\[(\bar{b}_j - \beta_j) = (\det \bar{M}_1)^{-1} \det \bar{M}_2(j) \quad j = 1, 2\]

on remembering the explanations in (3.11). We introduce the following notations to denote certain convergent series that we encounter in the sequel. Let for \(p \geq 0, q \geq 1\)

\[(3.23) \quad S_{pN}(q) = \sum_{U=0}^{N-1} \phi_0^U \sum_{U=1}^{\infty} \phi_0^{-qU} \sum_{U=1}^{\infty} \phi_0^{-qU} = S_p(q), \text{ say}\]

Recalling \(\phi(t)\) from (3.12), and from Lemma 3.1 (b) and (3.8), we have, if \(K^* = G_1 K_2 - G_2 K_1\)

\[(3.24) \quad \phi(t) = \phi_0 K^* + [G_2 \tilde{W}_1(t) - G_1 \tilde{W}_2(t)].\]

It can be checked on routine manipulations based on (3.8) that the expression within the square bracket on the right hand side of (3.24) has absolute expectation bounded by \(A_0\).

This observation together with a substitutional evaluation of the elements of the random determinants in (3.22), using the representation in Lemma 3.1 (b) will yield the following expressions on routine manipulations.

\[(3.25) \quad \text{i) } \sum \phi^2(t) = [K^* \phi_0^2 N S_{0N}(2)] + o_p(\phi_0^2)\]

\[(3.25) \quad \text{ii) } \sum \phi(t) Y(t) = (N + 1)[G_2 K^* \phi_0^2 N S_{0N}(2)] - \phi_0^2 [G_2 K^* S_{1N}(2) - K^* K_2 S_{0N}(2)] + o_p(\phi_0^2)\]

\[(3.25) \quad \text{iii) } \sum \phi(t) X(t) = (N + 1)[G_1 K^* \phi_0^2 S_{1N}(2)] - \phi_0^2 [G_1 K^* S_{1N}(2) - K^* K_1 S_{0N}(2)] + o_p(\phi_0^2)\]

\[(3.25) \quad \text{iv) } \sum Y^2(t) = (N + 1)^2 [\phi_0^2 G_2^2 S_{2N}(2)] - 2 \phi_0^2 (N + 1) [G_2^2 S_{1N}(2) + G_2 K_2 S_{0N}(2)]

+ \phi_0^2 [G_2^2 S_{2N}(2) + 2 K_2 G_2 S_{1N}(2) + K_2^2 S_{0N}(2)] + o_p(\phi_0^2)\]

\[(3.25) \quad \text{v) } \sum X(t) Y(t) = (N + 1)[\phi_0^2 G_1 G_2 S_{2N}(2)] - \phi_0^2 (N + 1) [G_1 G_2 S_{1N}(2) + K^* S_{1N}(2) + K_1 K_2 S_{0N}(2)] + o_p(\phi_0^2)\]

\[(3.25) \quad \text{vi) } \sum \phi(t) G'(t + 1) = [K^* \phi_0^N \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)] + o_p(\phi_0^N)\]

\[(3.25) \quad \text{vii) } \sum Y(t) G'(t + 1) = (N + 1) [\phi_0^N G_2 \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)]

- \phi_0^N G_2 \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)

+ \phi_0^N K_2 \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1) + o_p(\phi_0^N)\]

\[(3.25) \quad \text{viii) } \sum \phi(t) H'(t + 1) \text{ and } \sum Y(t) H'(t + 1) \text{ have simplifications similar to (vi) and (vii) above respectively with } G \text{ being replaced in } H.\]
We now substitute for the elements of the random determinants in (3.22) from (3.25) and use the following elementary row and column operations before evaluating the determinants.

i) For $\tilde{M}_1$, we adopt the following operations in that order

$$R_{12}(-K^{*-1}(N + 1)G_2)$$
$$C_{12}(-K^{*-1}(N + 1)G_2)$$
$$R_{12}(-K^{*-1}K_2)$$
$$C_{12}(-K^{*-1}K_2)$$

ii) For $\tilde{M}_1(i)$, we adopt the following operations in that order

$$R_{12}(-K^{*-1}(N + 1)G_2)$$
$$R_{12}(-K^{*-1}K_2)$$

iii) For $\tilde{M}_2$ and $\tilde{M}_2(j)$, operations are similar to those in (i) and (ii) with modifications $R_{ij}(\cdot)$ and $C_{ij}(\cdot)$ will be replaced by $R_{ji}(\cdot)$ and $C_{ji}(\cdot)$.

These simplifications will ultimately yield the following results.

$$G_2^2 \det \tilde{M}_1 = G_2^2 \det \tilde{M}_2 = \phi_0^{4N} K^{*} G_2^2 (S_0(2) S_2(2) - S_1^2(2)) + o_p(\phi_0^{4N})$$

$$G_2 \det \tilde{M}_1(1) = \phi_0^{3N} N K^* G_2^2 \left| \begin{array}{cc}
S_0(2) & \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1) \\
-S_1(2) & -\sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)
\end{array} \right| + o_p(\phi_0^{3N})$$

$$G_2 \det \tilde{M}_2(1) = \phi_0^{3N} N K^* G_2^2 \left| \begin{array}{cc}
S_0(2) & \sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1) \\
-S_1(2) & -\sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)
\end{array} \right| + o_p(\phi_0^{3N})$$

$$G_2 \det \tilde{M}_2(2) = \phi_0^{3N} N K^* G_2^2 \left| \begin{array}{cc}
\sum_{U=1}^{N-1} \phi_0^{-U} H'(N - U + 1) & S_0(2) \\
-S_1(2) & -\sum_{U=1}^{N-1} \phi_0^{-U} H'(N - U + 1)
\end{array} \right| + o_p(\phi_0^{3N})$$

The statements (a), (b) and (c) follow from (3.22) and (3.26) on closely adopting the arguments in the proof of Theorem 5 of Venkataraman (1974). Towards proving (d) we note that

$$\tilde{\theta}(3) = M_0(\tilde{\alpha}_1 - \alpha_1) + (\tilde{\alpha}_2 - \alpha_2)$$

$$= (\det \tilde{M}_1)^{-1} G_2^{-2} \det \begin{pmatrix}
0 & 0 & 0 & -1 \\
\sum G'(t + 1) \phi(t) & \sum \phi^2(t) & \sum \phi(t) Y(t) & \sum Y^2(t)
\end{pmatrix}$$
by virtue of (3.17). We substitute from (3.25) and effect the following transformations on the matrix on the right hand side of (3.27) in that order.

\begin{align*}
R_{12}(- (N + 1) K^{*-1} G_2) \\
C_{12}(- (N + 1) K^{*-1} G_2)
\end{align*}

These manipulations will finally yield that

\begin{equation}
\tilde{\theta}(3) = -\phi_0^{-N} G_2^{-1} (S_0(2) S_2(2) - S_1^2(2)) \left| \frac{\sum_{U=1}^{N-1} \phi_0^{-U} G'(N - U + 1)}{\sum_{U=1}^{N-1} U \phi_0^{-U} G'(N - U + 1)} \right| \left| \frac{S_0(2)}{-S_1(2)} \right| + o_p(1)
\end{equation}

Consequently, $\phi_0^N \tilde{\theta}(3)$ is bounded in probability. A similar proof holds for the boundedness in probability of $\phi_0^N \tilde{\theta}(4)$.

The proof of (e) is conceptually similar to that of statement (e) in Theorem 3.1. Hence the theorem.

In parallel, the asymptotic properties of the other estimator, namely $(\hat{\alpha}, \hat{\beta})$ defined in (3.10) b under the placement $\phi_1 = \phi_2 = \phi_0; |\phi_0| > 1$ can be derived on adopting similar arguments. We summarize these properties in the following theorem, stated without proof.

**THEOREM 3.4:** Let $\phi_1 = \phi_2 = \phi_0; |\phi_0| > 1$ and $P[G_i = 0] = 0$ for $i = 1, 2$. Then under the basic assumptions on (1.2) the following statements hold.

a) $(N^{-1} \phi_0^N (\hat{\alpha}_i - \alpha_i) i = 1, 2; N^{-1} \phi_0^N (\hat{\beta}_j - \beta_j) j = 1, 2)$ converges in law, as $N \to \infty$, to a random vector $(\xi_3(5), \xi_3(6), \xi_3(7), \xi_3(8))$, say, with mean zero.

b) If $M_0 = \frac{\alpha_2}{(\phi_0 - \alpha_1)} = \frac{\phi_0 - \beta_1}{\beta_2}$ then

\begin{align*}
M_0 \xi_3(5) + \xi_3(6) &= 0 \\
\xi_3(7) + M_0 \xi_3(8) &= 0
\end{align*}

c) $(\xi_3(5), \xi_3(7))$ is distributed like $(U_4, V_4)$, say, where $U_4$ and $V_4$ are ratios of two independent random variables which are themselves distributed like certain linear combinations of $\xi_i(t), t \geq 1, i = 1, 2$.

d) If $\hat{\theta}(3) = M_0(\hat{\alpha}_1 - \alpha_1) + (\hat{\alpha}_2 - \alpha_2)$ and $\hat{\theta}(4) = (\hat{\beta}_1 - \beta_1) + M_0(\hat{\beta}_2 - \beta_2)$ then each of $\phi_0^N \hat{\theta}(i), i = 3, 4$ is bounded in probability.

e) $N^{1/2} (\hat{\alpha}_3 - \alpha_3)$ and $N^{1/2} (\hat{\beta}_3 - \beta_3)$ are bounded in probability.
4. SECOND STAGE ESTIMATION: ESTIMATION OF \((\gamma, \delta)\)

Let us denote by \((\alpha^*, \beta^*)\) either of the estimators \((\hat{\alpha}, \hat{\beta})\) and \((\tilde{\alpha}, \tilde{\beta})\) proposed for \((\alpha, \beta)\) in Section 3 under the two possibilities \(|\phi_1| > |\phi_2| > 1\) and \(\phi_1 = \phi_2 = \phi_0; |\phi_0| > 1\). Let us introduce for \(t \geq 1\)

\[
G^*(t) = X(t) - \alpha_1^* X(t - 1) - \alpha_2^* Y(t - 1) - \alpha_3^* \\
H^*(t) = Y(t) - \beta_1^* Y(t - 1) - \beta_2^* X(t - 1) - \beta_3^*
\]
on interpreting \(X(t) = Y(t) = 0\) for \(t \leq 0\). Using these estimated residuals let us introduce the following covariance type functions.

\[
U_{GG}^*(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} G^*(t)G^*(t + m) \\
U_{GH}^*(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} H^*(t)G^*(t + m) \\
U_{HG}^*(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} G^*(t)H^*(t + m) \\
U_{HH}^*(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} H^*(t)H^*(t + m)
\]

In parallel, let also consider the following covariance type functions of the non-observable random variables \(G(t)\) and \(H(t)\).

\[
U_{GG}(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} G(t)G(t + m) \\
U_{GH}(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} H(t)G(t + m) \\
U_{HG}(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} G(t)H(t + m) \\
U_{HH}(m) = N^{-\frac{1}{2}} \sum_{t=1}^{N-m} H(t)H(t + m)
\]

These covariance type functions are similar to those introduced by Suresh Chandra and Gopal (1987). We now propose the Yule-Walker type estimator for \((\gamma, \delta)\) say \((\gamma^*, \delta^*)\) as the solution of the following system of equations.

\[
U_{GG}^*(L)\gamma_1^* + U_{HG}^*(L)\gamma_2^* = U_{GG}^*(L + 1) \\
U_{GH}^*(L)\gamma_1^* + U_{HH}^*(L)\gamma_2^* = U_{GH}^*(L + 1) \\
U_{HG}^*(K)\delta_1^* + U_{GG}^*(K)\delta_2^* = U_{HH}^*(K + 1) \\
U_{HH}^*(K)\delta_1^* + U_{GG}^*(K)\delta_2^* = U_{HH}^*(K + 1)
\]

The following lemma is useful in proving the asymptotic properties of \((\gamma^*, \delta^*)\).
LEMMA 4.1: Under the basic assumptions on (1.3) and under the assumptions underlying the
determination of \((\alpha^*, \beta^*)\), the following statement holds for both the cases: case (i) \(|\phi_1| > |\phi_2| > 1\)
and case (ii) \(\phi_1 = \phi_2 = \phi_0; |\phi_0| > 1\).

\[
\begin{align*}
N^{-\frac{1}{2}}(U^*_{GG}(m) - U_{GG}(m)) & \quad \text{P}_0 \\
N^{-\frac{1}{2}}(U^*_{GH}(m) - U_{GH}(m)) & \\
N^{-\frac{1}{2}}(U^*_{HG}(m) - U_{HG}(m)) & \\
N^{-\frac{1}{2}}(U^*_{HH}(m) - U_{HH}(m)) &
\end{align*}
\]

PROOF: Case (i): By definition of \(G(t), G^*(t)\) and on denoting by

\[
\begin{align*}
\theta^*(1) &= M_0(\alpha_1^* - \alpha) + (\alpha_2^* - \alpha_2) \\
\theta^*(2) &= (\beta_1^* - \beta_1) + M_0(\beta_2^* - \beta_2); \quad M_0 = \frac{G_1}{G_2}
\end{align*}
\]

we have

\[
\begin{align*}
U^*_{GG}(m) - U_{GG}(m) &= (\alpha_1 - \alpha_1^*)G_2^{-1}[\sum_{t=1}^{N-m} \phi(t - 1)G(t + m) + \sum_{t=1}^{N-m} \phi(t + m - 1)G(t)] \\
&\quad + (\alpha_1 - \alpha_1^*)G_2^{-2}[\sum_{t=1}^{N-m} \phi(t - 1)\phi(t + m - 1)] \\
&\quad - \theta^*(1)[\sum Y(t - 1)G(t + m) + \sum Y(t + m - 1)G(t)] \\
&\quad - \theta^*(1)(\alpha_1 - \alpha_1^*)G_2^{-1}[\sum \phi(t - 1)Y(t + m - 1) + \sum \phi(t + m - 1)Y(t - 1)] \\
&\quad + (\theta^*(1))^2[\sum Y(t - 1)Y(t + m - 1)] \\
&\quad + (\alpha_3 - \alpha_3^*)(\sum G(t) + \sum G(t + m)) \\
&\quad + (\alpha_3 - \alpha_3^*)(\alpha_1 - \alpha_1^*)G_2^{-1}[\sum \phi(t - 1) + \sum \phi(t + m - 1)] \\
&\quad - \theta^*(1)(\alpha_3 - \alpha_3^*)[\sum Y(t - 1) + \sum Y(t + m - 1)] \\
&\quad + (\alpha_3 - \alpha_3^*)^2(N - m)
\end{align*}
\]

By virtue of Theorems 3.1 and 3.2, the following terms are bounded in probability.

i) \(\phi_2^N(\alpha_1 - \alpha_1^*)\)

ii) \(\phi_1^N(\theta^*(1))\)

iii) \(N^{-\frac{1}{2}}(\alpha_3 - \alpha_3^*)\)
and again by virtue of (3.13), the following terms are bounded in probability.

i) \( \phi_2^{-N} \sum \phi(t - 1)G(t + m); \phi_2^{-N} \sum \phi(t + m - 1)G(t) \)

ii) \( \phi_2^{-2N} \sum \phi(t - 1)\phi(t + m - 1) \)

iii) \( \phi_1^{-N} \sum Y(t - 1)G(t + m); \phi_1^{-N} \sum Y(t + m - 1)G(t) \)

iv) \( \phi_1^{-N} \phi_2^{-N} \sum \phi(t - 1)Y(t + m - 1); \phi_1^{-N} \phi_2^{-N} \sum \phi(t + m - 1)Y(t - 1) \)

v) \( \phi_1^{-2N} \sum Y(t - 1)Y(t + m - 1) \)

vi) \( \phi_2^{-N} \sum \phi(t - 1); \phi_2^{-N} \sum \phi(t + m - 1) \)

vii) \( \phi_1^{-N} \sum Y(t - 1); \phi_1^{-N} \sum Y(t + m - 1) \)

Finally by virtue of (3.1), Lemma 2.1 of Suresh Chandra and Gopal (1987), we note that \( N^{-\frac{1}{2}} \sum G(t) \) and \( N^{-\frac{1}{2}} \sum G(t + m) \) are bounded in probability.

These observations will enable us to infer that \( U_{GG}^*(m) - U_{GG}(m) \) is bounded in probability. Consequently

\[
N^{-\frac{1}{2}} |U_{GG}^*(m) - U_{GG}(m)| \xrightarrow{p} 0.
\]

A similar proof holds for the other expressions, in the lemma. Case (ii) can also be proved on similar lines.

We now prove the following theorem towards establishing the asymptotic properties of \((\gamma^*, \delta^*)\).

**THEOREM 4.1:** Under the basic assumptions on (1.2) and under the assumptions underlying the determination of \((\alpha^*, \beta^*)\) the following statement holds for both the cases: case (i) \(|\phi_1| > |\phi_2| > 1\) and case (ii) \(\phi_1 = \phi_2 = \phi_0; \ |\phi_0| > 1\).

\[
(N^{\frac{1}{2}} (\gamma_1^* - \gamma_1), N^{\frac{1}{2}} (\gamma_2^* - \gamma_2), N^{\frac{1}{2}} (\delta_1^* - \delta_1), N^{\frac{1}{2}} (\delta_2^* - \delta_2))
\]

converges in law, as \(N \to \infty\), to a normal vector

\[
(\xi_4(1), \xi_4(2), \xi_4(3), \xi_4(4))
\]

say, with mean zero and well specified non-singular covariance matrix.
PROOF: The first equation in (4.1) can be written in the form

\[ (4.2) \quad N^{-\frac{1}{2}} U^*_G(L) N^\frac{1}{2} (\gamma^*_1 - \gamma_1) + N^{-\frac{1}{2}} U^*_H(L) N^\frac{1}{2} (\gamma^*_2 - \gamma_2) \]

\[ = N^{-\frac{1}{2}} (U^*_G(L + 1) - U^*_G(L)) - \gamma_1 N^{-\frac{1}{2}} (U^*_G(L) - U_G(L)) \]

\[ - \gamma_2 N^{-\frac{1}{2}} (U^*_H(L) - U_H(L)) + N^{-\frac{1}{2}} [U_G(L + 1) - \gamma_1 U_G(L) - \gamma_2 U_H(L)] \]

\[ = N^{-\frac{1}{2}} [U_G(L + 1) - \gamma_1 U_G(L) - \gamma_2 U_H(L)] + o_p(1) \]

on invoking Lemma 4.1.

It may be noted that the model (1.3) and the model (1.1) in Suresh Chandra and Gopal (1987) without its constant term have similar probabilistic structure. Consequently one can adopt the expressions in (2.8) of Suresh Chandra and Gopal (1987) to rewrite the right hand side of (4.2) in terms of \( D \) – functions defined in (2.7) of Suresh Chandra and Gopal (1987) to obtain that

\[ (4.3) \quad N^{-\frac{1}{2}} U^*_G(L) N^\frac{1}{2} (\gamma^*_1 - \gamma_1) + N^{-\frac{1}{2}} U^*_H(L) N^\frac{1}{2} (\gamma^*_2 - \gamma_2) \]

\[ = D_G(L + 1, 0) - \gamma_1 D_G(L, 0) - \gamma_2 D_H(L, 0) + o_p(1) \]

where

\[ D_G(P, 0) = N^{-\frac{1}{2}} \sum_{t=1}^{P} G(t + P) G(t) \]

\[ - N^{-\frac{1}{2}} Q_{11}(P) \sum_{t=1}^{N} e_1^2(t) - N^{-\frac{1}{2}} Q_{22}(P) \sum_{t=1}^{N} e_2^2(t) \]

\[ D_H(P, 0) = N^{-\frac{1}{2}} \sum_{t=1}^{P} H(t + P) H(t) \]

\[ - N^{-\frac{1}{2}} Q_{44}(P) \sum_{t=1}^{N} e_1^2(t) - N^{-\frac{1}{2}} Q_{32}(P - 1) \sum_{t=1}^{N} e_2^2(t) \]

\[ D_H(P, 0) = N^{-\frac{1}{2}} \sum_{t=1}^{P} H(t + P) H(t) \]

\[ - N^{-\frac{1}{2}} Q_{44}(P) \sum_{t=1}^{N} e_1^2(t) - N^{-\frac{1}{2}} Q_{33}(P) \sum_{t=1}^{N} e_2^2(t) \]

In a similar way, the other three equations in (4.1) will yield the following relations.
\[(4.4)\] 

i) \(N^{-1}U_{\text{GH}}^*(K)N^{\frac{1}{2}}(\gamma_1^* - \gamma_1) + N^{-1}U_{\text{HH}}^*(K)N^{\frac{1}{2}}(\gamma_2^* - \gamma_2)\) 
\[= D_{\text{GH}}(L + 1, 0) - \gamma_1 D_{\text{GH}}(L, 0) - \gamma_2 D_{\text{HH}}(L, 0) + o_p(1)\]

ii) \(N^{-1}U_{\text{HH}}^*(K)N^{\frac{1}{2}}(\delta_1^* - \delta_1) + N^{-1}U_{\text{GH}}^*(K)N^{\frac{1}{2}}(\delta_2^* - \delta_2)\) 
\[= D_{\text{HH}}(K + 1, 0) - \delta_1 D_{\text{HH}}(K, 0) - \delta_2 D_{\text{GH}}(K, 0) + o_p(1)\]

iii) \(N^{-1}U_{\text{GG}}^*(K)N^{\frac{1}{2}}(\delta_1^* - \delta_1) + N^{-1}U_{\text{GG}}^*(K)N^{\frac{1}{2}}(\delta_2^* - \delta_2)\) 
\[= D_{\text{GG}}(K + 1, 0) - \delta_1 D_{\text{GG}}(K, 0) - \delta_2 D_{\text{GG}}(K, 0) + o_p(1)\]

Now an appeal to the Theorem 2.1 of Suresh Chandra and Gopal (1987) as applied to the process \((G(t); t \geq 1)\) and \((H(t); t \geq 1)\) together with the arguments leading to the validity of the Theorem 4.1 of Suresh Chandra and Gopal (1987) will yield that 
\[
\begin{pmatrix}
D_{\text{GG}}(L + 1, 0) - \gamma_1 D_{\text{GG}}(L, 0) - \gamma_2 D_{\text{HG}}(L, 0) \\
D_{\text{GH}}(L + 1, 0) - \gamma_1 D_{\text{GH}}(L, 0) - \gamma_2 D_{\text{HH}}(L, 0) \\
D_{\text{HH}}(K + 1, 0) - \delta_1 D_{\text{HH}}(K, 0) - \delta_2 D_{\text{GH}}(K, 0) \\
D_{\text{HG}}(K + 1, 0) - \delta_1 D_{\text{HG}}(K, 0) - \delta_2 D_{\text{GG}}(K, 0)
\end{pmatrix}
\]

converges in law, as \(N \to \infty\), to 
\[
\begin{pmatrix}
\xi_{11}(L + 1, 0) - \gamma_1 \xi_{11}(L, 0) - \gamma_2 \xi_{21}(L, 0) \\
\xi_{12}(L + 1, 0) - \gamma_1 \xi_{12}(L, 0) - \gamma_2 \xi_{22}(L, 0) \\
\xi_{22}(K + 1, 0) - \delta_1 \xi_{22}(K, 0) - \delta_2 \xi_{12}(K, 0) \\
\xi_{21}(K + 1, 0) - \delta_1 \xi_{21}(K, 0) - \delta_2 \xi_{11}(K, 0)
\end{pmatrix}
\]

and

\[(4.5)\]

i) \(\begin{pmatrix}
N^{-1}U_{GG}(L) & N^{-1}U_{HG}(L) \\
N^{-1}U_{GH}(L) & N^{-1}U_{HH}(L)
\end{pmatrix} \xrightarrow{p} \begin{pmatrix}
\sigma_1^2 Q_{11}(L) + \sigma_2^2 Q_{22}(L) & \sigma_1^2 Q_{14}(L - 1) + \sigma_2^2 Q_{32}(L + 1) \\
\sigma_1^2 Q_{41}(L - 1) + \sigma_2^2 Q_{32}(L - 1) & \sigma_1^2 Q_{44}(L) + \sigma_2^2 Q_{33}(L)
\end{pmatrix}\)

ii) \(\begin{pmatrix}
N^{-1}U_{HH}(K) & N^{-1}U_{HG}(K) \\
N^{-1}U_{HG}(K) & N^{-1}U_{GG}(K)
\end{pmatrix} \xrightarrow{p} \begin{pmatrix}
\sigma_1^2 Q_{44}(K) + \sigma_2^2 Q_{33}(K) & \sigma_1^2 Q_{41}(K - 1) + \sigma_2^2 Q_{32}(K + 1) \\
\sigma_1^2 Q_{14}(K - 1) + \sigma_2^2 Q_{32}(K - 1) & \sigma_1^2 Q_{44}(K) + \sigma_2^2 Q_{33}(K)
\end{pmatrix}\)

It can be checked that the limiting matrices in (4.5) are non-singular by virtue of our basic assumptions. These observations together with (4.3) and (4.4) and standard convergence theorems establish the validity of the theorem.
5. ESTIMATION OF \((c, d)\)

In this section we seek to estimate \((c, d) = (c_i, i = 1, \ldots, 5; d_j, j = 1, \ldots, 5)\) based on the first and second stage estimators. Let \((\alpha^*, \beta^*)\) be either of the estimators \((\tilde{\alpha}, \tilde{\beta})\) and \((\check{\alpha}, \check{\beta})\). Let \((\gamma^*, \delta^*)\) be the second stage estimators of \((\gamma, \delta)\) using \((\alpha^*, \beta^*)\). We propose the estimators for \((c, d)\) using the relations (2.1). To be specific, let

\[
\begin{align*}
\text{i) } & \quad c_1^* = \alpha_1^* + \gamma_1^* \\
\text{ii) } & \quad c_2^* = -\left(\alpha_1^* \gamma_1^* + \beta_2^* \gamma_2^*\right) \\
\text{iii) } & \quad c_3^* = \alpha_2^* + \gamma_2^* \\
\text{iv) } & \quad c_4^* = -\left(\alpha_2^* \gamma_1^* + \beta_1^* \gamma_2^*\right) \\
\text{v) } & \quad c_5^* = \alpha_3^* - \left(\alpha_3^* \gamma_1^* + \gamma_2^* \beta_3^*\right)
\end{align*}
\]

\[
\begin{align*}
\text{vi) } & \quad d_1^* = \beta_1^* + \delta_1^* \\
\text{vii) } & \quad d_2^* = -\left(\beta_1^* \delta_1^* + \delta_2^* \alpha_2^*\right) \\
\text{viii) } & \quad d_3^* = \beta_2^* + \delta_2^* \\
\text{ix) } & \quad d_4^* = -\left(\beta_2^* \delta_1^* + \delta_2^* \alpha_2^*\right) \\
\text{x) } & \quad d_5^* = \beta_3^* - \left(\beta_3^* \delta_1^* + \delta_2^* \alpha_2^*\right)
\end{align*}
\]

The following theorem summarizes the asymptotic properties of \((c^*, d^*)\).

**THEOREM 5.1**: Let \(\left|\phi_1\right| > \left|\phi_2\right| > 1\) or \(\phi_1 = \phi_2 = \phi_0\); \(\phi_0 > 1\). Then under the basic assumptions on (1.1) together with the assumptions underlying the determination of \((\alpha^*, \beta^*)\), the following statements hold.

a) \(N^{\frac{1}{2}} (c_i^* - c_i)\ i = 1, \ldots, 4; \ N^{\frac{1}{2}} (d_j^* - d_j)\ j = 1, \ldots, 4\) converges in law, as \(N \to \infty\), to a normal vector \((\xi_5(1), i = 1, \ldots, 8)\), say with mean zero such that

\[
\begin{align*}
\xi_5(1) & = \xi_4(1); \quad \xi_5(3) = \xi_4(2) \\
\xi_5(5) & = \xi_4(3); \quad \xi_5(7) = \xi_4(4)
\end{align*}
\]

\((\xi_4(1), \xi_4(2), \xi_4(3), \xi_4(4))\) being as defined in the statement of Theorem 4.1

ii) \((\xi_5(2), \xi_5(4), \xi_5(6), \xi_5(8))\) are linear combinations of \(\xi_5(1), \xi_5(3), \xi_5(5), \xi_5(7)\), given by,

\[
\begin{align*}
\xi_5(2) & = -\left(\alpha_1 \xi_5(1) + \beta_2 \xi_5(3)\right) \\
\xi_5(4) & = -\left(\alpha_2 \xi_5(1) + \beta_1 \xi_5(3)\right) \\
\xi_5(6) & = -\left(\beta_1 \xi_5(5) + \alpha_2 \xi_5(7)\right) \\
\xi_5(8) & = -\left(\beta_2 \xi_5(5) + \alpha_1 \xi_5(7)\right)
\end{align*}
\]

b) \(N^{\frac{1}{2}} (c_5^* - c_5)\) and \(N^{\frac{1}{2}} (d_5^* - d_5)\) are bounded in probability and hence, as \(N \to \infty\), \(c_5^* \overset{p}{\to} c_5\) and \(d_5^* \overset{p}{\to} d_5\).
PROOF: A combined reading of (2.1) and (5.1) yields the following relations.

\begin{align*}
(5.2) & \\
\text{i)} & \quad N^{\frac{3}{4}}(c_1^* - c_1) = N^{\frac{3}{4}}(\gamma_1^* - \gamma_1) + [N^{\frac{3}{4}}(\alpha_1^* - \alpha_1)] \\
\text{ii)} & \quad N^{\frac{3}{4}}(c_2^* - c_2) = -\alpha_1 N^{\frac{3}{4}}(\gamma_1^* - \gamma_1) - \beta_2 N^{\frac{3}{4}}(\gamma_2^* - \gamma_2) \\
& \hspace{2cm} - [N^{\frac{3}{4}}(\gamma_1^* (\alpha_1^* - \alpha_1) + \gamma_2^* (\beta_2^* - \beta_2))] \\
\text{iii)} & \quad N^{\frac{3}{4}}(c_3^* - c_3) = N^{\frac{3}{4}}(\gamma_3^* - \gamma_2) + [N^{\frac{3}{4}}(\alpha_2^* - \alpha_2)] \\
\text{iv)} & \quad N^{\frac{3}{4}}(c_4^* - c_4) = -\alpha_2 N^{\frac{3}{4}}(\gamma_1^* - \gamma_1) - \beta_1 N^{\frac{3}{4}}(\gamma_2^* - \gamma_2) \\
& \hspace{2cm} - [N^{\frac{3}{4}}(\gamma_1^* (\alpha_2^* - \alpha_2) + \gamma_2^* (\beta_1^* - \beta_1))] \\
\text{v)} & \quad N^{\frac{3}{4}}(d_1^* - d_1) = N^{\frac{3}{4}}(\delta_1^* - \delta_1) + [N^{\frac{3}{4}}(\beta_1^* - \beta_1)] \\
\text{vi)} & \quad N^{\frac{3}{4}}(d_2^* - d_2) = -\beta_1 N^{\frac{3}{4}}(\delta_1^* - \delta_1) - \alpha_2 N^{\frac{3}{4}}(\delta_2^* - \delta_2) \\
& \hspace{2cm} - [N^{\frac{3}{4}}(\delta_1^* (\beta_1^* - \beta_1) + \delta_2^* (\alpha_2^* - \alpha_2))] \\
\text{vii)} & \quad N^{\frac{3}{4}}(d_3^* - d_3) = N^{\frac{3}{4}}(\delta_2^* - \delta_2) + [N^{\frac{3}{4}}(\beta_2^* - \beta_2)] \\
\text{viii)} & \quad N^{\frac{3}{4}}(d_4^* - d_4) = -\beta_2 N^{\frac{3}{4}}(\delta_1^* - \delta_1) - \alpha_1 N^{\frac{3}{4}}(\delta_2^* - \delta_2) \\
& \hspace{2cm} - [N^{\frac{3}{4}}(\delta_1^* (\beta_2^* - \beta_2) + \delta_2^* (\alpha_1^* - \alpha_1))]
\end{align*}

By virtue of Theorems 3.1, 3.2, 3.3 and 3.4, \((\alpha_i^* - \alpha_i)\) \(i = 1, 2\) and \((\beta_j^* - \beta_j)\) \(j = 1, 2\) converge in probability as \(N \to \infty\) to zero at a rate faster than \(N^{-\frac{1}{4}}\), whatever be the placements of the roots of \(P_i(z)\). Consequently, the expressions within the square brackets on the right hand side of the equations in (5.2) are rendered as \(o_p(1)\). This observation together with Theorem 4.1 establishes (a) through (5.2) and standard convergence theorems. Towards proving (b) we note that, by definition

\begin{align*}
(5.3) & \\
N^{\frac{3}{4}}(c_5^* - c_5) = (1 - \gamma_1^*) N^{\frac{3}{4}}(\alpha_2^* - \alpha_3) - \alpha_3 N^{\frac{3}{4}}(\gamma_1^* - \gamma_2) \\
& - \gamma_2^* N^{\frac{3}{4}}(\beta_2^* - \beta_3) - \beta_3 N^{\frac{3}{4}}(\gamma_2^* - \gamma_3) \\
N^{\frac{3}{4}}(c_6^* - c_6) = (1 - \delta_1^*) N^{\frac{3}{4}}(\beta_2^* - \beta_3) - \beta_3 N^{\frac{3}{4}}(\delta_1^* - \delta_2) \\
& - \delta_2^* N^{\frac{3}{4}}(\alpha_2^* - \alpha_3) - \beta_2^* N^{\frac{3}{4}}(\delta_2^* - \delta_3).
\end{align*}

The boundedness in probability of \(N^{\frac{3}{4}}(\alpha_3^* - \alpha_3)\) and \(N^{\frac{3}{4}}(\beta_3^* - \beta_3)\) as inferred from the statement (e) of Theorems 3.1, 3.2, 3.3 and 3.4 and an appeal to Theorem 4.1 establish the validity of (b). Hence the theorem.
References


