Solutions for a Class of Invex Programming Problems

by

Morgan A. Hanson

The Florida State University
Department of Statistics
Tallahassee, Florida 32306-3033


February, 1990
SOLUTIONS FOR A CLASS OF INVEX PROGRAMMING PROBLEMS

Morgan A. Hanson

The Florida State University
Department of Statistics
Tallahassee, Florida 32306-3033

Key Words: Invex Programming, Nonconvex Programming, Polynomial Constraints, Generalized Polynomial Constraints

ABSTRACT

A nonconvex constrained optimization problem is considered in which the constraints are of the form of generalized polynomials. An invexity kernel is established for this class of problem, and a consequent theorem gives sufficient conditions for the solutions of such problems.

1. INTRODUCTION

Most of the theory and computational procedures in mathematical programming have been developed in which the various functions are convex. This is a severe limitation in practical applications and much effort has been devoted to removing this limitation.

A very broad generalization of convexity, now known as invexity, was introduced by Hanson in (1981). This was generalized further by Hanson and Mond (1987) and applied to various theorems in mathematical programming. In particular, necessary and sufficient conditions for a constrained optimum were obtained in a very general context. Such results had previously been restricted to convex functions.

However, there are simple criteria for determining whether a function is convex. There do not appear to be corresponding simple criteria for determining whether a set of functions satisfy the requirements of Hanson and Mond, and various approaches to the problem have been made, for example by Hanson (1981), Craven (1981), Ben-Israel and Mond (1986), and Hanson and Rueda (1989).

The problem remains unsolved in general. It is easy to characterize any of these functions individually, but the difficulty arises in mathematical programming, as will be seen in the next section, that the same characterization must be applied to a whole set of functions simultaneously.

In this paper a solution is found for a particular class of mathematical functions.
2. GENERALIZED INVEXITY IN MATHEMATICAL PROGRAMMING

Consider the problem

\[
\begin{align*}
\text{Minimize} & \quad f(x) & \quad (2.1) \\
\text{Subject to} & \quad g(x) \leq 0 & \quad (2.2)
\end{align*}
\]

where \( x \in S \subseteq \mathbb{R}^n \), \( f \) is a differentiable scalar function over \( S \) and \( g \) is an \( m \)-dimensional differentiable vector function over \( S \).

For the problem (2.1) - (2.2) Hanson and Mond (1987) define \( f(x) \) to be a Type I objective function and \( g(x) \) to be a Type I constraint (vector) function at \( u \) with respect to the kernel \( \eta(x,u) \) if there exists a (vector) function \( \eta(x,u) \) for feasible \( x \) and \( u \) such that

\[
f(x) - f(u) \geq \eta'(x,u) \nabla f(u)
\]

and

\[-g_i(u) \geq \eta'(x,u) \nabla g_i(u), \quad i = 1, 2, \ldots, m,
\]

where \( \nabla \) is the \( n \)-dimensional differential operator with respect to \( u \). This is a slight generalization of invexity in mathematical programming. The Kuhn-Tucker necessary conditions for a minimum at \( u \), where \( u \) satisfies (2.2), are that there exists a vector \( v \), such that

\[
\nabla f(u) + \nabla v'g(u) = 0 \quad (2.3)
\]

\[
v'g(u) = 0 \quad (2.4)
\]

and

\[v \geq 0 \quad (2.5)
\]

**Theorem 1**

If in problem (2.1) - (2.2) \( f(x) \) and \( g(x) \) are Type I functions on the constraint set at a feasible point \( u \) with a common kernel \( \eta(x,u) \), and the Kuhn-Tucker conditions are satisfied at \( u \), then \( u \) is a global minimum for the problem.

**Proof:** For any feasible \( x \)

\[
f(x) - f(u) \geq \eta'(x,u) \nabla f(u), \text{ since } f \text{ is a Type I objective function}
\]

\[= \eta'(x,u) \left[ - \nabla v'g(u) \right], \text{ by (2.3)}
\]

\[\geq v'g(u), \text{ since } g \text{ is a Type I constraint function and } v \geq 0,
\]

\[= 0, \text{ by (2.4), which proves the theorem.}
\]

The difficulty arises in determining whether \( f(x) \) and \( g(x) \) have a common kernel \( \eta(x,u) \) in a given problem.
3. THE CONSTRAINTS

We consider the problem

\[
\text{minimize } f(x) \quad \text{(3.1)}
\]

subject to \( g_i(x) \leq 0, \ i = 1, \ldots, m \), \( \text{(3.2)} \)

where \( f(x) \) and \( g(x) \) are as defined in Section 2, with the additional condition that, for each \( i \), \( g_i(x) \) is a polynomial with nonpositive coefficients; or more generally \( g_i(x) \) is a sum of nonpositive homogeneous differentiable functions of various degrees of homogeneity. That is, \( g_i(x) \) can be written in the form:

\[
g_i(x) = \sum_{\nu} \gamma_{i\nu} (x)
\]

where \( \gamma_{i\nu} \) is a nonpositive homogeneous function of degree \( \alpha_i \geq 0 \), and \( p_i \) is some positive integer. Note that the constraint functions \( g_i(x), \ i = 1, 2, \ldots, m \), do not necessarily require their respective \( \gamma \)'s to be of the same degree.

Theorem 2

For the problem (3.1) - (3.2) with \( f(x) \) and \( g(x) \) defined as above, if \( u \) satisfies the Kuhn-Tucker conditions and \( u' \nabla f(u) < 0 \) then \( u \) is a global minimum.

**Proof:**

We show that the functions \( f(x) \) and \( g(x) \) are Type I at \( u \), and hence by Theorem 1, \( u \) is a global minimum.

For any fixed \( u \) divide the constraint set into two subsets \( A_u \) and \( B_u \) where

\[
A_u = \{ \ x | f(x) - f(u) < 0 \} \\
B_u = \{ \ x | f(x) - f(u) \geq 0 \}.
\]

For each pair of feasible values \( x, u \) in problem (3.1) - (3.2) define the kernel

\[
\eta(x,u) = \begin{cases} 
\frac{f(x) - f(u)}{u' \nabla f(u)} u & \text{if } f(x) - f(u) < 0 \\
0 & \text{if } f(x) - f(u) \geq 0
\end{cases}
\]

\eta(x,u) = \begin{cases} 
\frac{f(x) - f(u)}{u' \nabla f(u)} u & \text{if } f(x) - f(u) < 0 \\
0 & \text{if } f(x) - f(u) \geq 0
\end{cases}

In subset $A_u$

(i). $\eta'(x,u) \nabla f(u) = \frac{f(x) - f(u)}{u' \nabla f(u)} \ u' \nabla f(u)$

= $f(x) - f(u)$.

So, in $A_u$, $f(x)$ is Type I with respect to $\eta(x,u)$ at $u$.

(ii). for any of the constraint functions $g_i(x)$, $i = 1, 2, ..., m$,

$\eta'(x,u) \nabla g_i(u) = \frac{f(x) - f(u)}{u' \nabla f(u)} \ u' \nabla g_i(u)$

= $\frac{f(x) - f(u)}{u' \nabla f(u)} \sum_{l=1}^{p_i} \alpha_l \gamma_{il}(u)$

= $\frac{f(x) - f(u)}{u' \nabla f(u)} \sum_{l=1}^{p_i} \alpha_l \gamma_{il}$

since $\gamma_{il}$ is homogeneous of degree $\alpha_l$.

$\leq 0$, since $f(x) - f(u) < 0$, $u' \nabla f(u) < 0$,

$\alpha_l \geq 0$ and $\gamma_{il} \leq 0$.

$\leq -g_i(u)$, since $g_i(u) \leq 0$.

So, in $A_u$, $g_i(x)$, $i = 1, 2, ..., m$, is Type I with respect to $\eta(x,u)$ at $u$.

In subset $B_u$

(i). $\eta'(x,u) \nabla f(u) = 0$

$\leq f(x) - f(u)$.

So, in $B_u$, $f(x)$ is Type I with respect to $\eta(x,u)$ at $u$.

(ii). for any of the constraint functions $g_i(x)$, $i = 1, 2, ..., m$,

$\eta'(x,u) \nabla g_i(u) = 0$

$\leq -g_i(u)$, since $g_i(u) \leq 0$.

So, in $B_u$, $g_i(x)$, $i = 1, 2, ..., m$, is Type I with respect to $\eta(x,u)$ at $u$. 
Since all the functions in problems (3.1) - (3.2) have now been shown to be Type I with respect to the same kernel, then by Theorem 1 the Kuhn-Tucker conditions at $u$ for this problem are sufficient for $u$ to be a global minimum for the problem.

A simple example is

$$\text{Minimize } e^{(x_1+2)^2} + x_2$$

Subject to $-x_1 - 1 \leq 0$

and $-x_2 - 2 \leq 0$.

The solution is $u_1 = -1$, $u_2 = -2$.

Here $u' \nabla f(u) = 2u_1(u_1 + 2)e^{(u_1+2)^2} + u_2$

$$= -2e - 2$$

$$< 0.$$ 

The Kuhn-Tucker conditions are satisfied at $u_1 = -1$, $u_2 = -2$ with $v_1 = 2e$ and $v_2 = 1$. 
4. REFERENCES


