A Chi-Square Goodness-of-Fit Test for Randomly Censored Data

by

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ABSTRACT

Procedures analogous to Karl Pearson's well-known chi-square goodness-of-fit test for a simple null hypothesis are developed under the random censorship model. The procedures are motivated through the intuitive property of self-consistency of the product-limit estimator (PLE), and make use of the weak convergence of the PLE to a Gaussian process. It is shown that a straightforward analog of Pearson's test leads to a highly anticonservative test. This leads us to develop an asymptotically exact test based on a Wald-type test statistic. This test is then compared through a simulation with a test proposed by Akritas (1988). We compared their significance levels and powers, under three models where the true lifetime distributions are exponential (model I), Weibull (model II), and normal (model III), and where the sample size, censoring probability, and number of intervals in the partition of the real line are varied. The general conclusions from this simulation study are that the proposed test usually achieves the desired significance levels when the probability of observing a censored or an uncensored value in the last interval is not small, while Akritas' test tends to be a bit anticonservative. On the other hand, Akritas' test is more powerful than the proposed test in model II, while in models I and III it is more powerful when the true mean is smaller than the null hypothesis mean, and less powerful when the true mean is larger than the null hypothesis mean. These tests are then applied to a data set in Hollander and Proschan (1979) and their p-values are found to be in reasonable agreement.

Key Words and Phrases: Akritas' test; Gaussian process; Pearson's test; product-limit estimator; random censorship model; self-consistency; weak convergence.
1. INTRODUCTION AND SUMMARY

Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables from an unknown survivor function $F(x) = P(X_1 > x)$. Karl Pearson's test which rejects the null hypothesis $H_0: F = F_0$ (where $F_0$ is a completely specified survivor function) whenever

$$
\chi^2 = \sum_{i=1}^{M} \frac{(N_i - E_i)^2}{E_i} \geq \chi^2_{M-1; \alpha}
$$

and accepts otherwise, is one of the most commonly-used statistical methods. It is easy to apply, intuitive, adaptable, and uses the chi-square distribution whose percentile points are readily available. In the expression for $\chi^2$, $M$ is the number of intervals in the partition of $R$ — the real line, $N_i$ and $E_i$ ($i = 1, \ldots, M$) are the observed and expected (under $H_0$) frequencies of the $X_i$'s in the $i$th interval, respectively, and $\chi^2_{M-1; \alpha}$ is the $100(1-\alpha)$th percentile of the chi-square distribution with $M - 1$ degrees-of-freedom (df). This test was considered by the American Association for the Advancement of Science to be one of the 20 most important scientific discoveries of the 20th century (Science, 1984). Extended to the composite null hypothesis that $F$ belongs to a known parametric family of survivor functions $F = (F(\cdot; \theta) : \theta \in \Theta)$, where $\Theta$ is an open subset of $\mathbb{R}^K$, Pearson's test is also a source of intriguing and surprising results. For instance, Chernoff and Lehmann (1954) showed that if $\theta$ is estimated by the more efficient maximum likelihood estimator (MLE) instead of a minimum chi-square estimator, then the limiting distribution of the $\chi^2$ statistic is not a chi-square distribution but rather depends on the unknown parameter $\theta$ and is bounded by two chi-square distributions. This phenomenon led to variants of the $\chi^2$ statistic which are not of the simple form above but which have chi-square limiting distributions. An informative account of these results can be found in the review paper by Moore (1978).

A review of the literature shows that Pearson's procedures have not yet been satisfactorily extended to censorship models, and due to their success in the complete data setting, it is fitting and appealing to explore censored data analogs. This paper studies analogs of Pearson's goodness-of-fit procedures under the random censorship model when the
null hypothesis is simple, while a sequel will deal with the composite null hypothesis case. Specifically, in Section 3, it examines the limiting distribution of a natural analog of Pearson's $\chi^2$ statistic, denoted by $Q_1$, and it ascertains the effect of censoring on its limiting distribution. It is shown that the limiting distribution of $Q_1$ under the null hypothesis is not chi-square, and if a test is performed using the usual percentile points of the chi-square distribution, then the resulting test could be extremely and unacceptably anticonservative. A variant of the $Q_1$ statistic, denoted by $Q_2$ and which is a Wald-type statistic, is then considered. It is shown that this statistic has a limiting chi-square distribution with $df = M - 1$, and an asymptotically $\alpha$-level test based on $Q_2$ is proposed. Both statistics $Q_1$ and $Q_2$ are developed via the self-consistency property (Efron, 1967) of the product-limit estimator (PLE) (Kaplan and Meier, 1958). Hence the resulting procedures have intuitive motivations. The PLE and its self-consistency property are briefly reviewed in Section 2. Akritas (1988) presented an interesting Pearson-type procedure for testing a simple null and composite null hypotheses under the random censorship model, wherein he partitioned the two half-lines corresponding to the uncensored and censored values. This enabled him to count exact frequencies of observations falling in the intervals of his partition of $R$, but at the cost of forcing him to estimate the expected frequencies of these intervals, even in the simple null hypothesis case. As will be pointed out in Section 2, Akritas' test statistic may not be formally computable for some sample realizations when the partition is done independently of the sample data. In contrast, the proposed procedure partitions $R$ analogously to Pearson's procedure and estimates the "observed" frequencies of the intervals by appealing to the self-consistency property of the PLE. The exact expected frequencies of the intervals are, however, obtainable in the usual way. In Section 4 the proposed procedure and Akritas' procedure are assessed through a computer simulation comparing their finite sample significance levels and powers. Under the three simulation models considered, it can be concluded that the proposed procedure based on $Q_2$ usually achieves the desired significance levels under certain conditions on the number of intervals in the
partition, sample size, and censoring proportion, while Akritas’ procedure tends to be a bit anticonservative. Akritas’ procedure on the other hand has better power than the proposed test under the model with Weibull lifetimes, but does not dominate the proposed test when the lifetimes are exponentially and normally distributed. In Section 5, the proposed procedure and Akritas’ procedure are applied to a data set in Hollander and Proschan (1979). The p-values for these procedures, associated with the null hypothesis that the lifetime distribution is exponential with mean 100, are in reasonable agreement.

Aside from Akritas (1988) there are other goodness-of-fit procedures that have been presented for censored data. Habib and Thomas (1986) proposed chi-square goodness-of-fit procedures based on the PLE for the composite null hypothesis case under random censorship. In estimating the unknown parametric index θ they used the MLE, hence their results are the censored data analog of Chernoff and Lehmann’s (1954) results. Mihalko and Moore (1980) also considered chi-square tests under Type II censorship, while Turnbull and Weiss (1978), Gail and Ware (1979), and O’Neill (1984) considered tests when the data are grouped. Analogs of the Kolmogorov-Smirnov and Cramér-von Mises tests are also available, such as those by Barr and Davidson (1973), Koziol and Byar (1975), Koziol and Green (1976), Pettit and Stephens (1976), Koziol (1980), Csörgő and Horváth (1981) and an exact test by Guilbaud (1988). Of course, graphical methods, such as Nelson’s (1972) hazard plotting techniques, preceded most of these formal tests, and are still being developed; cf. Arjas (1988). Other specialized types of tests include those of Breslow (1975), Hollander and Proschan (1979) and Chen (1984). Diagnostic tools for assessing the adequacy of Cox’s proportional hazards model (Cox 1972) are discussed in Fleming and Harrington (1991, ch. 4). A major motivation for this paper is our belief that an omnibus test such as an analog of Pearson’s test would be most appealing to the general practitioner of statistics.
2. PRELIMINARIES

2.1 RANDOM CENSORSHIP MODEL

Let \( X_1, ..., X_n \) be iid nonnegative random variables with common continuous survivor function \( F(x) = P(X_1 > x) \), and let \( Y_1, ..., Y_n \) be another iid sequence of nonnegative random variables, independent of the \( X_i \)'s, with common continuous survivor function \( G(y) = P(Y_1 > y) \). Both \( F \) and \( G \) are assumed unknown. In clinical trials or life-testing studies the \( X_i \)'s typically denote failure times of units in the study, where failure may mean death, relapse, nonfunctioning of unit, or some other endpoint event. The \( Y_i \)'s on the other hand right-censor the \( X_i \)'s so the available observations are \( (Z_i, \delta_i) \) \( (i=1, ..., n) \), where \( Z_i = \min(X_i, Y_i) \) and \( \delta_i = I(X_i \leq Y_i) \) with \( I(\cdot) \) denoting the indicator function. The goodness-of-fit problem that we consider in this paper is to test the simple null hypothesis \( H_0: F = F_0 \), where \( F_0 \) is a completely specified survivor function. The tests are to be based on the censored data \( (Z_i, \delta_i) \) \( (i=1, ..., n) \).

2.2 PRODUCT-LIMIT ESTIMATOR AND SELF-CONSISTENCY

A major catalyst for the intense statistical and mathematical studies of the random censorship model and other censoring schemes is the fundamental paper of Kaplan and Meier (1958) which introduced the PLE of \( F \) together with its basic properties. Denoting by \( R_i \) \( (i=1, ..., n) \) the rank of \( Z_i \) among \( Z_1, ..., Z_n \), the PLE is defined by

\[
F_n(t) = \prod_{i: Z_i \leq t} \left( \frac{n - R_i}{n - R_i + 1} \right)^{\delta_i} I(t \leq Z(n))
\]

(2.1)

where \( Z(n) = \max(Z_1, ..., Z_n) \). Without censorship (2.1) reduces to the empirical survivor function. Based on intuitive considerations, Efron (1967) defined an estimator \( \hat{F} \) of \( F \) to be "self-consistent" if, for every \( t \), it satisfies
\[ n\hat{F}(t) = \sum_{i=1}^{n} \left\{ \delta_i I(Z_i > t) + (1 - \delta_i) P( X_i > t \mid X_i > Z_i, X_i - \hat{F}) \right\}. \]

Since \( \delta_i I(Z_i > t) = \delta_i I(X_i > t) \) for every \( t \), the above condition becomes

\[ n\hat{F}(t) = \sum_{i=1}^{n} \left\{ \delta_i I(X_i > t) + (1 - \delta_i) P( X_i > t \mid X_i > Z_i, X_i - \hat{F}) \right\}. \quad (2.2) \]

Efron (1967) proved that the unique self-consistent estimator of \( F \) is the PLE \( F_n \) in (2.1). He also established asymptotic properties of \( F_n \) such as its weak convergence to a Gaussian process. Note that (2.2) is equivalent to

\[ -n \int_{A} dF_n(x) = \sum_{i=1}^{n} \left\{ \delta_i I(X_i \in A) - (1 - \delta_i) \int_{A \cap (Z_i, \infty)} F_n(z) \, dF_n(z) \right\} \quad (2.3) \]

for any Borel set \( A \) on \( \mathbb{R} \). We will utilize this equation in counting the frequencies of observations in each of the intervals.

### 2.3 AKRITAS' PROCEDURE

The basic setting, which will also be the setting for the development of the proposed procedure in Section 3, is that there is a fixed partition of \( \mathbb{R}^+ = [0, \infty) \), given by \( I_1 = [a_0, a_1] \) and \( I_j = (a_{j-1}, a_j) \) (\( j = 2, ..., M \)), where \( 0 = a_0 < a_1 < ... < a_{M-1} < a_M = \infty \). Akritas defined the observed frequency \( N_{1j} \) (\( j = 1, ..., M \)) and the estimate of the expected frequency \( \hat{p}_{1j} \) (\( j = 1, ..., M \)) by, respectively,

\[ N_{1j} = \sum_{i=1}^{n} I(I_{ij} \in I_j, \delta_i = 1), \quad \hat{p}_{1j} = -\int_{I_j} \frac{G_n(u)}{F_0(u)} \, dF_0(u). \]

Here \( G_n(u) = H_n(u)/F_0(u) \) and \( H_n(u) = n^{-1} \sum_{i=1}^{n} I(Z_i > u) \), the latter being the empirical survivor function of \( Z_1, ..., Z_n \). Akritas' Pearson-type test statistic is then given by
\[ Q = \sum_{j=1}^{M} \frac{(N_{1j} - \hat{n}p_{1j})^2}{\hat{n}p_{1j}}, \] (2.4)

which he showed to converge in distribution to a chi-square distribution with df = M under 
\( H_0 \). His asymptotically \( \alpha \)-level test rejects \( H_0 \) if \( Q \geq \chi^2_M; \alpha \) and accepts \( H_0 \) otherwise. Two
remarks are in order: (1) Note that Akritas’ \( N_{1j} \)'s may not sum to \( n \), making it less a direct
analog of Pearson’s test, though this does not lead to any defects for his test; and (2)
Akritas’ Q statistic given in (2.4) is actually undefined whenever \( Z(n) < a_{M-1} \) since in that
situation \( \hat{p}_{1M} = 0 \) and \( N_{1M} = 0 \). (If \( Z(n) < a_{i-1} \) for some \( i \in \{1, ..., M\} \), then \( \hat{p}_{1j} = 0 \) and \( N_{1j} = 0 \)
for \( \forall j \geq i \).) Akritas does not mention this difficulty in his paper. This difficulty can arise
in practice since the partition of \( R \) may be determined prior to observing the sample. In fact
that is the modus operandi for performing computer simulations. To avoid having Akritas’
statistic undefined in the simulation study of Section 4 we used the convention \( 0/0 = 0 \).

3. DEVELOPMENT OF THE PROPOSED TEST

3.1 STANDARDIZED FREQUENCY VECTOR

Let \( p_{0j} = P(X_1 \in I_j \mid X_1 - \mu_0) = F_0(a_{j-1}) - F_0(a_j), \quad (j=1, ..., M), \) be the probabilities of
the intervals under \( H_0: \mu = \mu_0 \). The expected number of \( X_i \)'s that will fall in \( I_j \) is then \( E_j = np_{0j} \) \( (j=1, ..., M) \). For uncensored or complete data the observed frequency of \( X_i \)'s in \( I_j \) is given
by \( N_j = \sum_{i=1}^{n} I(X_i \in I_j) \), and Pearson’s \( \chi^2 \) statistic is \( \chi^2 = \sum_{j=1}^{M} (N_j - E_j)^2/E_j \) which, under
\( H_0 \), is asymptotically chi-square distributed with df = \( M - 1 \). With right-censored data,
however, \( N_j \) could not be determined, since when \( \delta_i = 0 \), \( I(X_i \in I_j) \) is not completely
determined. In lieu of the exact \( N_j \) we therefore estimate the frequency of \( X_i \)'s in \( I_j \)
conditional on \( (Z_i, \delta_i) \) \( (i=1, ..., n) \). Let us consider possibilities: (i) If \( \delta_i = 1 \) then \( X_i \in I_j \) if
and only if \( Z_i \in I_j \), while (ii) if \( \delta_i = 0 \) then we would only know that \( X_i > Z_i \). If \( a_j \leq Z_i \)
then \( X_i \in I_j \), while if \( a_j > Z_i \) then the conditional probability that \( X_i \in I_j \) is
\( P(X_i \in I_j \mid X_i > Z_i, X_i - F) \). This probability is unknown since \( F \) is unknown but a plausible estimator is obtained by replacing \( F \) by the PLE \( F_n \). This estimator is given by

\[
\hat{P}(X_i \in I_j \mid X_i > Z_i, X_i - F) = P(X_i \in I_j \mid X_i > Z_i, X_i - F_n).
\]

As an estimator of the observed frequency \( N_j \) (\( j = 1, \ldots, M \)) we could therefore use

\[
\hat{N}_j = \sum_{i=1}^{n} \left\{ \delta_i I(X_i \in I_j) + (1 - \delta_i) P(X_i \in I_j \mid X_i > Z_i, X_i - F_n) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ \delta_i I(X_i \in I_j) - (1 - \delta_i) \int_{I_j \cap (Z_i, \infty)} F_n(x)^{-1} dF_n(x) \right\}
\]

\[
= -n \int_{I_j} dF_n(x) = n\{F_n(a_{j-1}) - F_n(a_j)\}, \quad (3.1)
\]

where the second-to-last equality is obtained by appealing to the self-consistency property of \( F_n \) given by (2.3). Note that, in contrast to Akritas’ procedure, \( \sum_{j=1}^{M} \hat{N}_j = n \).

Let \( W_n(x) = \sqrt{n} [F_n(x) - F_0(x)], \ 0 \leq x < \infty, \) be the empirical product-limit process. If we define

\[
C(t) = -\int_{0}^{t} \frac{dF_0(u)}{F_0(u)^2 G(u)} = -\int_{0}^{t} \frac{dF_0(u)}{F_0(u) H(u)} \quad (3.2)
\]

where \( H = F_0 G, \) then from Efron (1967), (also see Breslow and Crowley (1974), and Gill (1983)) we may state Theorem 3.1. In Theorem 3.1, \( D[0, T] \) denotes Skorokhod’s space on \([0, T]\) (cf. Billingsley (1968), ch. 4), and \( \Rightarrow \) denotes weak convergence.

**THEOREM 3.1:** Let \( T > 0 \) with \( H(T) > 0 \). Under \( H_0, \ W_n \Rightarrow W \) on \( D[0, T], \) where \( W \) is a zero-mean Gaussian process with covariance function \( c(s, t) = F_0(s) F_0(t) C[\min(s, t)]. \)
Expressing \( \hat{N}_j \) in terms of \( W_n \), the vector of "observed" minus expected frequencies becomes

\[
\sqrt{n} \mathbf{V}_n = \left( \hat{N}_1 - np_{01}, ..., \hat{N}_M - np_{0M} \right)' = -\sqrt{n} \left( \int_{a_0}^{a_1} dW_n(x), ..., \int_{a_{M-1}}^{a_M} dW_n(x) \right)'.
\]

Letting \( D = \text{diag}(p_{01}^{-1/2}, ..., p_{0M}^{-1/2}) \) denote the \( M \times M \) diagonal matrix with diagonal elements \( p_{0j}^{-1/2} \), \( (j=1, ..., M) \), the standardized frequency vector is given by

\[
\mathbf{V}_n = D\mathbf{V}_n' = \left( p_{01}^{-1/2} \int_{a_0}^{a_1} dW_n(x), ..., p_{0M}^{-1/2} \int_{a_{M-1}}^{a_M} dW_n(x) \right)'.
\]

(3.3)

### 3.2 Asymptotics

Let \( B(t) = 1 - F_0(t) - F_0(t)C(t) \), and for notational convenience, write \( F_{0i} \) for \( F_0(a_i) \), \( C_i \) for \( C(a_i) \), and \( B_i \) for \( B(a_i) \) \( (i=0, ..., M-1) \). Furthermore, let \( q = (\sqrt{p_{01}}, ..., \sqrt{p_{0M}})' \) be the \( M \times 1 \) vector with elements \( \sqrt{p_{0j}} \) \( (j=1, ..., M) \), and denote by \( I_M \) the \( M \times M \) identity matrix.

Then we have the following result.

**Theorem 3.2:** If \( H(a_{M-1}) > 0 \) then, under \( H_0 \), \( V_n \) converges in distribution to an \( M \)-dimensional multivariate normal vector \( \mathbf{V} \) with mean vector \( 0 = (0, ..., 0)' \) and covariance matrix \( \Sigma = I_M - qq' - L \), where \( L = (l_{ij}) \) is the \( M \times M \) symmetric matrix with elements

\[
l_{ij} = B_i - B_{i-1} + \frac{F_{0i}}{p_{0i}} (B_i - B_{i-1}) \quad \text{for } i = j;
\]

\[
= -\sqrt{\frac{p_{0i}}{p_{0j}}} (B_i - B_{i-1}) \quad \text{for } i < j.
\]

**Proof:** Let \( (D, D) \) denote Skorokhod's measurable space on \([0, a_{M-1}]\), so that, since \( H(a_{M-1}) > 0 \), Theorem 3.1 holds. Let \( \pi_1, ..., \pi_{M-1} : (D, D) \rightarrow (\mathbb{R}_M, B_M) \) be the natural projection map defined by \( \pi_1, ..., \pi_{M-1}(w) = (w(a_1), ..., w(a_{M-1})) \); let \( \kappa : (\mathbb{R}_M, B_M) \rightarrow (\mathbb{R}_M, B_M) \) be defined by \( \kappa(x_1, ..., x_{M-1}) = (x_1, x_2 - x_1, ..., x_{M-1} - x_{M-2}, -x_{M-1})' \); and let \( \gamma : (\mathbb{R}_M, B_M) \rightarrow (\mathbb{R}_M, B_M) \) be defined by \( \gamma(x_1, ..., x_{M-1}) = (x_1 p_{01}, ..., x_M p_{0M})' \). Then by
letting \( h: (D, D) \to (R^M, B^M) \) be defined by \( h = \gamma \circ \kappa \circ \pi_1, \ldots, M-1 \), where \( \circ \) denotes function composition, \( V_n = h(W_n) \). Since \( h \) is a continuous map, then by the continuous mapping theorem (Theorem 5.1 of Billingsley (1968)) and by Theorem 3.1, it follows that \( V_n = h(W_n) \) converges weakly or in distribution to \( V = h(W) \), where \( W \) is given in Theorem 3.1. Since \( W \) is a Gaussian process, its finite-dimensional distributions are multivariate normal, and since \( \pi_1, \ldots, M-1 \) is a natural projection map and both \( \kappa \) and \( \gamma \) are linear transformations, then \( V \) has a multivariate normal distribution. Now, \( \pi_1, \ldots, M-1(w) = (w(a_1), \ldots, w(a_{M-1}))' \sim N_{M-1}(0, \Sigma^*) \), where \( \Sigma^* \) has elements \( c(a_i, a_j) \) \( (i, j = 1, \ldots, M-1) \) with the function \( c(s, :) \) given in Theorem 3.1. By the definitions of \( c(s, t) \) and \( B(t) \) however, note that \( c(s, t) = F_0[\max(s, t)][1 - F_0[\min(s, t)]] - F_0[\max(s, t)]B[\min(s, t)]. \) Hence if \( \Sigma^* = (\sigma_{ij}) \), then \( \sigma_{ij} = F_0(1 - F_{0i}) - F_0B_i \) for \( i \leq j \). The linear transformations \( \kappa \) and \( \gamma \) can be represented, respectively, by the \( M \times (M-1) \) matrix \( A = (a_{ij}) \) and the \( M \times M \) matrix \( D \), where \( a_{ij} = 1 \) if \( i = j; \ -1 \) if \( i = 1 \) and \( j = 1, \ldots, M-1; \) and 0 otherwise. Thus, \( V = h(W) = DA\pi_1, \ldots, M-1(W) \sim N_M(0, \Sigma) \) where \( \Sigma = DA\Sigma^*A'D \). A straightforward calculation then shows that \( \Sigma = I_M - qq' - L \), where the matrix \( L \) is as given in the statement of the theorem.

Note that if there is no censoring, \( G(t) = 1 \ \forall t \), which implies that \( B(t) = 0 \ \forall t \), hence \( L = 0 \). In this case the limiting covariance matrix of \( V_n \) is \( \Sigma = I_M - qq' \), the well-known limiting covariance matrix of the standardized frequency vector for complete data (see Moore (1978)). The matrix \( L \) thus represents the effect of censorship on the limiting distribution of \( V_n \). Below we examine further the limiting covariance matrix \( \Sigma \). We first present two standard results about matrices.

**Lemma 3.1:** Let \( \alpha_1, \ldots, \alpha_M \) and \( \beta_1, \ldots, \beta_M \) be real numbers, and let \( \Sigma = (\xi_{ij}) \) be the \( M \times M \) symmetric matrix with elements \( \xi_{ij} = \alpha_i \alpha_j \beta_i \) for \( i \leq j \). Then \( \det(\Sigma) = \prod_{i=1}^{M} \alpha_i^2 \beta_i - \beta_{i-1} \), where \( \beta_0 = 0 \).
PROOF: The assertion trivially holds when $M = 1$. Assume then that it holds for $M = k$.

The induction proof will be complete if we could show that it holds for $M = k$. Let $\Xi$ denote the associated $k \times k$ matrix. Then $\det(\Xi) = \sum_{i=1}^{k} a_1 a_i \beta_1 \Xi_{11}$, where $\Xi_{11}$ is the cofactor associated with $\xi_{11} = a_1 a_i \beta_1$. Using the induction hypothesis we obtain that

$$
\Xi_{11} = a_2^2 \cdots a_k^2 (\beta_3 - \beta_2) \cdots (\beta_k - \beta_{k-1})$$

and

$$
\Xi_{12} = -a_1 a_2 a_3 \cdots a_k^2 (\beta_3 - \beta_2) \cdots (\beta_k - \beta_{k-1}),
$$

while $\Xi_{1j} = 0$ for $j = 3, \ldots, k$ since the relevant submatrices are singular. Simplifying we then obtain that $\det(\Xi) = \prod_{i=1}^{k} a_i^2 (\beta_i - \beta_{i-1})$. \( \square \)

**LEMMA 3.2:** Let $A$ be an invertible matrix and $B$ be another matrix such that $AB$ is well-defined. Then $\text{rank}(AB) = \text{rank}(B)$.

**PROOF:** Assume that $A$ is $n \times n$, and $B$ is $n \times m$, and denote by $B$ the range space of $B$ and by $C$ the range space of $AB$. By definition of range spaces, $C = \{ y : y = ABx, \forall x \in \mathbb{R}^n \}$

$= \{ y : y = Az, \forall z \in B \}$. Since $A$ is invertible, then $B$ and $C$ are isomorphic, hence have equal dimensions. But $\text{rank}(C) = \text{dimension}(C) = \text{dimension}(B) = \text{rank}(B)$. \( \square \)

**THEOREM 3.3:** If $p_0 > 0$ (i = 1, ..., M) then $\text{rank}(\Sigma) = M - 1$.

**PROOF:** Recall from the proof of Theorem 3.2 that $\Sigma = DA \Sigma^* A'D$. Since $p_0 > 0$ (i = 1, ..., M), then $D$ is of rank $M$, while clearly, $A$ is of rank $M - 1$. By Lemma 3.2, $DA$ is therefore of rank $M - 1$. Since $\Sigma^* = (\sigma_{ij}^*)$ has elements $\sigma_{ij}^* = F_{0i} F_{0j} C_i$ for $i \leq j$, then by Lemma 3.1, $\det(\Sigma^*) = \prod_{i=1}^{M-1} F_{0i}^2 (C_i - C_{i-1})$. From (3.2) it is also easily seen that $C_i - C_{i-1} \geq p_0 > 0$ (i = 1, ..., M), hence $\det(\Sigma^*) > 0$ implying that $\Sigma^*$ is invertible. Furthermore, $\Sigma^*$ can be expressed as $TT'$ where $T$ is an invertible $(M - 1) \times (M - 1)$ matrix. Consequently, $\Sigma = (DAT)(DAT)'$, and the rank of $\Sigma$ equals the rank of $DAT$. Since $\text{rank}(DA) = M - 1$ and

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is invertible, then Lemma 3.2 implies that \( \text{rank}(DAT) = M - 1 \), completing the proof of the theorem. \( \| \)

3.3 ANALOG OF PEARSON'S STATISTIC AND EFFECT OF CENSORING

The censored data analog of Pearson's \( \chi^2 \) test statistic is now defined by \( Q_1 = V_n^T V_n \)
\[
= \sum_{j=1}^{M} \left( \hat{N}_j - np_{0j} \right)^2 / (np_{0j}).
\]
As pointed out following the proof of Theorem 3.2, when there is no censoring the limiting covariance matrix of \( V_n \) is \( \Sigma = I_M - qq' \) which is known to be idempotent of rank \( M - 1 \). Consequently, with complete data, \( Q_1 \overset{d}{\to} \sum_{i=1}^{M-1} Z_i^2 \), where \( Z_1, ..., Z_{M-1} \) are iid standard normal random variables, and "\( \overset{d}{\to} \)" means "converges in distribution". Thus, with complete data, \( Q_1 \) is asymptotically chi-square with \( df = M - 1 \).

With censored data, the limiting covariance matrix of \( V_n \) is \( \Sigma = I_M - qq' - L \) from Theorem 3.2. This covariance matrix is not idempotent, but is still of rank \( M - 1 \) from Theorem 3.3. From well-known results about multivariate normal distributions (cf. Moore (1978)), it follows that

\[
Q_1 \overset{d}{\to} \sum_{i=1}^{M-1} \lambda_i Z_i^2,
\]

where \( \lambda_1, ..., \lambda_{M-1} \) are the non-zero eigenvalues of \( \Sigma \). Determining closed-form expressions for the \( \lambda_i \)'s is not easy; however we get a glimpse of the effect of censorship on the limiting distribution of \( Q_1 \) by specializing to the case with two intervals, that is, with \( M = 2 \).

THEOREM 3.4: If \( M = 2 \) then, under \( H_0 \), \( Q_1 \overset{d}{\to} \lambda Z^2 \), where \( Z \) is standard normal, and \( \lambda = p_{02}c_1 / p_{01} \). Furthermore, \( 1 \leq \lambda \leq 1/G(a_1) \).

PROOF: The characteristic equation of \( \Sigma \), \( \det(\lambda I_M - \Sigma) = 0 \), is given by \( [\lambda - (1-p_{01}-l_{11})] \)
\[
[\lambda - (1-p_{02}-l_{22})] - [\sqrt{p_{01}p_{02}} + l_{12}]^2 = 0,
\]
where the \( l_{ij} \)'s are defined in Theorem 3.2. Simplifying, this equation becomes \( \lambda [\lambda - (1 - B_1/p_{01})] = 0 \). Consequently, the non-zero
eigenvalue is \( \lambda = 1 - \frac{B_1}{P_{01}} = \frac{P_{02}C_1}{P_{01}} \). From (3.4) the first assertion follows. By noting that

\[
\frac{P_{01}}{P_{02}} = \int_0^{a_1} \frac{dF_0(u)}{F_0(u)^2} \leq C_1 = \int_0^{a_1} \frac{dF_0(u)}{F_0(u)^2 G(u)} \leq \int_0^{a_1} \frac{dF_0(u)}{F_0(u)^2 G(a_1)} = \frac{P_{01}}{P_{02} G(a_1)},
\]

it follows that \( 1 \leq \lambda \leq \frac{1}{G(a_1)} \). \|

Suppose one employs the straightforward analog of Pearson's test, viz. reject \( H_0 \) if \( Q_1 \geq \chi_{M-1; \alpha/2} \) and accept \( H_0 \) otherwise. For the case \( M = 2 \), Theorem 3.4 shows that such a test will be asymptotically anticonservative, and it is our conjecture that such a property still holds true when \( M > 2 \). In fact, the simulation study in Section 4 shows that the anticonservatism of this test could be severe. Thus such a test should not be used in practice, even as an approximate test. Of course, an asymptotically exact test could still be based on the statistic \( Q_1 \) wherein the eigenvalues of \( \Sigma \) are consistently estimated and the distribution of the right-hand side of (3.4) determined. Such an approach, however, defeats the purpose of developing a practical test. For one thing the distribution of the right-hand side of (3.4) is not chi-square. Due to these disadvantages we turn to a variant of \( Q_1 \) which has a limiting chi-square distribution.

### 3.4 A WALD-TYPE STATISTIC AND THE PROPOSED TEST

From the theory of generalized inverses recall the following important characterization (cf. Moore (1978)): Let \( X \sim \mathcal{N}_{M}(0, \Sigma) \) where \( \text{rank}(\Sigma) = r \leq M \). Then \( X'CX \) has a chi-square distribution with \( df = r \) if and only if \( C = \Sigma^{-} \), where \( \Sigma^{-} \) is a generalized inverse of \( \Sigma \). Recall that a generalized inverse \( G \) of a matrix \( A \) satisfies \( AGA = A \). As defined such a generalized inverse is not unique; however, if we impose the additional requirements that \( AG \) and \( GA \) be symmetric, and \( GAG = G \), then the resulting \( G \) is unique and is referred to as the Moore-Penrose inverse. This is usually denoted by \( A^+ \). Since \( \Sigma^+ \) is
clearly a generalized inverse of \( \Sigma \), then it follows from the above characterization that 
\( X' \Sigma^+ X \) has a chi-square distribution with \( df = r \). We base our development of our proposed procedure on this result.

To construct a test statistic with a limiting chi-square distribution, there is now a need to estimate \( \Sigma = I_M - qq' - L \) in Theorem 3.2 since \( L \) depends on the unknown functional parameter \( G \). Under \( H_0 \), \( Z_1, ..., Z_n \) are iid from \( H = F_0 G \), hence by the Glivenko-Cantelli Theorem a uniformly strongly consistent estimator of \( H \) is provided by \( H_n \), the empirical survivor function of the \( Z_i \)'s. It follows that a natural estimator of \( C(t) \) in (3.2) is given by

\[
C_n(t) = - \int_0^t \frac{dF_0(u)}{F_0(u) H_n(u)}.
\]  

(3.5)

For computational purposes note that if \( Z(1) < Z(2) < ... < Z(n) \) are the order statistics associated with the \( Z_i \)'s, then for \( t \) with \( t \in [Z_{(k-1)}, Z_{(k)}) \),

\[
C_n(t) = \sum_{i=1}^{k-1} \left( \frac{n}{n-i+1} \right) \ln \left( \frac{F_0(Z_{i-1})}{F_0(Z_i)} \right) + \left( \frac{n}{n-k+1} \right) \ln \left( \frac{F_0(Z_{k-1})}{F_0(t)} \right).
\]

Lemma 3.3 gives the uniform strong consistency of \( C_n \). Its proof is straightforward, hence omitted.

**LEMMA 3.3:** If \( H(a_{M-1}) > 0 \) then, under \( H_0 \), \( \lim_{n \to \infty} \sup_{0 \leq t \leq a_{M-1}} |C_n(t) - C(t)| = 0 \) almost surely (a.s.).

By applying Lemma 3.3, the empirical function \( B_n(t) = 1 - F_0(t) - F_0(t)C_n(t) \) is then uniformly strongly consistent for \( B(t) \) on \([0, a_{M-1}]\), provided that \( H(a_{M-1}) > 0 \). Let \( b_{ni} = B_n(a_i) \) \( (i = 0, 1, ..., M-1) \), and \( \Sigma_n = I_M - qq' - L_n \), where \( L_n = (l_{nij}) \) is an \( M \times M \) symmetric matrix with elements

\[
l_{nij} = b_{ni-1} + \frac{F_0(i)}{P_{0i}} (b_{ni} - b_{ni-1}) \quad \text{for } i = j; \quad -\sqrt{\frac{P_{0i}}{P_{0i}}} (b_{ni} - b_{ni-1}) \quad \text{for } i < j.
\]
It follows from Theorem 3.2 and Lemma 3.3 that, under $H_0$, $\Sigma_n \to \Sigma$ a.s. as $n \to \infty$. Let

$$\Psi_n = 1(\Sigma_n \text{ has rank } M - 1) .$$

From Theorem 3.3, if $p_{0i} > 0$ ($i = 1, \ldots, M$), $\Psi_n \to 1$ a.s. as $n \to \infty$.

Since the elements of the Moore-Penrose inverse $A^+$ of $A$ are continuous functions of the elements of $A$, then for every sample realization with $\Psi_n = 1$, $\Sigma_n^+ \to \Sigma^+$. Consequently, $\Sigma_n^+ \to \Sigma^+$ a.s. as $n \to \infty$. From these results we immediately obtain the following theorem concerning the limiting distribution of the Wald-type statistic $Q_2$ defined by

$$Q_2 = V_n \Sigma_n^+ V_n' .$$

(3.6)

**THEOREM 3.5:** Assume that $p_{0i} > 0$ ($i = 1, \ldots, M$). Then, under $H_0$, $Q_2 \overset{d}{\to} \chi_{M-1}^2$ as $n \to \infty$.

The proposed asymptotically exact $\alpha$-level test of $H_0$ is: Reject $H_0$ if $Q_2 \geq \chi_{M-1}^2; \alpha$, and accept otherwise. In the next section we compare this test with that of Akritas in terms of their achieved significance levels and powers under three models and with varying values of $M, n$, and degrees of censoring.

### 4. COMPARISON OF AKRITAS' TEST AND TEST BASED ON $Q_2$

Program development of this simulation study was performed on a VAX 785 computer at Bowling Green State University, while the final computer runs were done on a Cray Y-MP8 supercomputer at the Ohio Supercomputer Center. Random numbers in this simulation were generated via IMSL routines, and the Moore-Penrose inverses needed in the test statistic $Q_2$ were computed via the IMSL routine LSGRR. Three models corresponding to different specifications of the true survivor functions were considered. In model (I), the true survivor function was exponential given by $F(x; \theta) = e^{-\theta x}$ for $x \geq 0$, and $\theta$ was set to the values $\theta = 2.00, 1.33, 1.00, 0.75,$ and $0.50$ — the value $\theta = 1.00$ corresponding to the null hypothesis. The censoring survivor function was of form $G(y; \theta, \eta) = e^{-\eta y}$ for $y \geq 0$. Given values of $\theta$ and the censoring probability (CP), the parameter $\eta$ was determined in order to
have \( CP = P(X_1 > Y_1) \), so that, \( \eta = \frac{\theta CP}{1 - CP} \). In model (II), the true survivor function was of the Weibull family given by \( F(x; \theta) = \exp(-x^\theta) \) for \( x \geq 0 \), and \( \frac{1}{\theta} \) was set to the values 0.65, 0.87, 1.00, 1.50, and 2.00, which correspond, respectively, to the specifications that the ratios between the true mean and the null hypothesis mean be (approximately) equal to 0.90, 0.95, 1.00, 1.33, and 2.00. The value of \( \theta = 1.00 \) corresponds to the null hypothesis. The censoring survivor function was also of the Weibull family given by \( G(y; \theta, \eta) = \exp(-\eta y^\theta) \) for \( y \geq 0 \), and where, for given values of \( \theta \) and \( CP \), \( \eta \) was chosen in order to have \( CP = P(X_1 > Y_1) \), so that, \( \eta = \left( \frac{CP}{1 - CP} \right)^{1/\theta} \). In model (III), the true survivor function was given by \( F(x; \theta) = 1 - \Phi(x - \theta) \), where \( \Phi \) is the standard normal cumulative distribution function. The parameter \( \theta \) was set to -1.0, -0.5, 0.0, 0.5, and 1.0, with \( \theta = 0.0 \) corresponding to the null hypothesis. The censoring survivor function was of form \( G(y; \theta, \eta) = (1 - \Phi(y - \theta))^\eta \). For given values of \( \theta \) and \( CP \), \( \eta \) was determined in order to have \( CP = P(X_1 > Y_1) \), so that \( \eta = \frac{CP}{1 - CP} \). For each model, \( CP \) was set to 33\%, 50\%, and 67\%, the "sample size" parameter \((n)\) was set to 50, 100, and 200, and the "number of intervals" parameter \((M)\) was set to 2, 5, and 10. The boundaries \( a_1, ..., a_{M-1} \) were determined in order to have equal probabilities for the intervals \( I_i = (a_{i-1}, a_i) \) \((i = 1, ..., M)\), under \( H_0 \). For each combination of \( \theta \), \( CP \), \( n \), and \( M \), 1000 replications of the following basic experiment were performed. Samples \( X_1, ..., X_n \) iid from \( F(x; \theta) \) and \( Y_1, ..., Y_n \) iid from \( G(y; \theta, \eta) \) were generated, and the associated censored data \((Z_i, \delta_i) \) \((i = 1, ..., n)\), obtained. Using these censored data, it was determined whether Akritas' test based on \( Q \) (described in Section 2.3), the proposed test based on \( Q_2 \) (described in Section 3.4), and the approximate test based on \( Q_1 \) (described in Section 3.3), reject \( H_0 \) for the significance levels \( \alpha = 1\%, 5\%, \) and \( 10\% \). Since the complete data \( X_1, ..., X_n \) were also available, Pearson's complete data test described in Section 1 was also applied (of course, in practice this is not possible since the \( X_i \)'s will be unobserved). For each of these tests, the percentages of rejections of \( H_0 \) were determined. We also kept track of the percentages of censored samples \((Z_i, \delta_i) \) \((i = 1, ..., n)\) for which \( Z_{(n)} < a_{M-1} \) (in Tables 1 and 2 these
percentages are denoted by LS for 'low sample'). Recall from the the description of Akritas' procedure in Section 2.3 that Q is not formally defined for such censored data. As mentioned in Section 2.3, for such samples we used the convention \(0/0 = 0\) in order to compute \(Q\).

Tables 1, 2 and 3 present summaries of the simulation results for models (I), (II) and (III), respectively. We did not include in these summaries the results associated with the approximate test based on \(Q_1\) since this test is simply too anticonservative. For example, under model (I) with \(CP = 33\%\), \(n = 100\), \(M = 5\), and \(\alpha = 10\%\), the achieved significance level of the \(Q_1\)-test is 23.6\%, while those of the \(Q\)-test and the \(Q_2\)-test are 11.2\% and 8.5\%, respectively. The simulation results suggest that the approximate test based on \(Q_1\) should not be used in practice, even for exploratory and/or approximate purposes. Since the conclusions for \(\alpha = 1\%\) are similar to those for \(\alpha = 5\%\) and \(\alpha = 10\%\), we have omitted the results for the former to save space. We also omit the results associated with Pearson's complete data test.

**INSERT TABLES 1-3 HERE**

Under the null hypothesis, models (I) and (II) are identical, hence discussions about the level of tests could be limited to models (I) and (III). Referring to Tables 1 and 3, we observe that whenever LS is less than 30\% the \(Q_2\)-test achieves the desired significance levels, while the \(Q\)-test tends to be anticonservative for small \(n\) (e.g., in Table 1, for \(M = 5\), \(CP = 33\%\), \(n = 50\), and \(\alpha = 10\%\), the simulated levels are 13.5\% for the \(Q\)-test and 9.2\% for the \(Q_2\)-test; while in Table 3, for \(M = 5\), \(CP = 50\%\), \(n = 100\), and \(\alpha = 5\%\), the simulated levels are 8.2\% for the \(Q\)-test and 5.9\% for the \(Q_2\)-test). Recall that the statistic LS denotes the percentage of the 1000 replications in which \(Z(n) < a_{M-1}\), thus estimates

\[
P(Z(n) < a_{M-1}) = \prod_{i=1}^{n} P(Z_i < a_{M-1}) = [1 - F(a_{M-1})G(a_{M-1})]^n. \tag{4.1}
\]
For fixed $a_{M-1}$, the probability in (4.1) approaches zero as $n \to \infty$, provided that $F(a_{M-1})G(a_{M-1}) > 0$. However, for fixed $n$, the probability in (4.1) could be large for some choices of $M$ and $a_{M-1}$. For instance, under $H_0$ of models (I) and (II), if $M = 10$ and CP = 67%, then since $a_9 = 2.3026$, the probability in (4.1) equals 0.9109 and 0.8298 for $n = 100$ and $n = 200$, respectively. Notice that the observed values of LS are consistent with these theoretical values. Suppose now that we have a censored sample with $Z(n) < a_{M-1}$. Then as pointed out earlier, we need the convention that $0/0 = 0$ in order to compute Akritas' $Q$-statistic. On the otherhand, $\hat{N}_M = \int_{a_{M-1}}^{\infty} dF_n(u) = 0$, hence the last component of the standardized frequency vector $V_n$ is $\sqrt{p_{0M}}$. The same thing happens with the $(M - 1)_{th}$ interval if $Z(n) < a_{M-2}$. Thus, if the probability in (4.1) is large, the sampling distributions of $Q_1$ and $Q_2$ are shifted to the right of zero. This explains the anticonservatism of the $Q_2$-test in the simulation when LS is large. Therefore, in practice, $M$ and $a_{M-1}$ should be chosen such that the probability in (4.1) is not large. In contrast, the anticonservatism of the $Q$-test seems somewhat unrelated to the value of LS.

With respect to the powers of the tests, by referring to Table 2, we find that the $Q$-test is more powerful than the $Q_2$-test under model (II). In fact, we mention that the $Q$-test even outperforms Pearson's complete data test under model (II) with $M = 2$, and this is surprising since the former test used the censored data while the latter test used the complete data. For $M = 5$ and $M = 10$, however, Pearson's test outperforms the $Q$-test. For models (I) and (III), the $Q$-test does not dominate the $Q_2$-test. The general conclusion that can be obtained by referring to Tables 1 and 3 is that the $Q$-test is more powerful than the $Q_2$-test when the true mean is smaller than the null hypothesis mean, while when the true mean is larger than the null hypothesis mean the $Q_2$-test is better than the $Q$-test. As expected, increasing $n$ leads to an increase in power for these tests and better achieved significance levels. Also, an increase in CP leads to a decrease in the power for these tests. On the otherhand, an increase in $M$ does not necessarily increase the power of these tests.
This is a perennial problem for these types of tests since a balance needs to be struck among \( n, M, \) the interval boundaries, and CP to achieve optimum results, and this problem still requires more study.

5. EXAMPLE

We applied the proposed test based on \( Q_2 \) and Akritas test based on \( Q \) to the prostate cancer data in Hollander and Proschan (1979). Those data are an updated version of the data used by Koziol and Green (1976). The data are from a Veterans Administration Cooperative Urological Group study of 211 state IV prostate cancer patients that were treated with estrogen. Of these 211 patients, 90 died of prostate cancer, 105 died of other diseases, and 16 were still alive at the March 1977 closing of the study. In the analysis, those patients that died of other causes and those that were still alive were considered censored. The null hypothesis tested was that the true survivor function was exponential with mean 100. Koziol and Green reported that it was postulated that if the patients had not been treated with estrogen, this particular exponential would describe their survival. The minimum \( Z \)-value and the maximum \( Z \)-value in the data set were 0 and 164, respectively. Another feature of this data set is the presence of tied values among the \( Z \)'s. To accommodate ties, we computed the PLE using the right-continuous version of formula (1.10) of Kalbfleisch and Prentice (1980). This version of the PLE is given by

\[
F_n(t) = \prod_{i: Z_i \leq t} \left( 1 - \frac{d_i}{n_i} \right) I(t \leq Z_{(n)}),
\]

where the \( Z_i \)'s are the distinct uncensored \( Z \)-values, \( d_i \) is the number of uncensored \( Z \)-values equal to \( Z_i \), and \( n_i \) is the number at risk just before \( Z_i \). We chose \( M = 5 \) and the intervals to have equal probabilities under \( H_0 \). These intervals are \( I_1 = [0, 22.31) \), \( I_2 = [22.31, 51.08) \), \( I_3 = [51.08, 91.63) \), \( I_4 = [91.63, 160.94) \), and \( I_5 = [160.94, \infty) \). For this partition, the value of the \( Q\)-
statistic is 13.9054 which has an associated p-value of 0.0162 based on a chi-square
distribution with df = 5. The Q2-statistic equals 10.3662 with associated p-value of 0.0347
based on a chi-square distribution with df = 4. Finally, for completeness, we mention that
the nonzero eigenvalues of the covariance matrix Σn are 1.1835, 1.6325, 2.4820, and 5.2416.

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\[ P(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \]

TABLE 8: Simulated Powers of Tests and Proposed Tests under a Model where the True

\[ P(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \]

Lifet ime Survivor Function is \( P(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \)
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<th>( \alpha )</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
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<tbody>
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<td>0.0013</td>
<td>0.012</td>
<td>0.075</td>
<td>0.22</td>
<td>0.59</td>
<td>1.37</td>
<td>5.00</td>
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<tr>
<td>( M = 5 )</td>
<td>0.0013</td>
<td>0.012</td>
<td>0.075</td>
<td>0.22</td>
<td>0.59</td>
<td>1.37</td>
<td>5.00</td>
</tr>
<tr>
<td>( M = 2 )</td>
<td>0.0013</td>
<td>0.012</td>
<td>0.075</td>
<td>0.22</td>
<td>0.59</td>
<td>1.37</td>
<td>5.00</td>
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Table 3 Continued