STOCHASTIC ORDER FOR REDUNDANCY ALLOCATIONS
IN SERIES AND PARALLEL SYSTEMS

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Stochastic Order for Redundancy Allocations in Series and Parallel Systems

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ABSTRACT

The problem of where to allocate a redundant component in a system in order to optimize the lifetime of a system is an important problem in reliability theory which also poses many interesting questions in mathematical statistics. We consider both active redundancy and standby redundancy, and investigate the problem of where to allocate a spare in a system in order to stochastically optimize the lifetime of the resulting system. Extensive results are obtained in particular for series and parallel systems.
§1. Introduction.

The problem of where to allocate a redundant component in a system in order to optimize the lifetime of the system is an important problem in reliability theory, which also poses many interesting questions in mathematical statistics. There are two common types of redundancy that are used, namely active or parallel redundancy (which stochastically leads to consideration of the maximum of random variables) and standby redundancy (which stochastically leads to consideration of the convolution of random variables). Suppose for example that $T_1$ and $T_2$ are two independent nonnegative random variables representing the lifetimes of two components in a series system. Assume the lifetime of a spare component is $T$ (independent of $T_1$ and $T_2$), and this component is available for active (or standby) redundancy with one of the components in the system. If $T_1$ is stochastically less than $T_2$ (that is $P[T_1 \leq t] \geq P[T_2 \leq t]$ for all $t$, and written $T_1 \overset{st}{\leq} T_2$), then under what conditions is it stochastically more favorable to make the redundancy with component 1 than with component 2? To put it another way, under what circumstances is

$$ (1) \quad \min\{\max(T_1, T), T_2 \} \overset{st}{\geq} \min\{T_1, \max(T_2, T)\} \quad \text{(active redundancy)} $$

and/or

$$ (2) \quad \min\{T_1 + T, T_2 \} \overset{st}{\geq} \min\{T_1, T_2 + T\} \quad \text{(standby redundancy)} $$

We will see that (1) holds generally, but that (2) holds for all such $T$ if and only if the densities or mass functions of $T_1$ and $T_2$ belong to a one parameter family with a monotone likelihood ratio property.

More generally we consider the problem of allocating a redundant spare in a $k$ out of $n$ system where the components are independent and stochastically ordered. A $k$ out of $n$ system is a system of $n$ components which functions if and only if at least $k$ of its components function. See Barlow and Proschan [2] for an extensive treatment of $k$ out of $n$ systems and coherent systems in general. We show that for active redundancy it is stochastically optimal to always allocate to the weakest component. In the case of standby redundancy the problem is more complex, but we give sufficient conditions to ensure that in a series system the allocation should go to the weakest component while in a parallel system it should go to the strongest.

In section 2 we present stochastic results for order statistics motivated by active and standby redundancy operations. Throughout we use $F$ to denote the distribution function of a random variable $X$, and let $\bar{F} = 1 - F$. In section 3 we discuss applications in reliability theory of the mathematical results of section 2. Section 4 concludes with consideration of a problem of allotment of many spares in a series or parallel system with i.i.d. component lifetimes.

Definition 2.1. If \( \{Y_1, \ldots, Y_n\} \) is a set of random variables, then we let \( Y[1] \geq \cdots \geq Y[n] \) be the order statistics for this set. We furthermore occasionally represent the \( k^{th} \) order statistic \( Y[k] \) by \( \{Y_1, \ldots, Y_n\}[k] \). We also use the symbols \( \lor \) and \( \land \) to represent max and min respectively, so that
\[
Y[1] = \lor\{Y_1, \ldots, Y_n\} = \max\{Y_1, \ldots, Y_n\}
\]
and
\[
Y[n] = \land\{Y_1, \ldots, Y_n\} = \min\{Y_1, \ldots, Y_n\}.
\]

Lemma 2.2. Let \( X_1, X_2 \) be independent random variables. Then
\[
(2.1) \quad \land\{\lor(X_1, X), X_2\} \leq \lor\{X_1, \lor(X_2, X)\}
\]
for all \( X \) independent of \( \{X_1, X_2\} \)
\[
\iff X_1 \leq X_2.
\]

Proof: Let \( F_i \) be the distribution function of \( X_i \), \( i = 1, 2 \), and \( F \) the distribution function of \( X \). Then (2.1) holds
\[
\iff [1 - F_1(t)F(t)]F_2(t) \geq \bar{F}_1(t)[1 - F_2(t)F(t)] \quad \text{for all } t
\]
\[
\iff F_1(t)\bar{F}(t) \geq F_2(t)\bar{F}(t) \quad \text{for all } t.
\]
Hence if \( X_1 \leq X_2 \), the result (2.1) follows for any \( X \) independent of \( \{X_1, X_2\} \). Conversely, if \( X \) is any random variable independent of \( \{X_1, X_2\} \) and with support equal to \( R \), then (2.1) for such an \( X \) implies \( X_1 \leq X_2 \).

Proposition 2.3. Let \( X_1, \ldots, X_n \) be independent random variables. Then
\[
(2.2) \quad \lor\{\lor(X_1, X), X_2, X_3, \ldots X_n\}[k] \geq \lor\{X_1, \lor(X_2, X), X_3, \ldots, X_n\}[k]
\]
for all \( k \) and all \( X \) independent of \( \{X_1, \ldots, X_n\} \)
\[
\iff X_1 \leq X_2.
\]

Proof: Note that if \( k = 1 \),
\[
\lor\{\lor(X_1, X), X_2, X_3, \ldots, X_n\}[i] = \lor\{X_1, \lor(X_2, X), X_3, \ldots, X_n\}[i].
\]
For \( k \geq 2 \) and \( X \) independent of \( \{X_1, \ldots, X_n\} \), the result follows by observing that (see for example [6])
\[
P[\lor\{\lor(X_1, X), X_2, X_3, \ldots, X_n\}[k] > t] = P[\lor\{X_1, \lor(X_2, X), X_3, \ldots, X_n\}[k] > t] - P[\lor\{X_1, \lor(X_2, X), X_3, \ldots, X_n\}[k] > t]
\]
\[
= ([1 - F_1(t)F(t)]F_2(t) - \bar{F}_1(t)[1 - F_2(t)F(t)]) - P[\text{exactly } k - 2 \text{ of } \{X_3, \ldots, X_n\} \text{ exceed } t]
\]
\[
= \bar{F}(t)[F_1(t) - F_2(t)]P[\text{exactly } k - 2 \text{ of } \{X_3, \ldots, X_n\} \text{ exceed } t].
\]
LEMMA 2.4. Let $X_1, X_2$ be independent nonnegative random variables with distribution functions $F_1, F_2$. Then

(a) $\wedge \{X_1 + X, X_2\} \overset{st}{\geq} \wedge \{X_1, X_2 + X\}$ for all nonnegative $X$ independent of $\{X_1, X_2\}$

\begin{equation}
\iff F_1(z)F_2(y) \geq F_2(z)F_1(y) \quad \text{whenever } z \leq y.
\end{equation}

(b) $\vee \{X_1 + X, X_2\} \overset{st}{\leq} \vee \{X_1, X_2 + X\}$ for all nonnegative $X$ independent of $\{X_1, X_2\}$

\begin{equation}
\iff F_1(z)F_2(y) \geq F_1(y)F_2(z) \quad \text{whenever } z \leq y.
\end{equation}

PROOF: We prove a) only, as b) may be proved in a similar manner.
a) Suppose (2.3) is satisfied. Then for any $z$ and $t \geq 0$, $\overline{F}_1(t-z)\overline{F}_2(t) \geq \overline{F}_1(t)\overline{F}_2(t-x)$.

Hence

$$P[\wedge \{X_1 + X, X_2\} > t] - P[\wedge \{X_1, X_2 + X\} > t]$$

$$= \left[ \int_0^\infty \overline{F}_1(t-x)dF(x) \right] \overline{F}_2(t) - \left[ \int_0^\infty \overline{F}_2(t-x)dF(x) \right] \overline{F}_1(t)$$

$$= \int_0^\infty [\overline{F}_1(t-z)\overline{F}_2(t) - \overline{F}_1(t)\overline{F}_2(t-z)]dF(x) \geq 0.$$

Now let us suppose that $\wedge \{X_1 + X, X_2\} \overset{st}{\geq} \wedge \{X_1, X_2 + X\}$ for all non-negative $X$ independent of $\{X_1, X_2\}$. We want to show that if $t > z \geq 0$, then

$$\overline{F}_1(t-z)\overline{F}_2(t) \geq \overline{F}_1(t)\overline{F}_2(t-x).$$

Taking $X$ to be degenerate at $z$, a) follows.

Initially one might naively suspect that if $X_1, X_2$ and $X$ are independent nonnegative random variables where $X_1 \overset{st}{\leq} X_2$, then both

$$\wedge \{X_1 + X, X_2\} \overset{st}{\geq} \wedge \{X_1, X_2 + X\}$$

and

$$\vee \{X_1 + X, X_2\} \overset{st}{\leq} \vee \{X_1, X_2 + X\}$$

should hold in general. Lemma 2.4 shows that this is not the case and allows one to easily find examples where it is not true. For example suppose $X_1$ is such that $P(X_1 = 3) = 2P(X_1 = 2) = 2P(X_1 = 1) = \frac{1}{2}$, $X_2$ is such that $P(X_2 = 2) = 3P(X_2 = 1) = \frac{5}{6}$, $P(X_2 = 3) = \frac{1}{6}$, and $X$ is degenerate at 1. Assuming $X_1, X_2$ and $X$ independent it is clear that $X_1 \overset{st}{\leq} X_2$ but $\wedge \{X_1 + X, X_2\} \overset{st}{\geq} \wedge \{X_1, X_2 + X\}$ since

$$P[\wedge \{X_1 + X, X_2\} > 2] < P[\wedge \{X_1, X_2 + X\} > 2].$$
COROLLARY 2.5. Let $X_1, \ldots, X_n$ be independent nonnegative random variables with respective distribution functions $F_1, \ldots, F_n$. Then

a) 1. $\bar{F}_1(x)\bar{F}_2(y) \geq \bar{F}_1(y)\bar{F}_2(z)$ whenever $x \leq y$

$\Rightarrow \land \{X_1 + X, X_2, X_3, \ldots, X_n\} \leq \land \{X_1, X_2 + X, X_3, \ldots, X_n\}$

for all nonnegative $X$ independent of $\{X_1, \ldots, X_n\}$.

2. $F_1(x)F_2(y) \geq F_1(y)F_2(x)$ whenever $x \leq y$

$\Rightarrow \lor \{X_1 + X, X_2, X_3, \ldots, X_n\} \geq \lor \{X_1, X_2 + X, X_3, \ldots, X_n\}$

for all nonnegative $X$ independent of $\{X_1, \ldots, X_n\}$.

and conversely

b) assume that $0 < F_i(t) < 1$ for all $t > 0$ and $i = 3, \ldots, n$. Then

1. $\land \{X_1 + X, X_2, X_3, \ldots, X_n\} \geq \land \{X_1, X_2 + X, X_3, \ldots, X_n\}$ for all nonnegative $X$

$\Rightarrow \bar{F}_1(x)\bar{F}_2(y) \geq \bar{F}_1(y)\bar{F}_2(z)$ whenever $x \leq y$, and

2. $\lor \{X_1 + X, X_2, X_3, \ldots, X_n\} \leq \lor \{X_1, X_2 + X, X_3, \ldots, X_n\}$ for all nonnegative $X$

$\Rightarrow F_1(x)F_2(y) \geq F_1(y)F_2(x)$ whenever $x \leq y$.

Remark 2.6. Note that if $X_1$ and $X_2$ are two nonnegative random variables with the property that $\bar{F}_1(x)\bar{F}_2(y) \geq \bar{F}_1(y)\bar{F}_2(z)$ (or $F_1(x)F_2(y) \geq F_1(y)F_2(x)$) whenever $x \leq y$, then $X_1 \leq X_2$. The converse is clearly not true.

Many one parameter families of density functions $\{f_\theta(x) : \theta \in \Theta\}$ or mass functions $\{p_\theta(x) : \theta \in \Theta\}$ of life distributions possess the following property

$\theta_1 > \theta_2$ and $z \leq y \Rightarrow f_{\theta_1}(z)f_{\theta_2}(y) \geq f_{\theta_1}(y)f_{\theta_2}(z)$ \hspace{1cm} (2.5)

(or similarly for mass functions). In the parlance of total positivity, this is equivalent to saying that the function $g(\theta, z) = f_{\theta}(z)$ has the reverse rule of order 2 (i.e. is $RR_2$) property in $\theta$ and $z$ (see [10] and [12] for more on $RR_2$ functions). When division by 0 does not occur, (2.5) is equivalent to saying that the ratio

$$\frac{f_{\theta_1}(x)}{f_{\theta_2}(x)}$$

is a decreasing function of $x$ whenever $\theta_1 > \theta_2$. Such families are deemed to have the decreasing monotone likelihood ratio property (see [9] pg. 607 for a discussion of the monotone likelihood ratio property).
Lemma 2.7. Let \( \{f_{\theta}(x) : \theta \in \Theta\} \) (or \( \{p_{\theta}(x) : \theta \in \Theta\} \)) be a one parameter family of life densities (mass functions) which has the reverse rule property \((RR_2)\) in \(\theta\) and \(x\). Then both of the families \(\{\overline{F}_{\theta}(x) : \theta \in \Theta\}\) and \(\{F_{\theta}(x) : \theta \in \Theta\}\) have the \(RR_2\) property in \(\theta\) and \(x\).

Proof: We indicate the proof for the family \(\{\overline{F}_{\theta}(x) : \theta \in \Theta\}\) in the continuous case, the other implications following in a similar way. Let \(\theta_1 > \theta_2\) and \(x < y\) be given. We want to show that
\[
\overline{F}_{\theta_1}(x) \overline{F}_{\theta_2}(y) \geq \overline{F}_{\theta_1}(y) \overline{F}_{\theta_2}(x),
\]
or equivalently
\[
\int_{x}^{\infty} f_{\theta_1}(s)ds \int_{y}^{\infty} f_{\theta_2}(t)dt \geq \int_{x}^{\infty} f_{\theta_2}(s)ds \int_{y}^{\infty} f_{\theta_1}(t)dt.
\]
After cancellation, this reduces to showing
\[
\int_{y}^{\infty} \left[ \int_{x}^{s} [f_{\theta_1}(s)s - f_{\theta_2}(s)s]ds \right] dt \geq 0,
\]
which follows from the \(RR_2\) property of \(\{f_{\theta}(x) : \theta \in \Theta\}\).

Example 2.8. The following are examples of families of densities of life distributions with the \(RR_2\) property:

a) Let \(f_{\theta}(x) = a(\theta)b(x) \exp[c(\theta)d(x)]\) for \(\theta \in \Theta\) and \(x > 0\), where \(c\) and \(d\) are monotonic functions, one nonincreasing and the other nondecreasing. Then the resulting family has the \(RR_2\) property in \(\theta\) and \(x\). Typical examples of such families are:

1) Gamma families. The gamma distribution \(\Gamma(\lambda, m)\) with parameters \(\lambda\) and \(m\) is given by the density \(f_{\lambda,m}(x) = \lambda^m x^{m-1} e^{-\lambda x}/\Gamma(m)\) for \(x > 0\). The families of distributions \(\{\Gamma(\theta, m) : \theta > 0\}\) (fixed shape parameter) and \(\{\Gamma(\lambda, \frac{1}{\theta}) : \theta > 0\}\) (fixed scale parameter) have densities with the \(RR_2\) property.

2) Weibull family of densities \(\{f_{\theta}(x) = a^{\alpha} z^{\alpha-1} e^{-\theta z^\alpha} : \theta > 0\}\) with fixed shape parameter \(\alpha\).

3) Pareto family of densities \(\{f_{\theta}(x) = \frac{\theta}{(1+x)^{\theta+1}} : \theta > 0\}\).

b) Let \(f_{\theta}(x)\) be the uniform density on \([0, \frac{1}{\theta}]\). Then the family \(\{f_{\theta}(x) : \theta > 0\}\) has the \(RR_2\) property in \(\theta\) and \(x\).

c) If the family \(\{f_{\theta}(x) : \theta > 0\}\) has the property that for some log concave \(g\) on \((0, +\infty)\), \(f_{\theta}(x) = g(\theta + x)\), then \(\{f_{\theta}(x) : \theta > 0\}\) has the \(RR_2\) property in \(\theta\) and \(x\).

Example 2.9. The following are examples of discrete families of life distributions with the \(RR_2\) property in \(\theta\) and \(x\):

a) The Poisson family \(\{p_{\theta}(x) = (\frac{1}{\theta})^x e^{-\frac{1}{\theta}}/x! : \theta > 0\}\).
b) The negative binomial distribution \( \text{NBD}(k, q) \) is given by the mass function 
\[
p(x) = \binom{x-1}{k-1} p^{x-k} q^k \quad \text{for } x = k, k+1, \ldots
\]
Then both of the families \( \{\text{NBD}(k, \theta) : \theta \in (0, 1)\} \) (for fixed \( k \)) and \( \{\text{NBD}(-\theta, q) : \theta \in \{-1, -2, \ldots\}\} \) (for fixed \( q \)) have the RR\(_2\) property in \( \theta \) and \( x \).

§3. Applications to Redundancy Allocation in Reliability Theory.

There are two common forms of redundancy in reliability theory, namely active (or parallel) redundancy and standby redundancy. In an active redundancy of a spare component to position \( i \) in a system, the position functions if either the original component or the spare is functioning. In standby redundancy of a spare component to position \( i \) the spare is only put into operation after the original component there ceases to function. We will be concerned mainly with the problem of where to allocate a spare component (by either an active or standby redundancy) in a \( k \) out of \( n \) system in order to optimize improvement of system reliability.

Herein we shall assume that the component lifetimes \( T_1, \ldots, T_n \) of the \( n \) components are stochastically independent. If \( \tau(T_1, \ldots, T_n) \) represents the lifetime of a given system with component lifetimes \( T_1, \ldots, T_n \), then for a \( k \) out of \( n \) system we write the lifetime as

\[
\tau_k(T_1, \ldots, T_n) = \{T_1, \ldots, T_n\}[k] = T_k.
\]

In particular for a series system \( \tau = \wedge\{T_i\} = T[n] \) and for a parallel system \( \tau = \vee\{T_i\} = T[1] \).

Suppose that one has a system of components and that a spare component with lifetime \( T \) (which is stochastically independent of \( \{T_1, \ldots, T_n\} \)) is available for active redundancy with component \( i \). The resulting lifetime of a \( k \) out of \( n \) system is \( \tau_k \{T_1, \ldots, T_i \vee T, T_{i+1}, \ldots, T_n\} \equiv \tau_k^{(i)} \). When the random variables \( T_1, \ldots, T_n \) are stochastically ordered, Proposition 2.3 implies the following result (see also [5]).

**Theorem 3.1.** Let \( \{T_1, \ldots, T_n\} \) be the stochastically independent lifetimes of a \( k \) out of \( n \) system where \( T_1 \leq T_2 \leq \cdots \leq T_n \). Suppose \( T \) is the lifetime of an independent spare. Then

\[
\tau_k^{(1)} \geq \tau_k^{(2)} \geq \cdots \geq \tau_k^{(n)}
\]

for all \( k = 1, \ldots, n \).

Note therefore that in such a stochastically ordered \( k \) out of \( n \) system, it is always stochastically preferable to perform active redundancy on weaker components. This is not surprising for series systems where it is generally felt that the weakest component is the most important. It is however perhaps surprising for \( k \) out of \( n \) systems when \( k < n \).

We have been considering the situation where a "common" spare with lifetime \( T \) is to be placed in active redundancy with one of the components in the system. In many situations, however, we have the possibility of placing a component with lifetime \( T_i' \) in active
redundancy with component $i$, but where the distribution of $T_{i}^t$ may depend on $i$. In the case where only one such active redundancy allocation is permitted, a reasonable problem is to determine where the allocation should be made in order to give the greatest improvement. Even in the situation where $T_{i}^{d} = T_{i}^{t}$, all $T_{1}, \ldots, T_{n}, T_{1}^{t}, \ldots, T_{n}^{t}$ are independent and $T_{1}^{t} \leq \cdots \leq T_{n}^{t}$ the answer is not obvious. For example in a 2 out of 3 system the answer might be either at position 1, 2 or 3 depending on the specific distributions of $T_{1}, T_{2}$ and $T_{3}$ (see [6]). In order to address this problem various studies have been made of what might be called the (active) redundancy importance of component $i$ (see [3], [4], [5], [6], [15] and [16]). Råde [15] obtains results for some series parallel systems when the components are exponentially distributed.

We now turn to the problem of optimally allocating a standby redundancy in $k$ out of $n$ systems. Let us initially consider the problem of where to allocate a (common) spare component with lifetime $T$. Corollary 2.5 and Lemma 2.7 yield the following result for series and parallel systems.

**Theorem 3.2.** Let $T_{1}, \ldots, T_{n}$ be stochastically independent lifetimes with respective densities $f_{1}, \ldots, f_{n}$ where $T_{1}^{st} \leq T_{2}^{st} \cdots \leq T_{n}^{st}$. Assume that there exists a one parameter family $\mathcal{S}_{\theta}$ of life distributions whose densities (or mass functions) possess the RR property in $\theta$ and $x \geq 0$, and where $f_{i} \in \mathcal{S}_{\theta}$ for each $i$. Then if $T$ represents the lifetime of an independent spare component and $r_{k}\{T_{1}, \ldots, T_{i-1}, T_{i} + T, T_{i+1}, \ldots, T_{n}\} = r_{k}^{[i]}$ is the lifetime of the $k$ out of $n$ system with components $\{T_{1}, \ldots, T_{i-1}, T_{i} + T, T_{i+1}, \ldots, T_{n}\}$, then

a) **Series System:**

$$r_{n}^{[1]} \geq r_{n}^{[2]} \geq \cdots \geq r_{n}^{[n]}$$

and

b) **Parallel System:**

$$r_{1}^{[1]} \leq r_{1}^{[2]} \leq \cdots \leq r_{1}^{[n]}.$$

For example suppose that $T_{i} \sim \Gamma(\lambda_{i}, m)$ where $\lambda_{i} \downarrow$ in $i$, $i = 1, \ldots, n$. Assume that a (common) spare component with lifetime $T$ is available for standby redundancy with a component and that $T, T_{1}, \ldots, T_{n}$ are independent. It follows from Theorem 3.2 that for a series (parallel) system, standby redundancy of $T$ with $T_{i}$ yields a system which is stochastically decreasing (increasing) in $i$.

Unfortunately results like Theorem 3.2 do not extend (beyond series and parallel systems) to more general $k$ out of $n$ systems. The following example helps illustrate this point.
Example 3.3. Let $T_i, i = 1, 2, 3$ be the independent exponentially distributed lifetimes of the components in a 2 out of 3 system. Assume that $T_i \sim \text{Exp}(\lambda_i)$, where $\lambda_1 > \lambda_2 > \lambda_3$, and hence $T_1 \leq T_2 \leq T_3$. Suppose $T_s$ is a spare component with degenerate lifetime $= \epsilon > 0$ which we can place in standby redundancy with one of the components. The question is where to place the standby redundancy in order to make the 'best' improvement in the system. Unfortunately even when one only wants to maximize expected system life as a result of the redundancy, the answer depends on the relative values of $\lambda_1, \lambda_2, \lambda_3$. This may be seen by considering what is termed the Barlow-Proshan time independent measure of component importance $I_{BP}^{(i)}$ of component $i$ (see [1] and [7]). $I_{BP}^{(i)}$ is actually the probability that system life coincides with the life of component $i$, but it is also the limit as $\epsilon \to 0$ of the expected improvement in system life as a result of extending the lifetime of component $i$ by $\epsilon$, all divided by $\epsilon$. Hence $I_{BP}^{(i)} > I_{BP}^{(j)}$ means that for small $\epsilon$ a larger expected system life is obtained as a result of allocating a standby redundancy (with degenerate lifetime $\epsilon$) to component $i$ rather than to component $j$.

Now one may show that in the 2 out of 3 system above, $I_{BP}^{(2)} \geq I_{BP}^{(3)}$ in general, but that depending on $(\lambda_1, \lambda_2, \lambda_3)$, $I_{BP}^{(1)}$ may either $> \max\{I_{BP}^{(2)}, I_{BP}^{(3)}\}$, or $< \min\{I_{BP}^{(2)}, I_{BP}^{(3)}\}$, or $\in [I_{BP}^{(2)}, I_{BP}^{(3)}]$. For example if $\lambda_1 > \lambda_2 > \lambda_3$, then $I_{BP}^{(3)} > I_{BP}^{(1)}$, while if $\lambda_1 > \lambda_2 > \lambda_3 \sim 0$, then $I_{BP}^{(2)} > I_{BP}^{(1)} > I_{BP}^{(3)}$. When $\lambda_1 = 1.01, \lambda_2 = 1, \lambda_3 = .9999$, $I_{BP}^{(1)} > I_{BP}^{(2)} > I_{BP}^{(3)}$.

Another practical situation where one might consider standby redundancy allocation is where the spare being considered for allocation in position $i$ has life distribution $T'_i$ independent of $T_i$ but such that $T'_i \leq T_i$. Natvig [14] showed that when $T_1 \leq \cdots \leq T_n$ and the $T_i$ are independent Gamma distributed with a common shape parameter $m$, then $E(r_n^{(i)})$ is decreasing in $i$ while $E(r_1^{(i)})$ is increasing in $i$. More particularly he showed that if $T_i \leq T_j$ where the $i^{th}$ and $j^{th}$ components are in series with the rest of the system, then $E(r_n^{(i)}) \geq E(r_n^{(j)})$. Here

$$r_n^{(i)} = \{T_1, \ldots, T_{i-1}, T_i + T'_i, T_{i+1}, \ldots, T_n\}_{[k]}.$$ 

For a parallel system, the following more general result follows from Theorem 3.2.

**Corollary 3.3.** Suppose that $T_1, \ldots, T_n, T'_1, \ldots, T'_n$ are stochastically independent lifetimes where $T_i \leq T'_i$, and $T_1 \leq T_2 \leq \cdots \leq T_n$. Furthermore assume that there exists a one parameter family $S_\theta$ of densities or mass functions with the $RC_2$ property in $\theta$ and $x$, and where $f_i \in S_\theta$ for each $i$. Let $r_1^{(i)} = \vee\{T_1, \ldots, T_{i-1}, T_i + T'_i, T_{i+1}, \ldots, T_n\}$ be the lifetime of a parallel system where a standby redundancy (with life length $T'_i$) has been made with component $i$. Then

$$r_1^{(1)} \leq r_1^{(2)} \leq \cdots \leq r_1^{(n)}.$$
PROOF: From Theorem 3.2, it follows that when \( i < j \),
\[
\tau_1^{(i)} \leq \tau^* \leq \{ T_1, \ldots, T_i, \ldots, T_j + T_i', \ldots, T_n \},
\]
which in turn is \( \leq \tau_1^{(j)} \) as \( T_i' \leq T_j' \). 

§4. Optimal Allotment of Standby Redundancies.

Finally in this section we consider the particular problem of optimally allotting a number of standby redundancies to the components of a coherent system when the component lifetimes are i.i.d.. More specifically, consider a coherent system with \( n \) components whose lifetimes \( T_1, \ldots, T_n \) are independent and identically distributed with common density function \( f \). Suppose that we have \( N \) spares whose lifetimes \( S_1, \ldots, S_N \) are independent with the same common density function \( f \) available for standby redundancy with any of the \( n \) components of the system. Assume further that we have to divide those \( N \) spares among the \( n \) components whereby \( k_i \) spares will be assigned to component \( i \), \( 1 \leq i \leq n \) and \( k_1 + \cdots + k_n = N \). A natural question to ask is: what is the optimal choice \( k^* = (k_1^*, \ldots, k_n^*) \), \( k_1^* + \cdots + k_n^* = N \), that will stochastically maximize the lifetime of the system? We answer this question for the case of series and parallel systems under the additional assumption that the density function \( f \) is log concave on \( (0, \infty) \) (gamma and Weibull densities with shape parameter \( \alpha \geq 1 \) are examples of such densities). The result for the series case is implicitly contained in the work of Karlin and Proschan [11].

**Lemma 4.1.** Let \( f \) be a log concave density on \( (0, \infty) \) and let \( F \) be the corresponding distribution function. Let \( F^{(k)} \) denote the \( k \)-fold convolution of \( F \) with itself and \( F^{(k)}(x) = 1 - F^{(k)}(x) \) the corresponding survival distribution. Then:

(a) \( n_1 \leq n_2 \Rightarrow F^{(n_1+1)}(x)F^{(n_2)}(x) \geq F^{(n_1)}(x)F^{(n_2+1)}(x) \) for all \( x \).

(b) \( n_1 \leq n_2 \Rightarrow \overline{F^{(n_1+1)}(x)F^{(n_2)}(x)} \geq \overline{F^{(n_1)}(x)F^{(n_2+1)}(x)} \) for all \( x \).

**Proof:** (a) By theorem 1 of Karlin and Proschan [11], \( f^{(n)}(x) \), the \( n \)-fold convolution of the density \( f \) is totally positive of order 2 \( (TP_2) \) in \( n, x \)
(i.e. \( f^{(n_1)}(x)f^{(n_2)}(y) \geq f^{(n_1)}(y)f^{(n_2)}(x) \) when \( x \leq y \) and \( n_1 \leq n_2 \)). A proof similar to that of lemma 2.7 will show that \( F^{(n)}(x) \) is also \( TP_2 \) in \( n, x \). Now let \( n_1 \leq n_2 \) and \( z \geq 0 \). Then \( F^{(n_1)}(x - z)F^{(n_2)}(z) \geq F^{(n_1)}(x)F^{(n_2)}(x - z) \), and integrating both sides of the preceding inequality with respect to the distribution function \( F(x) \) yields the result in (a).

(b) The proof is similar to (a) and is therefore omitted. 

We now turn to the problem of optimally allotting the \( N \) standby redundancies to the \( n \) components of a coherent system. The nonnegative random variables \( T_1, \ldots, T_n, S_1, \ldots, S_N \)
are assumed to be independent with common log concave density function \( f \). Let \( r_1^{\{k\}} (r_n^{\{k\}}) \) denote \( \max \) (\( \min \)) of \( \{T_1 + \sum_{i=1}^{k_1} S_i, T_2 + \sum_{i=k_1+1}^{k_1+k_2} S_i, \ldots, T_n + \sum_{i=k_1+\cdots+k_{n-1}+1}^{N} S_i\} \), where \( k = (k_1, \ldots, k_n) \) is the vector indicating that \( k_i \) spares have been assigned to component \( i, 1 \leq i \leq n \), and \( k_1 + \cdots + k_n = N \). We now assume the familiarity of the reader with the concept of majorization, a partial order on \( \mathbb{R}^n \) denoted here by \( \succeq^m \) (see for example [13]).

**Theorem 4.2.** Let \( k \) and \( k' \) be two vectors such that \( k \succeq^m k' \) and \( \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} k'_i = N \) (that is the components of \( k \) are more dispersed than those of \( k' \)). Then

(a) \( r_1^{\{k\}} \geq r_1^{\{k'\}} \) and

(b) \( r_n^{\{k\}} \leq r_n^{\{k'\}} \).

**Proof:** (a) Without loss of generality we may assume \( k_1 > k_2 \) and \( k'_1 = k_1 - 1, k'_2 = k_2 + 1, k'_j = k_j \) for \( 3 \leq j \leq n \). Now \( P[r_1^{\{k\}} \leq t] - P[r_1^{\{k'\}} \leq t] = [F^{(k_1+1)}(t)F^{(k_2+1)}(t) - F^{(k'_1+1)}(t)F^{(k'_2+1)}(t)] \prod_{j=3}^{n} F^{(k_j+1)}(t)] \leq 0 \) by (a) of lemma 4.1. (b) is proved in an analogous manner.

Theorem 4.2 indicates that for a parallel (series) system the more heterogeneous (homogeneous) the coordinates of \( k \) are the better (stochastically) is the lifetime of the system. Thus the optimal allotment of the \( N \) spares to the components of a parallel system is to assign all of them to a single component. However in the case of a series system the optimal allotment vector is

\[
\underbrace{(m + 1, \ldots, m + 1)}_{r \text{ times}} \quad \underbrace{, m, \ldots, m}_{n-r}
\]

where \( N = mn + r, 0 \leq r < n, \ m \geq 0. \)
References.


The problem of where to allocate a redundant component in a system in order to optimize the lifetime of a system is an important problem in reliability theory which also poses many interesting questions in mathematical statistics. We consider both active redundancy and standby redundancy, and investigate the problem of where to allocate a spare in a system in order to stochastically optimize the lifetime of the resulting system. Extensive results are obtained in particular for series and parallel systems.