MONOTONICITY OF RANK ORDER LIKELIHOOD RATIOS

by

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FSU Statistics Report M85

September, 1965
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Work supported by the Office of Naval Research Contract No.
N00014-66-A-0188(13)

Reproduction in whole or in part is permitted for any purpose of the United States Government.
1. **Introduction and summary.** Suppose \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \) are mutually independent random variables. The \( X_i \)'s and the \( Y_i \)'s have the continuous distribution functions \( F(x, \theta_1) \) and \( F(x, \theta_2) \) respectively. Let the observed values of \( X_1, \ldots, X_m \) be called the first sample and the observed values of \( Y_1, \ldots, Y_n \) be called the second sample. Consider the joint ranking of the two samples in ascending order. Form the sequence \( z = (z_1, \ldots, z_N) \), where \( N = m + n \), by defining \( z_i = 0 \) or \( 1 \) according as the \( i^{th} \) observation is from the first or the second sample. The random variable \( Z \) corresponding to \( z \) is called a rank order. So long as there is no confusion, \( z \) will be called a rank order rather than an observed value of the rank order \( Z \). Denote the probability that \( Z = z \) by \( P_{\theta_1, \theta_2}(z) \). Suppose this probability depends on \( \theta_1 \) and \( \theta_2 \) through a function \( h(\theta_1, \theta_2) \), so that it can be written as \( P_{h(\theta_1, \theta_2)}(z) \) or simply \( P_h(z) \).

From the Bayesian viewpoint, \( h(\theta_1, \theta_2) \) is a given value of a random variable \( H = H(\theta_1, \theta_2) \) and \( P_h(z) \) is the conditional probability that \( Z = z \) given \( h(\theta_1, \theta_2) \). Let the prior density for \( H \) be \( g(h) \). Then the posterior density of \( H \) given the rank order \( z \), denoted by \( g(h|z) \), is given by

\[
g(h|z) = \frac{P_h(z)g(h)}{\int_{-\infty}^{\infty} P_h(z)g(h) \, dh}
\]

(1.1)

The present paper has been motivated by the study of the monotonicity of the likelihood ratio for the family of posterior
densities \( g(h|z) \). For this family, the indexing set is the set of all rank orders \( z \). Unlike the case when the indexing set is a set of real numbers, no natural ordering prevails over the set of all rank orders. Therefore, the notion of monotone likelihood ratio depends on which ordering of the \( z \)'s is of interest.

Consider any two rank orders \( z \) and \( z' \). It is seen from (1.1) that

\[
(1.2) \quad \frac{g(h|z)}{g(h|z')} = \frac{C P_h(z)}{P_h(z')} ,
\]

where the constant \( C \) does not depend on \( h \). In the rank order likelihood ratio \( P_h(z) \) \( P_h(z') \) in (1.2), \( z \) and \( z' \) play the role of parameters. Thus, given an ordering for the set of rank orders \( z \), the study of the monotonicity of the likelihood ratio for the family of posterior densities involves the study of the monotonicity of the rank order likelihood ratio as a function of \( h \).

In many cases \( h(\theta_1, \theta_2) \) is a monotone function of one argument when the other argument is fixed.

From a non-Bayesian viewpoint, in the rank order likelihood ratio \( \frac{P_{\theta_1, \theta_2}(z)}{P_{\theta_1, \theta_2}(z')} \) \( z \) and \( z' \) are two values of the vector random variable \( Z \) and \( \theta_1, \theta_2 \) play the role of parameters. Given an ordering defined over the sample space of \( Z \), one can investigate the monotonicity behavior of the rank order likelihood ratio with
respect to one of the two parameters \( \theta_1 \) and \( \theta_2 \), keeping the other fixed. Thus the study of the rank order likelihood ratio besides having the Bayesian motivation described earlier, is a special case of the study of the likelihood ratio for a family of distributions of vector valued random variables.

The notation \( zRz' \) denotes the following relationship:

\[
z_i = z_i' \text{ for all } i = 1, \ldots, N \text{ except } j \text{ and } j + 1 \text{ and } z_j = z_{j+1}' = 0, \\
z_{j+1} = z_j' = 1.
\]

The notation \( zRz' \) is also used if there exist \( z_1, z_2, \ldots, z^k \) such that \( zRz' \). For example, \( (0011)R(1001) \) since \( (0011)R(0101) \) and \( (0101)R(1001) \). The notation \( z^* R z' \)
denotes either of the following relationships: (a) \( z_i = z_i' \)
for \( i = 3, \ldots, N \) and \( z_1 = z_2' = 0, z_2 = z_1' = 1 \); (b) \( z_i = z_i' \)
for \( i = 1, \ldots, N - 2 \) and \( z_{N-1} = z_N' = 0, z_N = z_{N-1}' = 1 \). The notation \( z^* R z' \) is also used if there exists \( z^1 \) such that \( z^R z^1 R z' \). For example \( (0101101)R^*(1001110) \) since \( (0101101)R^*(0101110) \) and \( (0101110)R^*(1001110) \). The relationship \( R^* \) implies the relationship \( R \). The relationship \( R \) is a partial ordering (see Savage [1964]).

With the partial ordering of the rank orders defined by the \( R \) relationship the monotonicity of the rank order likelihood ratio has been exhibited for: (a) a Lehmann family of distributions in section 3, (b) the uniform family of distributions over \( (0, \theta) \) in section 4, and (c) one observation from one population and an arbitrary number of observations from some
other population in section 5. In (a) and (b) monotonicity of the rank order likelihood ratio has been obtained also for certain rank orders not in R-relation for example the rank orders (0110) and (1001). In (c) only one population is assumed to belong to a family of distributions with densities having increasing likelihood ratio. In section 6, the partial ordering defined by the $R^*$ relationship is considered. The monotonicity of the rank order likelihood ratio is exhibited for a family of distributions with densities having increasing likelihood ratio and further satisfying the following condition: for all $\theta_2 \geq \theta_1$

$$\frac{f(x, \theta_2) F(x, \theta_1)}{f(x, \theta_1) F(x, \theta_2)}$$

is an increasing function of $x$. A normal family and a logistic family of distributions with translation parameters are found to satisfy the conditions of section 6. When $m = n = 2$, the $R$-relationship defines a simple ordering of the rank orders. In section 7, monotonicity of the rank order likelihood ratio is exhibited for this simple ordering for a normal family, a logistic family and a double exponential family of distributions, each family with a translation parameter. Section 8 shows local monotonicity of the rank order likelihood ratio in a neighborhood of $\theta_1 = \theta_2$ for families with densities having increasing likelihood ratio. Section 9 gives a conjecture regarding monotonicity of the rank order likelihood ratio for families of densities with increasing likelihood ratio. Different cases considered
in sections 3 through 8 provide some evidence in support of this conjecture. It may be added that, even though no counter example is available at present, there is no stronger evidence in support of the conjecture. Section 9 also gives the motivation for a conjecture regarding monotonicity of the ratio of density functions of $Y_1$ and $Y_2$, where

$$Y_1 = \sum_{i=1}^{N_i} z_{i} W_{i}, \quad Y_2 = \sum_{i=1}^{N_i} z_{i} W_{i},$$

$W_{i}$ being the $i^{th}$ order statistic in a sample of size $N$ from a standard normal population and $zRz'$. The appendix gives two results which were obtained during this investigation for likelihood ratios.

2. Notation and assumptions. All through this work, the distribution functions are assumed to be continuous unless explicit mention is made to the contrary. The $X$'s correspond to the first sample and the $Y$'s correspond to the second sample. Whenever there is a need to emphasize that the two samples could have been taken from populations belonging to two different families of distributions, the distribution function for the $X$'s is denoted by $F(x, \theta)$ and the distribution function for the $Y$'s is denoted by $G(x, \eta)$. The corresponding density functions are denoted by $f(x, \theta)$ and $g(x, \eta)$. The probability that the rank order $Z = z$ is denoted by $P_{\theta, \eta}(z)$. It will be denoted
by \( P_{\theta, \eta}(z^N) \) whenever the sum of the sizes of the two samples needs explicit mentioning. Whenever no association of a parameter with a distribution function is needed, the corresponding subscript is deleted from \( P_{\theta, \eta}(z) \). For example, \( P_{\eta}(z) \) denotes the probability of the rank order \( z \), when the \( X \)'s have the distribution function \( F(x) \) and the \( Y \)'s have the distribution function \( G(x, \eta) \). The distribution functions for the \( X \)'s and the \( Y \)'s are assumed to be \( F(x, \theta_1) \) and \( F(x, \theta_2) \) respectively when and only when the populations, from which the two samples are taken, belong to the same family of distributions.

A single parameter family of density functions \( f(x, \theta) \) is said to have an increasing likelihood ratio if, for all \( x' < x \) and \( \theta' < \theta \), the following inequality holds:

\[
(2.1) \quad f(x, \theta) f(x', \theta') - f(x, \theta') f(x', \theta) \geq 0 .
\]

If the inequality \( (2.1) \) is strict, the family is said to have a strictly increasing likelihood ratio.

The parameter \( \theta \) of a density function \( f(x, \theta) \) is said to be a translation parameter if \( f(x, \theta) = f(x - \theta) \). Further, the density function \( f(x - \theta) \) is said to be symmetric if \( f(x - \theta) = f(\theta - x) \).

Let \( (u : z : v) \) denote the event that \( Z = z \) and all the observed values of \( X_1, \ldots, X_n; Y_1, \ldots, Y_n \) are between \( u \) and \( v \). The probability of the event \( (u : z : v) \) is denoted by \( P_{\theta, \eta}(u : z : v) \). For example \( P_{\theta, \eta}(u : 0.1 : v) = \text{Prob.}(u \leq X \leq Y \leq v) \).
The transpose and the complement of a rank order $z$ are defined by $z^t = (z_1^t, \ldots, z_N^t)$ and $z^c = (z_1^c, \ldots, z_N^c)$ respectively, where $z_i^t = z_{i+1} - 1$ and $z_i^c = 1 - z_i$, $i = 1, \ldots, N$.

Now think of the distribution of the random variable $Z$ (the rank order defined in section 1.) as depending on the two parameters $\theta$ and $\eta$. Our interest is centered on the likelihood ratio $\frac{P_{\theta, \eta}(z)}{P_{\theta', \eta}(z')}$. The rank order likelihood ratio is said to be increasing if, for each $\theta$, it is an increasing function of $\eta$ whenever $z R z'$. The rank order likelihood ratio is said to be decreasing if, for each $\eta$, it is a decreasing function of $\theta$ whenever $z R z'$. The rank order likelihood ratio is said to be increasing ($\succ$) if, for each $\theta$, it is an increasing function of $\eta$ whenever $z R^* z'$. The rank order likelihood ratio is said to be decreasing ($\prec$) if, for each $\eta$, it is a decreasing function of $\theta$ whenever $z R^* z'$.

An increasing rank order likelihood ratio implies that, for each $\theta$ and for all $\eta' < \eta$, the following inequality holds whenever $z R z'$:

$$P_{\theta, \eta}(z) P_{\theta', \eta'}(z') - P_{\theta, \eta}(z') P_{\theta', \eta}(z) \geq 0.$$ 

A decreasing rank order likelihood ratio implies that, for each $\eta$ and for all $\theta' < \theta$, the following inequality holds whenever $z R z'$:

$$P_{\theta, \eta}(z) P_{\theta', \eta'}(z') - P_{\theta, \eta}(z') P_{\theta', \eta}(z) \leq 0.$$
Similar implications follow when the rank order likelihood ratio is increasing (*) and decreasing (*).

Lemma 2.1 Suppose properties $\pi_1$ and $\pi_2$ of the two populations sampled imply that the rank order likelihood ratio is increasing (decreasing) whenever $zRz'$ with the interchange between a pair of adjacent components of $z$. Then, the rank order likelihood ratio is increasing (decreasing) whenever $z$ and $z'$ are in the general $R$-relationship.

The proof is immediate.

The lemma shows that, for exhibiting the monotonicity of the rank order likelihood ratio, it is sufficient to consider $z$ and $z'$ whenever $zRz'$ with the interchange between a pair of adjacent components of $z$.

Remark to lemma 2.1 The lemma holds when the partial ordering of the rank orders is defined by the $R_\prec$-relationship.

3. Lehmann family. A family of distribution functions $F(x, \theta)$ is called a Lehmann family if $F(x, \theta) = H^\theta(x)$, where $H(x)$ is a continuous distribution function and $\theta > 0$. These families include as particular cases (see Savage (1956)) the family of exponential distributions defined by

$$F(x, \theta) = \begin{cases} e^{\theta x} & \text{for } x < 0, \theta > 0, \\ 1 & \text{for } x \geq 0 \end{cases}$$

and the family of extreme value distributions defined by

$$F(x, \theta) = \exp[-e^{-(x-\theta)}], x \in (-\infty, \infty), \theta \in (-\infty, \infty).$$
It is easily seen that a Lehmann family has increasing likelihood ratio.

**Theorem 3.1** Suppose \((X_1, \ldots, X_m) \) and \((Y_1, \ldots, Y_n)\) are independent random samples respectively from two populations with distribution functions \(F(x, \theta_1)\) and \(F(x, \theta_2)\) belonging to a Lehmann family. Then, the rank order likelihood ratio is increasing in \(\theta_2\) and decreasing in \(\theta_1\).

**Proof.** Corollary 7.a.1 of Savage (1956) gives

\[
(3.1) \quad P_{\theta_1, \theta_2}(z) = \frac{m!n! \theta_1^m \theta_2^n}{(m+n) \prod_{i=1}^{m+n} (\theta_1 v_i + \theta_2 v_i)} ,
\]

where \(v_i = \sum_{j=1}^i z_j\) and \(u_i = 1 - v_i\).

Consider \(z\) and \(z'\) such that \(z \# z'\) with the interchange at the \(j^{th}\) and the \((J+1)^{st}\) positions. Then,

\[
(3.2) \quad u_i' = u_i \quad \text{for all } i \text{ except } J, \quad u_J' = u_J - 1
\]

and

\[
(3.3) \quad v_i' = v_i \quad \text{for all } i \text{ except } J, \quad v_J' = v_J + 1.
\]

From (3.1) - (3.3), we obtain,

\[
(3.4) \quad \frac{P_{\theta_1, \theta_2}(z)}{P_{\theta_1, \theta_2}(z')} = 1 + \frac{\theta_2 - \theta_1}{\theta_1 u_J + \theta_2 v_J} .
\]

It follows from (3.4) that for \(\theta_1\) positive and fixed, the rank order likelihood ratio is an increasing function of \(\theta_2\) if and
only if \( u_j + v_j > 0 \) which is true since \( u_j + v_j = J > 0 \). Further, for \( \theta_2 \) positive and fixed, the rank order likelihood ratio is a decreasing function of \( \theta_1 \) if and only if \( u_j + v_j > 0 \). An application of lemma 2.1 completes the proof of the theorem.

**Corollary 3.1.** If

\[
F(x, \theta_1) = \begin{cases} 
0 & \text{for } x \leq 0 \\
-x/\theta_1 & \text{for } x \geq 0, \theta_1 > 0, i = 1, 2,
\end{cases}
\]

then the rank order likelihood ratio is increasing in \( \theta_2 \) and decreasing in \( \theta_1 \).

**Proof.** Consider the following transformation of the random variables \( X_1, \ldots, X_m ; Y_1, \ldots, Y_n \):

\[
U_i = -X_i, \ i = 1, \ldots, m,
\]

and

\[
V_j = -Y_j, j = 1, \ldots, n.
\]

The distribution functions \( G(x, \eta_1) \) for the \( U_i \)'s and \( G(x, \eta_2) \) for the \( V_i \)'s are given by

\[
G(x, \eta_1) = \begin{cases} 
\eta_1^x & \text{for } x \leq 0 \\
1 & \text{for } x \geq 0, \ i = 1, 2,
\end{cases}
\]

where \( \eta_i = 1/\theta_i \). The distribution functions \( G(x, \eta_1) \) thus belong to a Lehmann family and consequently theorem 3.1 applies.

It should be noted that there are other pairs of rank orders than those of theorem 3.1 such that \( P_{\theta_1, \theta_2}(z)/P_{\theta_1', \theta_2'(z')} \) is increasing in \( \theta_2 \) and decreasing in \( \theta_1 \). An example is given below.
Example 3.1 Consider $z$ and $z'$ defined below:

$$z = (z_1, \ldots, z_{I-1}, 0110z_{I+4}, \ldots, z_N),$$

$$z' = (z_1, \ldots, z_{I-1}, 1001z_{I+4}, \ldots, z_N).$$

Suppose

$$v = \sum_{i=1}^{I} z_i, \quad u = I - v.$$

Then denoting $\frac{\theta_2}{\theta_1}$ by $\theta$, it follows from (3.1) that

$$\frac{P_{\theta}(z)}{P_{\theta}(z')} = \frac{(u-1+\theta v+\theta)(u+1+\theta v+\theta)}{(u+\theta v)(u+\theta v+2\theta)}.$$ 

Differentiating $P_{\theta}(z)/P_{\theta}(z')$ with respect to $\theta$, it is seen that $P_{\theta}(z)/P_{\theta}(z')$ is an increasing function of $\theta$ if and only if

$$u(v+1) + \theta(u^2+v^2+2v) + \theta^2(u+uv) > 0.$$

Since $\theta > 0$, the above inequality holds. Therefore

$$P_{\theta_1,\theta_2}(z)/P_{\theta_1,\theta_2}(z')$$

is increasing in $\theta_2$ and decreasing in $\theta_1$.

It is to be noticed that $P_{\theta}(z)/P_{\theta}(z')$ may not be an increasing function of $\theta$ in general for $z$ and $z'$ of the following form:

$$z = (z_1, \ldots, z_{I-1}, 01z_{I+2}, \ldots, z_{J-1}, 10z_{I+2}, \ldots, z_N),$$

$$z' = (z_1, \ldots, z_{I-1}, 10z_{I+2}, \ldots, z_{J-1}, 01z_{I+2}, \ldots, z_N).$$

An example is provided by the following rank orders:

$$z = (00111110), \quad z' = (01011110).$$

It is seen that the derivative of $P_{\theta}(z)/P_{\theta}(z')$ with respect to $\theta$ evaluated at $\theta = 0$ is negative.
4. **Uniform family.** A family of distribution functions $F(x, \theta)$ is called the uniform family, if $F(x, \theta)$ is as defined below:

$$
F(x, \theta) = \begin{cases} 
0 & \text{for } x \leq 0 \\
\frac{x}{\theta} & \text{for } 0 \leq x \leq \theta \\
1 & \text{for } \theta \leq x 
\end{cases}
$$

It is seen that the uniform family has increasing likelihood ratio.

**Lemma 4.1** Suppose $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_n)$ are independent random samples respectively from two populations with distribution functions $F(x, \theta_1)$ and $F(x, \theta_2)$ belonging to the uniform family. Then, (a) the probability of any $z$ depends on $\theta_1$ and $\theta_2$ through the ratio $\theta = \frac{\theta_2}{\theta_1}$ and (b) the probability of a particular rank order depends on the number of ones at the extreme right when $\theta_2 \geq \theta_1$ and on the number of zeroes at the extreme right when $\theta_2 \leq \theta_1$.

**Proof.** Consider

$$(4.1) \quad z^1 = (z_1, \ldots, z_{m+n-r-1} 01(r)) \ , \ r \geq 1,$$

and

$$(4.2) \quad z^2 = (z'_1, \ldots, z'_{m+n-s-1} 10(s)) \ , \ s \geq 1,$$

where $l(r)$ and $0(s)$ denote runs of $r$ ones and $s$ zeroes respectively. It is seen that.

$$(4.3) \quad P_{\theta}(z^1) = \begin{cases} 
\frac{\min \theta^m}{(m+n)!} , & \text{if } \theta \leq 1 \\
\frac{1}{\theta} \left\{ \frac{\min \theta^m}{r! (m+n-r-1)!} \int_0^{1-x} x^{m+n-r-1} dx \right\} , & \text{if } \theta \geq 1
\end{cases}$$
and

\begin{align}
(4.4) \quad P_\theta(z^2) &= \begin{cases} \frac{m!n!}{s!(m+n-s-1)!\theta^n} \int_0^\theta (1-x)^s x^{m+n-s-1} \, dx, & \text{if } \theta \leq 1 \\
\frac{m!n!}{(m+n)!\theta^n} & \quad \text{if } \theta \geq 1,
\end{cases}
\end{align}

where \( \theta = \theta_2/\theta_1 \), (for an alternative form of \( P_\theta(z) \) see Savage (1956)). Since any \( z \) is either of the form \( z^1 \) or of the form \( z^2 \), (b) is established.

From (a) of lemma 4.1 it is clear that for any \( z \) and \( z' \) the rank order likelihood ratio is increasing in \( \theta_2 \) and decreasing in \( \theta_1 \) if and only if it is increasing in \( \theta \).

**Theorem 4.1** Suppose \( (X_1, \ldots, X_m) \) and \( (Y_1, \ldots, Y_n) \) are independent random samples respectively from two populations with distribution functions \( F(x, \theta_1) \) and \( F(x, \theta_2) \) belonging to the uniform family. Consider

\[ z = (z_1, \ldots, z_{m+n-r-1}) \]

and

\[ z' = (z_1', \ldots, z_{m+n-s-1}') \]

where \( r > s \geq 0 \). Then \( P_\theta(z)/P_\theta(z') \) is increasing in \( \theta \).

**Proof.** First consider \( s > 0 \). Then, for \( \theta \leq 1 \), from (4.3), \( P_\theta(z)/P_\theta(z') \) is unity. For \( \theta \geq 1 \), from (4.3)

\begin{align}
(4.5) \quad \frac{P_\theta(z)}{P_\theta(z')} &= \frac{s!(m+n-s-1)!}{r!(m+n-r-1)!} \cdot \frac{1}{\int_0^\theta (1-x)^r x^{m+n-r-1} \, dx} \cdot \frac{1}{\int_0^\theta (1-x)^s x^{m+n-s-1} \, dx}
\end{align}
Proof. If $s = 0$, the result immediately follows from theorem 4.1.

Consider $s > 0$. Then for $\theta \geq 1$, from (4.4), \( P_{\theta}(z)/P_{\theta}(z') \) is unity. For $\theta \leq 1$, from (4.4)

\[
\frac{P_{\theta}(z)}{P_{\theta}(z')} = \frac{\frac{\theta}{s!(m+n-s-1)!} \int_{0}^{\theta} (1-x)^s x^{m+n-s-1} \, dx}{\frac{\theta}{s!(m+n-s-1)!} \int_{0}^{1-x} x^{m+n-s-1} \, dx}.
\]

Comparing (4.6) with (4.5) it is seen that $P_{\theta}(z)/P_{\theta}(z')$ is increasing in $\theta$.

Corollary 4.2 The rank order likelihood ratio $P_{\theta_1, \theta_2}(z)/P_{\theta_1, \theta_2}(z')$ is increasing in $\theta_2$ and decreasing in $\theta_1$.

Proof. Consider any $z$ and $z'$ such that $zRz'$. If the interchange of components affects the last run, the result follows from theorem 4.1 and corollary 4.1. Suppose the interchange of components does not affect the last run, then $P_{\theta}(z)/P_{\theta}(z')$ is unity from lemma 4.1.

Corollary 4.3 Suppose that

\[
F(x, \theta_i) = \begin{cases} 0 & \text{for } x \leq \theta_i \\ 1 - \exp[-(x-\theta_i)] & \text{for } x \geq \theta_i, \ i=1,2, \end{cases}
\]

then the rank order likelihood ratio is increasing in $\theta_2$. 

and decreasing in $\theta_i$.

**Proof.** Consider the following transformation of the random variables $X_1, \ldots, X_m; Y_1, \ldots, Y_n$:

$$U_i = e^{-X_i}, \quad i = 1, \ldots, m,$$

and

$$V_j = e^{-Y_j}, \quad j = 1, \ldots, n.$$

The distribution functions $G(x, \eta_i)$ for $U_i$'s and $G(x, \eta_2)$ for $V_j$'s are given by

$$G(x, \eta_i) = \begin{cases} \frac{x}{\eta_i} & \text{for } x \leq \eta_i \\ 1 & \text{for } x \geq \eta_i, \end{cases}$$

where $\eta_i = \exp(-\theta_i)$. The distribution functions $G(x, \eta_i)$ thus belong to a uniform family and the results of this section apply.

5. **Minimum $(m, n) = 1$.** In this section the monotonicity of the rank order likelihood ratios is considered when the first sample size $m$ is arbitrary and the second sample size $n$ is equal to 1. Similar results follow, from lemma 5.1, for the case when the first sample size $m$ is equal to one and the second sample size $n$ is arbitrary.

**Theorem 5.1** Suppose $X_1, \ldots, X_m$ and $Y$ are mutually independent random variables. The $X_i$'s have the distribution function $F(x)$ and $Y$ has the density function $g(x, \theta)$ with increasing
likelihood ratio. Then the rank order likelihood ratio is increasing.

**Proof.** Consider $z$ and $z'$ defined below:

$$z = \begin{pmatrix} 0^{(k)} & 1 & 0^{(m-k)} \end{pmatrix}$$

and

$$z' = \begin{pmatrix} 0^{(k-1)} & 1 & 0^{(m-k+1)} \end{pmatrix}.$$  

Then $zRz'$. The probabilities of the rank order $z$ and $z'$ are given by

$$P_\theta(z) = c \int_{-\infty}^{\infty} F_k(x)(1-F(x))^{m-k} g(x, \theta) \, dx$$

and

$$P_\theta(z') = c' \int_{-\infty}^{\infty} F_{k-1}(x)(1-F(x))^{m-k+1} g(x, \theta) \, dx,$$

where $c$ and $c'$ are constants not depending on $\theta$. By virtue of lemma 2.1, to prove the theorem, it is sufficient to show that, for all $\theta > \theta'$, the following inequality holds:

$$P_\theta(z)P_{\theta'}(z') - P_\theta(z')P_{\theta'}(z) \geq 0. \quad (5.3)$$

From (5.1) and (5.2) the inequality (5.3) can be written as:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ (1-F(x))(1-F(y)) \right]^{m-k} [F(x)-F(y)] \, dx \, dy \geq 0. \quad (5.4)$$

Denote the double integral in (5.4) by $I(\theta, \theta')$ and divide the region of integration into two parts as follows:

- $A$: $x > y$ and $A'$: $x < y$.

The points in the $(x,y)$ plane for which $x$ is equal to $y$ are ignored.
since the integrand of \( I(\theta, \theta') \) is zero for such points. Thus, symbolically the integral \( I(\theta, \theta') \) is written as

\[
I(\theta, \theta') = \int_A \int_A' F(x) F(y) \left\{ [1-F(x)](1-F(y)) \right\}^{m-k} [F(x)-F(y)]^{n-k} [g(x, \theta)g(y, \theta') - g(x, \theta')g(y, \theta)] \, dx \, dy.
\]

(5.5)

Since \( \frac{g(x, \theta)}{g(y, \theta')} \) has an increasing likelihood ratio, for \( \theta > \theta' \) and \( x > y \), the following inequality holds:

\[
g(x, \theta)g(y, \theta') - g(x, \theta')g(y, \theta) \geq 0.
\]

(5.6)

Further, for \( x > y \)

\[
F(x) \geq F(y).
\]

(5.7)

Therefore, from (5.6) and (5.7), the integrand of (5.5) is non-negative. Hence \( I(\theta, \theta') \) is non-negative and the proof of the theorem is completed.

It is to be noted that if \( g(x, \theta) \) has strictly increasing likelihood ratio and \( F(x) \) does not represent a single point or a two point distribution, then \( I(\theta, \theta') \) is positive. Then, the rank order likelihood ratio is strictly increasing.

**Corollary 5.1** Let \( X \) and \( Y \) be independent random variables. The random variable \( X \) has the distribution function \( F(x) \) and \( Y \)
has the density function $g(x, \eta)$ having increasing likelihood ratio. Then for $\eta > \eta'$,

$$P_{\eta}(01:x)P_{\eta}(10:x) - P_{\eta}(10:x)P_{\eta}(01:x) \geq \Theta$$

for any $x$.

**Proof.** If

$$H(y) = \begin{cases} \frac{F(y)}{F(x)} & \text{for } y \leq x \\ 1 & \text{for } y > x \end{cases}$$

and

$$k(y, \eta) = \begin{cases} \frac{g(y, \eta)}{G(x, \eta)} & \text{for } y \leq x \\ 0 & \text{for } y > x \end{cases}$$

then the density function $k(y, \eta)$ has increasing likelihood ratio and theorem 5.1 applies. That is, $P_{\eta}(01) / P_{\eta}(10)$ is increasing in $\eta$ when sampling under $H(*)$ and $k(*, \eta)$. The last ratio is identical to $P_{\eta}(01:x) / P_{\eta}(10:x)$ when sampling under $F(*)$ and $g(*, \eta)$.

The above corollary is used in the next section.

**Lemma 5.1** Suppose $X_{11}, \ldots, X_{1m}; Y_{11}, \ldots, Y_{1n}$ are mutually independent random variables. The $X_{1i}$'s have the distribution function $F(x, \theta)$ satisfying a property $\pi_1$ and the $Y_{1i}$'s have the distribution function $G(x, \eta)$ satisfying a property $\pi_2$.

Further, suppose $X_{21}, \ldots, X_{2n}; Y_{21}, \ldots, Y_{2m}$ are mutually independent random variables. The $X_{2i}$'s have the distribution function $H(x, \theta)$ satisfying the property $\pi_2$ and the $Y_{2i}$'s have the distribution function $K(x, \theta)$ satisfying the property $\pi_1$.

Consider the following four statements:

(a) For each $\theta$, $P_{\theta, \eta}(z) / P_{\theta, \eta'}(z')$ is an increasing function of
\( \eta \) whenever \( zRz' \);

(b) For each \( \eta \), \( P_{\theta, \eta}(z)/P_{\theta, \eta}(z') \) is a decreasing function of \( \theta \) whenever \( zRz' \);

(c) For each \( \gamma \), \( P_{\gamma, \delta}(z)/P_{\gamma, \delta}(z') \) is an increasing function of \( \delta \) whenever \( zRz' \);

(d) For each \( \delta \), \( P_{\gamma, \delta}(z)/P_{\gamma, \delta}(z') \) is a decreasing function of \( \gamma \) whenever \( zRz' \).

Then \( (a) \iff (d) \) and \( (b) \iff (c) \).

**Proof.** Only the implication \( (a) \Rightarrow (d) \) is proved below. Other implications are obtained in exactly similar manner.

Consider the following transformation:

\[ X_{2i} = Y_{2i}', \; i = 1, \ldots, n \]

and

\[ Y_{2j} = X_{2j}', \; j = 1, \ldots, m. \]

Then the \( X_{2j}'s \) have the distribution function \( K(x, \delta) \) satisfying the property \( \pi_1 \) and the \( Y_{2j}'s \) have the distribution function \( H(x, \gamma) \) satisfying the property \( \pi_2 \). Therefore if \( (a) \) is true, then

\[ P_{\delta, \gamma}(z)/P_{\delta, \gamma}(z') \]

is an increasing function of \( \gamma \) whenever \( zRz' \).

Now for the random variables \( X_{21}', \ldots, X_{2n}; \; Y_{21}', \ldots, Y_{2m} \), consider any \( z, z' \) such that \( zRz' \). Then,

\[ \frac{P_{\gamma, \delta}(z)}{P_{\gamma, \delta}(z')} = \frac{P_{\delta, \gamma}(z'^C)}{P_{\delta, \gamma}(z'^C)}, \]

where \( z'^C Rz'^C \). Therefore, \( P_{\delta, \gamma}(z'^C)/P_{\delta, \gamma}(z^C) \) is an increasing function of \( \gamma \) for each \( \delta \) which implies that \( P_{\gamma, \delta}(z)/P_{\gamma, \delta}(z') \) is a decreasing function of \( \gamma \) for each \( \delta \) whenever \( zRz' \).
Remark to lemma 5.1  The lemma holds if everywhere $zRz'$ is replaced by $zR^*z'$. This restricted ordering is considered in section 6.

**Corollary 5.2**  Let $X; Y_1, \ldots, Y_n$ be mutually independent random variables. The $Y_i$'s have the distribution function $G(x)$ and $X$ has the density function $f(x, \theta)$ having an increasing likelihood ratio. Then the rank order likelihood ratio is decreasing.

The proof immediately follows by applying lemma 5.1 to theorem 5.1.

**Corollary 5.3**  Let $X_1, \ldots, X_m; Y$ be mutually independent random variables. The $X_i$'s have the distribution function $F(x-\eta)$ and $Y$ has the density function $g(x-\eta)$ having increasing likelihood ratio. Then the rank order likelihood ratio is increasing in $\eta$ and decreasing in $\theta$.

**Proof.** A direct application of theorem 5.1 shows that the rank order likelihood ratio is increasing in $\eta$. The rank orders would have the same probabilities if the $X_i$'s had the distribution function $F(x)$ and $Y$ had the density function $g(x-\delta)$ where $\delta = \eta - \theta$. Therefore from theorem 5.1, the rank order likelihood ratio is decreasing in $\theta$.

**Corollary 5.4**  Let $X; Y_1, \ldots, Y_n$ be mutually independent random variables. The $Y_i$'s have the distribution function $G(x-\eta)$ and $X$ has the density function $f(x-\theta)$ having increasing likelihood ratio. Then the rank order likelihood ratio is increasing in $\eta$.
and decreasing in $\theta$.

The proof is immediate by applying lemma 5.1 to corollary 5.3.

**Corollary 5.5** Let $X_1, \ldots, X_m$; $Y$ be mutually independent random variables. The $X_i$'s have the distribution function $F(\theta x)$ and $Y$ has the density function $\eta g(\eta x)$ having increasing likelihood ratio, $\theta, \eta > 0$. Then the rank order likelihood ratio is increasing in $\eta$ and decreasing in $\theta$.

**Proof.** From theorem 5.1 it is immediate that the rank order likelihood ratio is increasing in $\eta$. The rank orders would have the same probabilities if the $X_i$'s had the distribution function $F(x)$ and $Y$ had the density function $(\eta/\theta)g(\eta x/\theta)$. Therefore, from theorem 5.1 the rank order likelihood ratio is decreasing in $\theta$.

**Corollary 5.6** Let $X; Y_1, \ldots, Y_n$ be mutually independent random variables. The $Y_i$'s have the distribution function $G(\eta x)$ and $X$ has the density function $\theta f(\theta x)$ having increasing likelihood ratio, $\theta, \eta > 0$. Then, the rank order likelihood ratio is increasing in $\eta$ and decreasing in $\theta$.

The proof is immediate by applying lemma 5.1 to corollary 5.5.

The following conjecture has not been proved: Suppose $X_1, \ldots, X_m$; $Y$ are mutually independent random variables. The $X_i$'s have the density function $f(x, \theta)$ having increasing likelihood ratio and $Y$ has the distribution function $G(x)$. Then the
rank order likelihood ratio is decreasing in $\theta$.

6. **Monotone properties of the likelihood ratio of rank orders for the $R^*$ relationship.** In this section the partial ordering considered is defined by the relation $z R^* z'$. The results of this section apply to normal families and logistic families of distributions with translation parameters. Both the families have increasing likelihood ratios. The following lemmas are used for proving theorem 6.1 of this section.

**Lemma 6.1** Let $X_1, \ldots, X_m; Y_1, \ldots, Y_n$ be mutually independent random variables. The $X_i$'s have the density function $f(x)$ and the $Y_i$'s have the density function $g(x, \eta)$ having increasing likelihood ratio. Then for all $\eta > \eta'$, $P_{\eta}(z^N : x)/P_{\eta'}(z^N : x)$ is an increasing function of $x$, where $N = m + n$.

**Proof.** Suppose the lemma is true for $N$. Then, consider

$$
\frac{P_{\eta}(z^{N+1} : x)}{P_{\eta'}(z^{N+1} : x)} = \frac{\int_{-\infty}^{x} f(y) \frac{g(y, \eta')}{g(y, \eta)} \, dy}{\int_{-\infty}^{x} f(y) \frac{g(y, \eta')}{g(y, \eta)} \, dy},
$$

where $c$ is a constant not depending on $x$. The derivative

$$
\frac{\partial}{\partial x} \left[ \frac{P_{\eta}(z^{N+1} : x)}{P_{\eta'}(z^{N+1} : x)} \right] \geq 0
$$

if and only if the following inequality holds:
\[ \int_{-\infty}^{x} \left\{ f(x)f(y) \right\} ^{1-z_{N+1}} \left\{ g(x, \eta)g(y, \eta') \right\} ^{z_{N+1}} \, dy \geq 0. \]

(6.1)

As \( g(x, \eta) \) has increasing likelihood ratio the inequality

(6.2) \[ g(x, \eta)g(y, \eta') \geq g(x, \eta')g(y, \eta) \]

holds for \( x > y \) and \( \eta > \eta' \). From the induction hypothesis the inequality

\[ P_{\eta}(z_{N}^{N}:x)P_{\eta}(z_{N}^{N}:y) \geq P_{\eta}(z_{N}^{N}:y)P_{\eta}(z_{N}^{N}:x) \]

holds for \( x > y \) and \( \eta > \eta' \). Thus the integrand of (6.1) is non-negative and consequently (6.1) is true.

Consider \( N \) equal to one. Then,

\[ \frac{P_{\eta}(z_{1}^{1}:x)}{P_{\eta}(z_{1}^{1}:x)} = \frac{x \int_{-\infty}^{x} \left\{ f(y)g(y, \eta) \right\} ^{1-z_{1}} \, dy}{x \int_{-\infty}^{x} \left\{ f(y)g(y, \eta') \right\} ^{1-z_{1}} \, dy} \].

Therefore,

\[ \frac{\partial}{\partial x} \left[ \frac{P_{\eta}(z_{1}^{1}:x)}{P_{\eta}(z_{1}^{1}:x)} \right] \geq 0 \]

if and only if the following inequality holds:

(6.3) \[ \int_{-\infty}^{x} \left\{ f(x)f(y) \right\} ^{1-z_{N+1}} \left\{ g(x, \eta)g(y, \eta') - g(x, \eta')g(y, \eta) \right\} ^{z_{1}} \, dy \geq 0. \]

From (6.2) the integrand of (6.3) is non-negative. Therefore (6.3) is true. This completes the proof of the lemma.
Lemma 6.2 Let $X_1, \ldots, X_m; Y_1, \ldots, Y_n$ be mutually independent random variables. The $X_i$'s have the density function $f(x)$ and the $Y_i$'s have the density function $g(x, \eta)$ having increasing likelihood ratio. Then for all $\eta > \eta'$, $\frac{P_\eta(x:z^N)}{P_{\eta'}(x:z^N)}$ is an increasing function of $x$, where $N = m + n$.

Proof. Relabel $z_i$'s in $z^N$ and write it as $z^N = (z_N z_{N-1} \ldots z_1)$.

Suppose the lemma is true for $N$. Consider

$$\frac{P_\eta(x:z^{N+1})}{P_{\eta'}(x:z^{N+1})} = \frac{\mathcal{C} \int_{x}^{\infty} P_\eta(y:z^N) f^{1-z_N^{N+1}}(y) g^{z_N^{N+1}}(y, \eta) \, dy}{\int_{x}^{\infty} P_{\eta'}(y:z^N) f^{1-z_N^{N+1}}(y) g^{z_N^{N+1}}(y, \eta') \, dy},$$

where $\mathcal{C}$ is a constant not depending on $x$. Now the proof of the lemma is obtained by the same procedure as for the proof of lemma 6.1.

Lemma 6.3 Let $X$ and $Y$ be independent random variables having the distribution function $F(x), G(x)$ and the density functions $f(x), g(x)$ respectively. If $g(x)F(x)/f(x)G(x)$ is an increasing function of $x$, then $P(01:x)/P(10:x)$ is an increasing function of $x$.

Proof. Consider

$$\frac{P_{01:x}}{P_{10:x}} = \frac{\int_{x}^{\infty} F(y) g(y) \, dy}{\int_{-\infty}^{x} G(y) f(y) \, dy}.$$
where $S$ is a subset of the real line,

$$(iii) \quad z = (0z_3, \ldots, zn) \text{ and } z' = (10z_3, \ldots, zn)$$

where $N = m + n$, then for all $\eta > \eta'$ where $\eta, \eta' \in S$, the following
inequality holds:

$$(6.5) \quad P_{\eta}(z)P_{\eta'}(z') - P_{\eta}(z')P_{\eta'}(z) \geq 0.$$

Proof. Denote the expression in (6.5) by $I(\eta, \eta')$. Then,

$$I(\eta, \eta') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ f(x)g(y) \right] \frac{1-z_3}{g(x, \eta)g(y, \eta')} \cdot z_3 \cdot P_{\eta}(x: z_4, \ldots, zn)$$

$$(6.6) \quad P_{\eta}(y: z_4, \ldots, zn) \{ P_{\eta}(0l: x)P_{\eta'}(10: y) - P_{\eta'}(0l: y)P_{\eta}(10: x) \} \, dx \, dy.$$

Break up the region of integration in (6.6) into the following
two parts

$A: \ x \geq y \quad \text{and} \quad A': \ x \leq y.$

The inclusion of equality sign in both the regions does not create
any problem as the integrand is a continuous function. Thus
symbolically

$$I(\eta, \eta') = \int \int + \int \int \ \ .$$

In $\int \int$ make the transformation: $x = y'$ and $y = x'$. Since $x'$ and
$y'$ are dummy variables, the relabelling of them as $x$ and $y$ after
the transformation has been made, is in order. Therefore,
\[ I(\eta, \eta') = \text{def} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) f(y) \left[ \left\{ g(x, \eta) g(y, \eta') \right\} \right]^{1-\zeta} \left[ g(x, \eta) g(y, \eta') \right]^{\zeta} P_{\eta}(x; z_4, \ldots, z_N) \]

(6.7)

\[
\left\{ P_{\eta}(01:x) P_{\eta}(10:y) - P_{\eta}(01:y) P_{\eta}(10:x) \right\} \]

dx dy .

Since \( g(x, \eta) \) has increasing likelihood ratio, it follows that

(6.8) \[ g(x, \eta) g(y, \eta') \geq g(y, \eta) g(x, \eta'), \text{ for } x > y \text{ and } \eta > \eta'. \]

From lemma 6.2, the following inequality holds:

(6.9) \[ P_{\eta}(x; z_4, \ldots, z_N) P_{\eta}(y; z_4, \ldots, z_N) \geq P_{\eta}(y; z_4, \ldots, z_N) P_{\eta}(x; z_4, \ldots, z_N), \]

for \( x > y \) and \( \eta > \eta' \).

Using (6.8) and (6.9) in (6.7) it is seen that

\[ I(\eta, \eta') \geq 0 \]

if the following two inequalities hold:

(6.10) \[ P_{\eta}(01:x) P_{\eta}(10:y) - P_{\eta}(01:y) P_{\eta}(10:x) \geq 0 \]

and

\[ P_{\eta}(01:x) P_{\eta}(10:y) - P_{\eta}(01:y) P_{\eta}(10:x) \]

(6.11) \[ -P_{\eta}(01:x) P_{\eta}(10:y) + P_{\eta}(01:y) P_{\eta}(10:x) \geq 0. \]

From lemmas 6.1, 6.3 and corollary 5.1 the following inequalities hold:

(6.12) \[ P_{\eta}(10:x) P_{\eta}(10:y) - P_{\eta}(10:y) P_{\eta}(10:x) \geq 0 \quad (\text{lemma 6.1}), \]
(6.13) \( P_\eta(01:x)P_\eta(10:y) - P_\eta(01:y)P_\eta(10:x) \geq 0 \) (lemma 6.3),

(6.14) \( P_\eta(01:x)P_\eta(10:x) - P_\eta(10:x)P_\eta(01:x) \geq 0 \) (corollary 5.1),

(6.15) \( P_\eta(01:y)P_{\eta'}(10:y) - P_\eta(10:y)P_{\eta'}(01:y) \geq 0 \) (corollary 5.1).

Using (6.13) and (6.14), (6.10) is found to be true. The inequalities (6.14) and (6.15) give

(6.16) \( P_\eta(01:x)P_\eta(10:y) \leq \frac{P_\eta(01:x)P_\eta(01:y)P_\eta(10:x)P_{\eta'}(10:y)}{P_\eta(10:x)P_{\eta'}(01:y)} \).

Denote the left hand side of (6.11) by \( J(\eta, \eta', x, y) \). Then, from (6.16) it follows that

\[
J(\eta, \eta', x, y) \geq P_\eta(01:x)P_{\eta'}(10:y) - P_{\eta'}(01:y)P_\eta(10:x)
\]

\[
- \frac{P_\eta(01:y)P_{\eta'}(10:x)}{P_\eta(10:x)P_{\eta'}(01:y)} \left[ P_\eta(01:x)P_{\eta'}(10:y) - P_{\eta'}(10:x)P_\eta(01:y) \right]
\]

\[
= \left[ P_\eta(01:x)P_{\eta'}(10:y) - P_{\eta'}(10:x)P_\eta(01:y) \right] \left[ P_{\eta'}(10:x)P_\eta(01:y) \right]^{-1}
\]

(6.17) \( P_{\eta'}(01:y)P_{\eta'}(10:x) \left[ P_{\eta'}(10:x)P_\eta(01:y) \right]^{-1} \).

If

\( P_{\eta'}(10:x)P_\eta(01:y) - P_{\eta'}(01:y)P_\eta(10:x) \geq 0 \),

then using (6.10) in (6.17) it follows that \( J(\eta, \eta', x, y) \geq 0 \).

Alternatively suppose

(6.18) \( P_{\eta'}(10:x)P_\eta(01:y) - P_{\eta'}(01:y)P_\eta(10:x) \leq 0 \).

From (6.12) and (6.14) it follows that

(6.19) \( P_\eta(01:x)P_{\eta'}(10:y) - P_{\eta'}(01:x)P_{\eta'}(10:y) \geq 0 \).
Then (6.18) and (6.19) give
\[ J(\eta, \eta', x, y) \geq 0. \]
Thus the inequalities (6.10) and (6.11) are found to be true implying that
\[ I(\eta, \eta') \geq 0. \]

In the above theorem, the density function for the \( X_i \)'s and the density function for the \( Y_i \)'s did not need to belong to the same family. For the following corollaries, it is assumed that both the density functions belong to a translation parameter family and that the density functions are symmetric about the translation parameter. If the \( X_i \)'s have the density function \( f(x-\theta_1) \) and the \( Y_i \)'s have the density function \( f(x-\theta_2) \), then the probabilities of the rank orders depend on \( \theta_1 \) and \( \theta_2 \) through the difference \( \theta = \theta_2 - \theta_1 \). Therefore the probabilities of the rank orders would be the same if the \( X_i \)'s had the density function \( f(x) \) and the \( Y_i \)'s had the density function \( f(x-\theta) \). The rank order likelihood ratio is increasing in \( \theta_2 \) and decreasing in \( \theta_1 \) if and only if it is increasing in \( \theta \).

**Corollary 6.1** Let \( X_1, \ldots, X_m; Y_1, \ldots, Y_n \) be mutually independent random variables. The \( X_i \)'s have the density function \( f(x) \) and the \( Y_i \)'s have the density function \( f(x-\theta) \). If

(i) \( f(x-\theta) = f(\theta-x) \),

(ii) \( f(x-\theta) \) has increasing likelihood ratio,

(iii) \( f(x-\theta)F(x)/f(x)F(x-\theta) \) is an increasing function of \( x \) for \( \theta \geq 0 \),

(iv) \( z = (0z_2, \ldots, z_N) \) and \( z' = (10z_2, \ldots, z_N) \), \( N = m + n \),
then $P_\theta(z)/P_\theta(z')$ is an increasing function of $\theta$ for all $\theta$.

**Proof.** It is clear from theorem 6.1 that $P_\theta(z)/P_\theta(z')$ is an increasing function of $\theta$ for $\theta \geq 0$. Consider $\eta \leq 0$. Theorem 1 of Savage, Sobel and Woodworth (1965) gives

\[
(6.20) \quad \frac{P_\eta(0z_3^c, \ldots, z_N^c)}{P_\eta(10z_3^c, \ldots, z_N^c)} = \frac{P_\eta(10z_3^c, \ldots, z_N^c)}{P_\eta(0z_3^c, \ldots, z_N^c)},
\]

where $\eta = -\theta \geq 0$. Applying theorem 6.1 to the reciprocal of the right hand side of (6.20), the corollary is proved.

**Corollary 6.2** If the first three conditions of corollary 6.1 are true and

\[
z = (z_1, \ldots, z_{N-2}01) \text{ and } z' = (z_1, \ldots, z_{N-2}10), \quad N = m + n,
\]

then $P_\theta(z)/P_\theta(z')$ is an increasing function of $\theta$.

**Proof.** Theorem 1 of Savage, Sobel and Woodworth (1965) gives

\[
(6.21) \quad \frac{P_\eta(z_1^t, \ldots, z_{N-2}01^t)}{P_\eta(z_1^t, \ldots, z_{N-2}10^t)} = \frac{P_\eta(10z_3^c, \ldots, z_N^c)}{P_\eta(0z_3^c, \ldots, z_N^c)},
\]

where $\eta = -\theta$. The result follows by applying corollary 6.1 to the reciprocal of the right hand side of (6.21).

From corollaries 6.1, 6.2 and the remark to lemma 2.1, the following theorem immediately follows.

**Theorem 6.2** Let $X_1, \ldots, X_m; Y_1, \ldots, Y_n$ be mutually independent random variables. The $X_i$'s have the density function $f(x)$ and
the $Y_i$'s have the density function $f(x - \theta)$. If

(i) $f(x - \theta) = f(\theta - x)$,

(ii) $f(x - \theta)$ has increasing likelihood ratio,

(iii) $f(x - \theta)F(x)/f(x)F(x - \theta)$ is an increasing function of $\chi$ for $\theta \geq 0$,

then the rank order likelihood ratio is increasing (*) in $\theta$.

**Example 6.1**. Consider

$$f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} (x - \theta)^2 \right\}, \quad -\infty \leq x \leq \infty.$$  

It has been found that conditions (i) and (ii) of theorem 6.2 are checked. Condition (iii) of theorem 6.2 holds due to lemma 1, appendix I of Savage, Sobel and Woodworth (1965).

Therefore, theorem 6.2 applies.

**Example 6.2**. Consider

$$f(x, \theta) = \frac{e^{-(x - \theta)}}{[1+e^{-(x - \theta)}]^2}, \quad -\infty \leq x \leq \infty.$$  

It has been found that conditions (i) and (ii) of theorem 6.2 are checked. Since

$$F(x, \theta) = \frac{1}{1+e^{-(x - \theta)}},$$

it follows that

$$\frac{f(x, \theta_2)F(x, \theta_1)}{f(x, \theta_1)F(x, \theta_2)} = e^{\theta_2 - \theta_1} \frac{F(x, \theta_2)}{F(x, \theta_1)}.$$  

For $\theta_2 \geq \theta_1$, lemma 1 of the appendix tells that $F(x, \theta_2)/F(x, \theta_1)$
is an increasing function of $x$. Therefore condition (iii) of theorem 6.2 holds. Consequently theorem 6.2 applies.

7. $m = n = 2$. Suppose $X_1, X_2; Y_1, Y_2$ are mutually independent random variables. The $X_i$'s have the distribution function $F(x - \theta_1)$ and the density function $f(x - \theta_1)$. The $Y_i$'s have the distribution function $F(x - \theta_2)$ and the density function $f(x - \theta_2)$. Assume that

$$f(x - \theta) = f(\theta - x).$$

The possible rank orders are: (0011), (0101), (1001), (0110), (1010) and (1100). The probabilities of these rank orders depend on $\theta_1$ and $\theta_2$ through the difference $\theta = \theta_2 - \theta_1$. Therefore the probabilities of the rank orders would be the same if the $X_i$'s had the distribution function $F(x)$ and the $Y_i$'s had the distribution function $F(x - \theta)$. The rank order likelihood ratio is increasing in $\theta_2$ and decreasing in $\theta_1$ if and only if it is increasing in $\theta$.

The following relations are obtained from theorem 1 and 2 of Savage, Sobel and Woodworth (1965):

$$P_{\theta}(0011) = P_{-\theta}(1100)$$

(7.1) $$P_{\theta}(0101) = P_{-\theta}(1010)$$

$$P_{\theta}(1001) = P_{\theta}(0110).$$

Therefore, to show that the rank order likelihood ratio is increasing in $\theta$ it is sufficient to show that the likelihood
ratios \( P_\theta(0011)/P_\theta(0101) \) and \( P_\theta(0101)/P_\theta(1001) \) are increasing in \( \theta \).

In this section three families of distributions, namely (a) a normal family, (b) a logistic family and (c) a double exponential family, are shown to have the rank order likelihood ratio increasing in \( \theta \) with \( m = n = 2 \). Theorem 6.2 of the previous section has been used for interchanges at the extreme ends in (a) and (b).

I. A normal family: Let \( F(\cdot) \) and \( f(\cdot) \) denote the standard normal distribution function and the density function respectively. From example 6.1 it follows that \( P_\theta(0101)/P_\theta(1001) \) is an increasing function of \( \theta \).

Theorem 9 of the appendix IV of Savage, Sobel and Woodworth (1965) shows that

\[
(7.2) \quad P(0101) = 2P^2(01) - 2P(0011),
\]

with arbitrary densities \( h(\cdot) \) and \( k(\cdot) \) for the \( X_i \)'s and the \( Y_i \)'s respectively. Therefore, \( P_\theta(0011)/P_\theta(0101) \) is an increasing function of \( \theta \) if and only if \( d[P_\theta(0011)/P_\theta^2(01)]/d\theta \geq 0 \), for all \( \theta \), or if and only if

\[
(7.3) \quad P_\theta(01) \frac{d}{d\theta} [P_\theta(0011)] - 2P_\theta(0011) \frac{d}{d\theta} [P_\theta(01)] \geq 0,
\]

for all \( \theta \). Denote the expression in (7.3) by \( G(\theta) \). The following result is well known.

\[
(7.4) \quad \int_{-\infty}^{\infty} F(ax + b) f(x) \, dx = F(b/\sqrt{1+a^2}).
\]

Using (7.4) it is seen that

\[
P_\theta(01) = F(\theta/\sqrt{2}) \quad \text{and} \quad d[P_\theta(01)]/d\theta = (1/\sqrt{2})f(\theta/\sqrt{2})
\]
\[ P_\theta(0011) = 2 \int_{-\infty}^{\infty} (1-F(x-\theta))^2F(x)f(x)dx, \]

\[ \frac{d}{d\theta}[P_\theta(0011)] = \frac{4}{\sqrt{2}} f\left(\frac{\theta}{\sqrt{2}}\right) \left[ F\left(\frac{\theta}{\sqrt{6}}\right) - \int_{-\infty}^{\infty} F\left(\frac{x}{\sqrt{2}} + \frac{\theta}{2}\right)F\left(\frac{x}{\sqrt{2}} - \frac{\theta}{2}\right)f(x)dx \right] \]

After some simplification it is found that

\[ G(\theta) = 2\sqrt{2} f(\theta/\sqrt{2})G_1(\theta) \quad \text{where} \]

\[ G_1(\theta) = \mathbb{E}\left(\frac{\theta}{\sqrt{2}}\right) \left[ F\left(\frac{\theta}{\sqrt{6}}\right) - \int_{-\infty}^{\infty} F\left(\frac{x}{\sqrt{2}} - \frac{\theta}{2}\right)F\left(\frac{x}{\sqrt{2}} + \frac{\theta}{2}\right)f(x)dx \right] \]

(7.5) \[ \int_{-\infty}^{\infty} (1-F(x-\theta))^2F(x)f(x)dx. \]

Therefore, \( G(\theta) \geq 0 \) for all \( \theta \), if and only if \( G_1(\theta) \geq 0 \)

for all \( \theta \). Notice that \( G_1(-\infty) = 0 \). Therefore \( G_1(\theta) \geq 0 \)

if the derivative of \( G_1(\theta) \) is non negative for all \( \theta \).

After some computations it is found that

\[ \frac{d}{d\theta}G_1(\theta) = \frac{f(\theta/\sqrt{2})}{\sqrt{6}} \left[ 2e^{\theta^2/6}F\left(\frac{\theta}{\sqrt{2}}\right)F\left(\frac{\theta}{\sqrt{3}}\right) \right. \]

(7.6) \[ -\sqrt{3} \int_{-\infty}^{\infty} F\left(\frac{\theta}{2} + \frac{x}{\sqrt{2}}\right)F\left(\frac{\theta}{2} - \frac{x}{\sqrt{2}}\right)f(x)dx \]

Denote the expression within the big brackets in (7.6) by

\( G_2(\theta) \). Then the derivative of \( G_1(\theta) \) is non negative if and only if \( G_2(\theta) \geq 0 \) for all \( \theta \). Notice that

\( G_2(-\infty) = 0 \). Further, it is found that


\[
\frac{d}{d\theta} G_2(\theta) = 2 e^{\theta^2/6} F\left(\frac{\theta}{\sqrt{3}}\right) \left[ f\left(\frac{\theta}{\sqrt{3}}\right) + \frac{\theta}{\sqrt{3}} F\left(\frac{\theta}{\sqrt{3}}\right) \right].
\]

The term \( f\left(\frac{\theta}{\sqrt{3}}\right) + \frac{\theta}{\sqrt{3}} F\left(\frac{\theta}{\sqrt{3}}\right) \) is clearly positive for \( \theta \geq 0 \) and it is positive for \( \theta < 0 \) from the Feller's inequality:

\[
\frac{F(x)}{f(x)} < -\frac{1}{x}
\]

for \( x < 0 \). Therefore \( d[G_2(\theta)]/d\theta \geq 0 \) for all \( \theta \). Consequently \( P_\theta(0011)/P_\theta(0101) \) is an increasing function of \( \theta \).

II. Logistic family: Consider

\[
f(x-\theta) = \frac{e^{-(x-\theta)}}{[1+e^{-(x-\theta)}]^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.
\]

From example 6.2 it follows that \( P_\theta(0101)/P_\theta(1001) \) is an increasing function of \( \theta \). For showing that \( P_\theta(0011)/P_\theta(0101) \) is an increasing function of \( \theta \), equivalently, it will be shown that \( P_\theta(0011)/P_\theta^2(01) \) is an increasing function of \( \theta \).

Denote \( e^\theta \) by \( c \) for simplicity sake. Then

\[
P_\theta(01) = \frac{c(c-1-\log c)}{(c-1)^2}
\]

and

\[
P_\theta(0011) = \frac{2c^2}{(c-1)^4} \left[ (5+c)(c-1) - 2(1+2c) \log c \right].
\]
Therefore

$$\frac{d}{d\theta} \left[ \frac{P_{\theta}(0011)}{P_{\theta}(01)} \right] \geq 0$$

for all $\theta$ if and only if

$$(7.7) \quad 2\theta^2 e^\theta + \theta (e^{2\theta} - 1) - 4(e^{\theta} - 1)^2 \geq 0$$

for all $\theta$. Denote the expression in (7.7) by $G(\theta)$ and define

$$G_1(\theta) = e^\theta G(\theta) = 2\theta^2 + \theta e^\theta - \theta e^{-\theta} - 4e^\theta - 4e^{-\theta} + 8.$$  

Then $G(\theta) \geq 0$ for all $\theta$ if and only if $G_1(\theta) \geq 0$ for all $\theta$.

Notice that $G_1(0)$ is zero. Therefore $G_1(\theta) \geq 0$ for all $\theta$ if $d[G_1(\theta)]/d\theta \geq 0$ for $\theta \geq 0$ and $d[G_1(\theta)]/d\theta \leq 0$ for $\theta \leq 0$.

By repeated differentiation it is seen that the first three derivatives of $G_1(\theta)$ evaluated at $\theta = 0$ are zero and the fourth derivative is

$$\frac{d^4}{d\theta^4} [G_1(\theta)] = \theta (e^\theta - e^{-\theta}) \geq 0$$

for all $\theta$. Therefore $G_1(\theta) \geq 0$ for all $\theta$. Consequently $P_{\theta}(0011)/P_{\theta}(0101)$ is an increasing function of $\theta$.

III. Double exponential family: Consider

$$f(x;\theta) = \frac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$  

As pointed out in (I), $P_{\theta}(0011)/P_{\theta}(0101)$ is an increasing function of $\theta$ if and only if $P_{\theta}(0011)/P_{\theta}^2(01)$ is an increasing function of $\theta$.  

Theorem 9 of the appendix IV of Savage, Sobel and Woodworth (1965) shows that

\begin{equation}
(7.8) \quad P_\theta(1001) = 2 \, P_\theta(011) - 2 \, P^2_\theta(01) .
\end{equation}

Then, from (7.2) and (7.8)

\[
\frac{P_\theta(0101)}{P_\theta(1001)} = \frac{\frac{P^2_\theta(01) - P_\theta(0011)}{P_\theta(01) - P^2_\theta(01)}}{P_\theta(011) - P^2_\theta(01)} .
\]

Thus, the following probabilities are needed to show that the rank order likelihood ratio is increasing in \( \theta \) when \( m = n = 2 \):

\[
P_\theta(01) = \begin{cases} 
1 - \frac{e^{-\theta}}{2} - \frac{\theta e^{-\theta}}{4} & \text{for } \theta \geq 0 \\
\frac{e^{-\theta}}{2} - \frac{\theta e^{-\theta}}{4} & \text{for } \theta \leq 0
\end{cases}
\]

\[
P_\theta(011) = \begin{cases} 
1 - \frac{7}{12} e^{-\theta} - \frac{1}{12} e^{-2\theta} - \frac{1}{2} \theta e^{-\theta} & \text{for } \theta \geq 0 \\
\frac{5 e^{-\theta}}{12} - \frac{2}{12} & \text{for } \theta \leq 0
\end{cases}
\]

and

\[
P_\theta(0011) = \begin{cases} 
1 - \frac{e^{-\theta}}{3} - \theta e^{-\theta} - \frac{\theta e^{-2\theta}}{8} - \frac{e^{-2\theta}}{2} & \text{for } \theta \geq 0 \\
\frac{1}{6} e^{2\theta} - \frac{\theta e^{2\theta}}{8} & \text{for } \theta \leq 0
\end{cases}
\]

It is seen by repeated differentiation that

\[
\frac{P_\theta(0011)}{P^2_\theta(01)} \quad \text{and} \quad \frac{P^2_\theta(01) - P_\theta(0011)}{P_\theta(011) - P^2_\theta(01)}
\]

are increasing function of \( \theta \) for all \( \theta \).
It should be noted that if independent random samples of sizes \( m \) and \( n \) are taken from populations with densities \( f(x) \) and \( f(x - \theta) \) respectively, then

\[
\frac{P_\theta(0^{(m)} 1^{(n)})}{P_\theta(1^{(n)} 0^{(m)})}
\]

is an increasing function of \( \theta \). This is so because \( P_\theta(0^{(m)} 1^{(n)}) \) is an increasing function of \( \theta \) for all \( \theta \) and \( P_\theta(1^{(n)} 0^{(m)}) \) is a decreasing function of \( \theta \) for all \( \theta \).

8. **Monotonicity of the rank order likelihood ratio** in a neighborhood of \( \theta_1 = \theta_2 \). The following theorem shows the monotonicity of the rank order likelihood ratio in a neighborhood of \( \theta_1 = \theta_2 \) when the two populations sampled belong to a family of densities with increasing likelihood ratio.

**Theorem 8.1** Let \( X_1, \ldots, X_m; Y_1, \ldots, Y_n \) be mutually independent random variables. The \( X_i \)'s have the density function \( f(x, \theta_1) \) and the \( Y_i \)'s have the density function \( f(x, \theta_2) \). Suppose \( f(x, \theta) \) has increasing likelihood ratio. Then, in a neighborhood of \( \theta_1 = \theta_2 \), the rank order likelihood ratio is increasing in \( \theta_2 \) and decreasing in \( \theta_1 \).

**Proof.** Consider \( z \) and \( z' \) such that \( z R z' \). For fixed \( \theta_1 \)
\[
\frac{\partial}{\partial \theta_2} \left\{ \frac{\theta_1, \theta_2 (z)}{\theta_1, \theta_2 (z')} \right\} \theta_2 = \theta_1
\]

\[
= \lim_{\theta_2 \to \theta_1} \frac{1}{\theta_2 - \theta_1} \left[ \frac{\theta_1, \theta_2 (z)}{\theta_1, \theta_2 (z')} - \frac{\theta_1, \theta_1 (z)}{\theta_1, \theta_1 (z')} \right]
\]

\[
(8.1) = \lim_{\theta_2 \to \theta_1} \frac{1}{\theta_2 - \theta_1} \left[ \frac{\theta_1, \theta_2 (z) - \theta_1, \theta_2 (z')}{\theta_1, \theta_2 (z') \theta_1, \theta_1 (z')} \right]
\]

as

\[
P_{\theta_1, \theta_2} (z) = P_{\theta_1, \theta_1} (z').
\]

From theorem 6.1 of Savage (1956)

\[
(8.2) \quad P_{\theta_1, \theta_2} (z) - P_{\theta_1, \theta_2} (z') \geq 0
\]

according as \( \theta_2 \geq \theta_1 \) or \( \theta_2 \leq \theta_1 \). Using (8.2) in (8.1) it follows that

\[
\left[ \frac{\partial}{\partial \theta_2} \left\{ \frac{\theta_1, \theta_2 (z)}{\theta_1, \theta_2 (z')} \right\} \right] \theta_2 = \theta_1 \geq 0.
\]

In a similar manner it can be shown that

\[
\left[ \frac{\partial}{\partial \theta_1} \left\{ \frac{\theta_1, \theta_2 (z)}{\theta_1, \theta_2 (z')} \right\} \right] \theta_1 = \theta_2 \leq 0.
\]

9. Conjectures. The following conjecture has not been proved:

Suppose \( X_1, \ldots, X_m \), \( Y_1, \ldots, Y_n \) are mutually independent random variables. The \( X_i \)'s have the density function \( f(x, \theta_1) \), the \( Y_i \)'s have the density function \( f(x, \theta_2) \) and \( f(x, \theta) \) has
increasing likelihood ratio. Then the rank order likelihood ratio is increasing in $\theta_2$ and decreasing in $\theta_1$.

Tables have been prepared by Milton (1965) for the probabilities of rank orders for a normal family with translation parameter, with various combinations of sample sizes from 1 through 7. These tables provide some numerical evidence in support of the above conjecture with respect to the normal family for the values of the parameter considered. The following conjecture has been motivated by an unsuccessful attempt to obtain monotonicity of the rank order likelihood ratio for a normal family:

Suppose $W_1, \ldots, W_N$ are the order statistics in an independent random sample of size $N$ from a standard normal population.

Consider $z$ and $z'$ such that $zRz'$. Define $Y_1 = \sum_{i=1}^{N} z_i W_i$ and

$Y_2 = \sum_{i=1}^{N} z'_i W_i$. Let $G_1(y)$ and $G_2(y)$ be the distribution functions and $g_1(y)$ and $g_2(y)$ be the density functions of $Y_1$ and $Y_2$ respectively. Then (i) $g_1(y)/g_2(y)$ is an increasing function of $y$ and (ii) $G_1(y)/G_2(y)$ is an increasing function of $y$.

The above conjecture related to the normal family is motivated by the following consideration. Suppose $X_1, \ldots, X_m$; $Y_1, \ldots, Y_n$ are mutually independent variables. The $X_i$'s have the density function $f(x)$ and the $Y_i$'s have the density function $f(x-\theta)$ where $f(\cdot)$ denotes the standard normal density.
Consider $z, z'$ such that $zRz'$. It is seen that
\[
\frac{P_\theta(z)}{P_\theta(z')} = \frac{\int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} \prod_{i=1}^{N} \exp \left[ \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \theta \sum_{i=1}^{N} z_i x_i \right] dx_i}{\int_{\infty}^{\infty} \cdots \int_{\infty}^{\infty} \prod_{i=1}^{N} \exp \left[ \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \theta \sum_{i=1}^{N} z_i' x_i \right] dx_i},
\]
where $N = m + n$. The equation (9.1) can be written as
\[
\frac{P_\theta(z)}{P_\theta(z')} = \frac{\int_{-\infty}^{\infty} e^{\theta y} g_1(y) dy}{\int_{-\infty}^{\infty} e^{\theta y} g_2(y) dy}.
\]
Then, $P_\theta(z)/P_\theta(z')$ is an increasing function of $\theta$ if and only if for all $\theta > \theta'$, the following inequality holds:
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ g_1(x)g_2(y) - g_2(x)g_1(y) \right] \exp(\theta x + \theta' y) \, dx \, dy \geq 0.
\]
The inequality (9.3) can be written as
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ g_1(x)^2 - g_2(x)g_1(y) \} \{ \exp(\theta x + \theta' y) - \exp(\theta y + \theta' x) \} \, dx \, dy \geq 0.
\]
As
\[
\exp(\theta x + \theta' y) - \exp(\theta y + \theta' x) > 0
\]
for $x > y$ and $\theta > \theta'$, (9.4) will hold if
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ g_1(x)g_2(y) - g_2(x)g_1(y) \} \geq 0
\]
for all $x$ and $y$. The inequality (9.5) will hold if the first part of the conjecture for the normal family is true.
To motivate the second part of the conjecture for the normal family, which of course will be true if the first part is true (see lemma 1 of the appendix), consider the behavior of \( P_\theta(z)/P_\theta(z') \) given by (9.2) for \( \theta \leq 0 \).

Write
\[
\int_{-\infty}^{\infty} e^{\theta y} g_1(y) \, dy = \left[ e^{\theta y} G_1(y) \right]_{-\infty}^{\infty} - \theta \int_{-\infty}^{\infty} e^{\theta y} G_1(y) \, dy
\]
and
\[
\int_{-\infty}^{\infty} e^{\theta y} g_2(y) \, dy = \left[ e^{\theta y} G_2(y) \right]_{-\infty}^{\infty} - \theta \int_{-\infty}^{\infty} e^{\theta y} G_2(y) \, dy.
\]

If it can be assumed that
\[
\left[ e^{\theta y} G_i(y) \right]_{-\infty}^{\infty} = 0, \quad i = 1,2
\]
for \( \theta \leq 0 \), then

\[
\frac{P_\theta(z)}{P_\theta(z')} = \frac{\int_{-\infty}^{\infty} e^{\theta y} G_1(y) \, dy}{\int_{-\infty}^{\infty} e^{\theta y} G_2(y) \, dy}.
\]

Now as above, it is seen that \( P_\theta(z)/P_\theta(z') \) is an increasing function of \( \theta \) for \( \theta \leq 0 \) if

\[
(9.6) \quad G_1(x)G_2(y) - G_1(y)G_2(x) \geq 0
\]
for all \( x > y \). The inequality (9.6) will hold if the second part of the conjecture for the normal family is true.
APPENDIX

Lemma 1. Consider two distribution functions $F(x)$ and $G(x)$ and the corresponding density functions $f(x)$ and $g(x)$. If $f(x)/g(x)$ is an increasing function of $x$, then the following results hold:

(a) $F(x)/G(x)$ is an increasing function of $x$,
(b) $(1 - F(x))/(1 - G(x))$ is an increasing function of $x$,
(c) $G(x)(1 - F(y)) \geq F(x)(1 - G(y))$ for all $x \leq y$.

Proof. (a) Consider:

$$
\frac{d}{dx} \left[ \frac{F(x)}{G(x)} \right] = \frac{f(x)G(x) - g(x)f(x)}{G^2(x)}
$$

$$
= \frac{1}{G^2(x)} \int_x^\infty [f(x)g(y) - g(x)f(y)] \, dy \geq 0,
$$

since the integrand is positive for all $y$. Therefore (a) is true.

(b) Consider:

$$
\frac{d}{dx} \left[ \frac{1-F(x)}{1-G(x)} \right] = \frac{g(x)(1-F(x)) - f(x)(1-G(x))}{(1-G(x))^2}
$$

$$
= \frac{1}{(1-G(x))^2} \int_x^\infty [f(y)g(x) - f(x)g(y)] \, dy \geq 0,
$$

since the integrand is positive for all $y$. Therefore (b) is true.

(c) Consider:

$$
G(x)(1-F(y)) - F(x)(1-G(y)) = \int_y^\infty \int_x^\infty [f(v)g(u) - f(u)g(v)] \, du \, dv
$$

$$
\geq 0 \text{ for all } x \leq y,
$$
as the integrand is positive for all \( u \) and \( v \) when \( x \leq y \). Therefore (c) is true.

**Corollary to lemma 1.** If the density function \( f(x, \theta) \) has increasing likelihood ratio, then for \( \theta_1 < \theta_2 \), the following results hold:

(a) \( F(x, \theta_2)/F(x, \theta_1) \) is an increasing function of \( x \),

(b) \((1 - F(x, \theta_2))/(1 - F(x, \theta_1))\) is an increasing function of \( x \),

(c) \( F(x, \theta_1)(1 - F(y, \theta_2)) \geq F(x, \theta_2)(1 - F(y, \theta_1)) \) for all \( x \leq y \).

The proof is immediate.

The following example shows that if \( F(x)/G(x) \) is an increasing function of \( x \), it does not imply that \( f(x)/g(x) \) is an increasing function of \( x \).

**Example.** Consider the density functions \( f(x) \) and \( g(x) \) given below:

<table>
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<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>( f(x) )</td>
<td>1/100</td>
<td>1/50</td>
<td>3/25</td>
<td>17/20</td>
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<tr>
<td>( f(x) )</td>
<td>1/10</td>
<td>1/20</td>
<td>59/100</td>
<td>13/50</td>
</tr>
<tr>
<td>( F(x)/G(x) )</td>
<td>1/10</td>
<td>3/15</td>
<td>15/74</td>
<td>1</td>
</tr>
<tr>
<td>( f(x)/g(x) )</td>
<td>1/10</td>
<td>2/5</td>
<td>12/59</td>
<td>85/26</td>
</tr>
</tbody>
</table>

Clearly \( F(x)/G(x) \) is increasing with \( x \) and \( f(x)/g(x) \) is not increasing with \( x \).

**Lemma 2.** The family of densities of the order statistic for a fixed sample size \( n \) from a population with an arbitrary density
function $f(x)$ has increasing likelihood ratio with the set of subscripts of the order statistics as the indexing set.

**Proof.** Let $g_i(x)$ denote the density function of the $i^{\text{th}}$ order statistic. Then

$$g_i(x) = \frac{n!}{(n-i)!(i-1)!} F^{i-1}(x)(1-F(x))^{n-i} f(x),$$

and

$$\frac{g_j(x)}{g_i(x)} = \text{const.} \left[ \frac{F(x)}{1-F(x)} \right]^{j-i}.$$

Since $F(x)/(1-F(x))$ is an increasing function of $x$, it follows that $g_j(x)/g_i(x)$ is an increasing function of $x$ for $j > i$.

**Acknowledgement.** The author is thankful to Professor I. R. Savage for his inspiring guidance all through this work. The author also wishes to acknowledge the help of Mr. Bradley Russell in checking the computations.
References.


Suppose \( X_1, \ldots, X_m; Y_1, \ldots, Y_n \) are mutually independent random variables. The \( X_i \)'s and the \( Y_i \)'s have the continuous distribution functions \( F(x, \theta_1) \) and \( F(x, \theta_2) \) respectively. Denote the probability of the rank order \( z \) by \( P_{\theta_1, \theta_2}(z) \).

The notation \( z \leq R \leq z' \) denotes the following relationship: \( z_i = z'_i \) for all \( i = 1, \ldots, N \) (\( = m+n \)) except \( j \) and \( k(j < k), z_j = z'_k = 0, z_k = z'_j = 1 \). If the interchange of components of \( z \) is only at the extreme ends, the relationship is denoted by \( z \leq R \leq z' \).

Whenever \( z \leq R \leq z' \), the monotonicity of the rank order likelihood ratio

\[
P_{\theta_1, \theta_2}(z)/P_{\theta_1, \theta_2}(z')
\]

is exhibited in \( \theta_1 \) and \( \theta_2 \) for (a) a Lehmann family, (b) a uniform family over \((0, \theta)\) and (c) one observation from one population and an arbitrary number of observations from the other population. In (c) only one population is assumed to belong to a family having \( m \cdot 1.R \). Whenever \( z \leq R \leq z' \), the monotonicity is exhibited when the families satisfy a special condition. This condition is satisfied by a normal and a logistic family. With \( m=n=2 \), the monotonicity is exhibited whenever \( z \leq R \leq z' \) for a normal logistic and a double exponential family. Two conjectures are given regarding the monotonicity of the rank order likelihood ratio.
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<th>Link B</th>
<th>Link C</th>
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