On Identifiability In The
Autopsy Model of Reliability Theory

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On Identifiability In The Autopsy Model of Reliability Theory.

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Abstract

A coherent system is observed until it fails. At the instant of system failure, the set of failed components and the failure time of the system are noted. The failure times of the components are not known. We consider whether the component life distributions can be determined from the distributions of the observed data.

Meilijson (1981) gave a condition on the structure of the system that was sufficient for the identifiability of the component distributions, under the assumption that the component life distributions are continuous and have common essential extrema. Nowik (1990) gave necessary and sufficient conditions for identifiability under the more restrictive condition that the component distributions have atoms at their common essential infimum and are mutually absolutely continuous. We give a necessary condition for identifiability, which we show to be equivalent to Nowik's condition, under the assumption that the distributions are continuous and strictly increasing. We derive a sufficient condition for identifiability, more general than Meilijson's, for the case in which the component distributions are assumed to be analytic. We also show that our necessary condition for identifiability is both necessary and sufficient when the component life distributions are assumed to belong to certain parametric families.

Key words and phrases: Coherent system, autopsy model, fatal set, incidence matrix, identifiable.

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1 Introduction and Summary

Consider a coherent system $S$ consisting of $m$ components which act independently. Assume that each of the components of $S$ occupies one of two states, functioning or failed, and that $S$ itself occupies one of these two states. We consider the statistical model in which each element of a sample of $n$ replicates of $S$ is observed until it fails. The observed data consist of the set of components that are dead at the instant of system failure and the failure time of the system. The failure times of the dead components are not known. These are the so-called autopsy statistics of the machine. This model is usually called the autopsy model.

Two important statistical problems arise in considering this model — the problems of estimating the life distributions of the components and that of the system itself. The system life distribution may always be estimated using the system failure times (by the empirical distribution function, for example). However, this procedure ignores the information contained in the data on component failures. Note that, if the structure of $S$ is known, the system life distribution may be calculated from a knowledge of the component life distributions. Therefore, one strategy would be to estimate the component life distributions and then to use the known structure of $S$ to estimate the system life distribution. This strategy has been implemented by Doss, Freitag and Proschan (1989) in the case of a related statistical model in which the failure times of the dead components are also known. However, they pointed out that the autopsy model was not amenable to their approach. But the autopsy model provides so little information about component failures that it is valid to ask why an alternative approach, such as testing components separately, cannot be used. The autopsy model is important in those situations in which this cannot be done, either because it is difficult to reproduce the conditions prevailing in the functioning system when components are tested separately or, as in the case of living things (biological systems), it is impossible to do so. Alternatively, the machine and its components may be so extremely expensive that few copies are available and large scale destructive testing of individual components is not feasible. In this case it is important to get every little bit of information about the behaviour of components when they are parts of the system.

Watelet (1990) proposed estimators of the component life distributions that are applicable to some, but not all, coherent systems. However, before attempting to estimate the component distributions, it is worthwhile to examine the more fundamental question of whether, and in what circumstances, the component distributions can be estimated. A closely related question is whether the component distributions can be recovered from a knowledge of the true distributions of the autopsy statistics. If this can be done, we say that the component distributions are identifiable and call $S$ an identifiable system.

It is easy to see that the question of identifiability is not a frivolous one and that not every system is identifiable. Consider a parallel system of order two. Let the component distributions be $F_1$ and $F_2$. Both components will always be dead when the system is observed. Thus, under the autopsy model, one in effect observes only the system failure time, which has distribution $F_1 F_2$. Hence it is not possible to determine $F_1$ and $F_2$ from a knowledge of the distribution of the observed data.
Meilijson (1981) gave a condition on the structure of the system for identifiability of the component life distributions, under the assumption that the distributions are independent, continuous and have the same essential extrema.

Nowik (1990) gave necessary and sufficient conditions for identifiability, but made the more restrictive assumption that the component distributions were absolutely continuous and had atoms at their common infimum.

The problem of characterizing the set of identifiable systems is a very difficult one. The present paper does not completely solve that problem. Rather, our results contribute towards such a complete solution by focusing on a class of functions which are, in at least one sense, on the opposite side of the spectrum from those considered by Nowik (1990). Instead of having atoms at their common essential infimum, the functions we look at — the analytic functions — are as smooth as can be. A large proportion of the distribution functions commonly used to model data in reliability theory are of this type. We derive a condition on the structure of the system that is sufficient for identifiability under the assumption that the distributions are analytic. This condition is shown to be more general than that of Meilijson, though it pertains to a smaller class of functions. We also obtain a necessary and sufficient condition for identifiability for the case in which it is assumed that the component distributions belong to certain parametric families, including the exponential distribution, the half-normal distribution, and the gamma and Weibull distributions with integer shape parameters. This condition is shown to be equivalent to the condition of Nowik (1990).

Before describing in more detail the results of Meilijson (1981), Nowik (1990) and of this article, we need to give some basic definitions and notation. The notion of a coherent system and other fundamental ideas in reliability can be found in Barlow and Proschan (1981). The idea of a fatal set is due to Meilijson (1981).

Throughout this article, the symbol $S$ will denote a coherent system of $m$ components. Where there is no danger of confusion, the set of components of $S$ will also be denoted by the symbol $S$. The $j^{th}$ component of $S$ will be denoted by the symbol $s_j$ or, when there is no danger of confusion, simply by the integer $j$. The life distribution of the system and of the $j^{th}$ component will be denoted by $F$ and $F_j$ respectively.

**Definition 1** A set of components $E$ forms a cut set of $S$ if failure of every element of $E$ leads to failure of the system.

A set $E$ is a minimal cut set of $S$ if $E$ is a cut set and no proper subset of $E$ is a cut set.

**Example 1** Consider the following system:
The sets \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\} and, of course, \{1, 2, 3\} are all cut sets. The minimal cut sets are \{1\} and \{2, 3\}. Note that although the set \{1, 2, 3\} is a cut set, it can never be observed. This is because \( S \) will fail as soon as either \{1\} or \{2, 3\} fails. This last observation leads to the notion of a fatal set.

**Definition 2** Let \( I \) be the random set of components that are dead at the instant at which \( S \) fails. A set \( D \) is said to be fatal if \( P(I = D) > 0 \).

A set \( M \) is said to be a minimal fatal set if \( M \) is fatal and no proper subset of \( M \) is fatal.

Evidently every fatal set is a cut set but, as the example shows, there are cut sets which can never be observed and are thus not fatal sets. However, a set is a minimal fatal set if and only if it is a minimal cut set.

It is clear that if the fatal set \{1\} is observed then the component that was the last to fail, and thus caused the failure of the system, is known. But if the fatal set \{1, 2\} is observed this is also true. The system fails as soon as component 1 fails so, if \{1, 2\} is observed, 1 must have been the last to fail. On the other hand, if \{2, 3\} is observed either 2 or 3 may have caused system failure, so the last component to fail cannot be known. These considerations lead to the following definition.

**Definition 3** Let \( D \) be a fatal set of \( S \). The critical set of \( D \) is the set of those elements of \( D \) which may have failed at the instant \( S \) failed and thus may have caused failure of the system. The critical set of \( D \) will be denoted by \( C_D \) or, where there is no danger of confusion, simply by \( C \).

**Definition 4** Let \( E \) be a subset of the components of \( S \). The incidence vector, \( v \), of \( E \) is the vector of zeros and ones such that

\[
v_j = \begin{cases} 
1 & \text{if } s_j \in E, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \mathcal{M} = \{M_1, \ldots, M_q\} \), \( \mathcal{D} = \{D_1, \ldots, D_l\} \) and \( \mathcal{C} = \{C_1, \ldots, C_l\} \) be the collections of minimal, fatal and critical sets respectively.

**Definition 5** The fatal incidence matrix of \( S \) is the \( l \times m \) matrix \( D \) defined by

\[
D_{ij} = \begin{cases} 
1 & \text{if } s_j \in D_i, \\
0 & \text{otherwise}.
\end{cases}
\]

The minimal and critical incidence matrices of \( S \) are defined similarly. The word ‘incidence’ will usually be omitted.

Consider once again Example 1. The critical sets corresponding to the fatal sets \{1\}, \{1, 2\}, \{1, 3\} and \{2, 3\} are \{1\}, \{1\}, \{1\} and \{2, 3\} respectively. Thus the minimal, fatal and critical matrices are

\[
\begin{align*}
M &= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}, \\
D &= \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}, \\
C &= \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}.
\end{align*}
\]
We will often speak interchangeably of a coherent system $S$, its minimal matrix $M$, and its structure function $\phi$.

**Definition 6** The series and parallel systems of two components, $s_1$ and $s_2$, are denoted by $s_1 \sqcap s_2$ and $s_1 \sqcup s_2$ respectively.

**Definition 7** A proper subset of components of $S$ which forms a coherent system is called a module of $S$.

Meilijson (1981) showed, under the assumption that the distributions are independent, continuous and have the same essential extrema, that $S$ is identifiable if the rank of its minimal incidence matrix is $m$. This is an important result, but the condition is by no means necessary. Meilijson himself provided a counterexample. Indeed, two of the five possible coherent systems of order three neither satisfy this condition nor are parallel systems (recall that parallel systems are not identifiable), so Meilijson’s result leaves their status undecided. Thus Meilijson’s result goes only a short way towards achieving the goal of characterizing those systems which are identifiable, but it has the great virtue of assuming only continuity.

Nowik (1990) assumed that the component life distributions were independent, mutually absolutely continuous, had the same extrema and each distribution possessed an atom at their common infimum. His results are best understood by looking at a simple system. Consider Example 1. Let $T$ be the system failure time and $D$ be the random fatal set that is observed. Let $G_i(t) = P(D = D_i, T \leq t)$. Suppose that the component distributions are mutually absolutely continuous. Let the density of $F_j$ be $f_j$ and that of $G_i$ be $g_i$. Recall that the fatal sets $\{1\}$ and $\{1, 2\}$ have the same critical set $\{1\}$. Then, from the fatal set $\{1\}$ we get

$$g_1(t) = f_1(t)(1 - F_2(t))(1 - F_3(t)), \quad (1)$$

and from $\{1, 2\}$ we get

$$g_2(t) = f_1(t)F_2(t)(1 - F_3(t)). \quad (2)$$

Hence, by dividing (2) by (1) one gets

$$\frac{g_2(t)}{g_1(t)} = \frac{F_2(t)}{1 - F_2(t)}.$$ 

Thus, $F_2$ can be determined. Similarly, $F_3$ can be found. Notice that the method works because the fatal sets $\{1\}$ and $\{1, 2\}$ have the same critical set $\{1\}$ and differ by only the single component $\{2\}$. Nowik (1990) defines sets of components $E$ and $J$ and a matrix $C_J$ as follows.

**Definition 8** Let $s$ be a component with the property that there exists a fatal set $D$ such that:

(i) $s \in D$ and $D_1 = D - s$ is also a fatal set,

(ii) $D$ and $D_1$ have the same critical sets.
Let $E$ be the set of components of $S$ with this property. Let $J$ be the set $S - E$. The matrix $C_J$ is defined to be the submatrix of $C$ formed by the columns corresponding to elements of $J$.

The distributions of components in $E$ can be found by the technique used in the example. Thus $S$ is identifiable if the components in $J$ can be determined. Nowik (1990) showed that the following two conditions are equivalent:

A1 The matrix $C_J$ has full rank.

A2 There do not exist components, $s_i$ and $s_j$, and a disjoint module $S_1$ such that $S$ has the form

$$S = s_1 \coprod s_2 \coprod S_1.$$  

This form can be represented diagrammatically as follows:

```
    S1
   /   \
S2   S1
```

Note that if $S = s_1 \coprod s_2 \coprod S_1$ then the life distributions of components 1 and 2 cannot be determined. Nowik uses methods essentially similar to Meilijson (1981), together with the assumption that the distributions have atoms at their common infimum, to show that, if $C_J$ is of full rank, then the distributions of the components in $J$ are also identifiable. Since it is clear that Condition A2 is a necessary condition for identifiability, (if A2 does not hold then there would be two components in parallel with each other and the rest of the system), it follows that A2 is both necessary and sufficient for identifiability.

**Throughout this article we assume that:**

C1 The components of $S$ act independently,

C2 The component life distributions are continuous with support over $[0, \infty)$.

The major result of Section 2 (Theorem 3) is that, if C1 and C2 hold, the following condition is necessary for identifiability.

A3 The rank of the fatal incidence matrix $D$ is $m$.

This is proved by showing that Conditions A2 and A3 are equivalent. Since A2 is a necessary condition for identifiability, it follows that A3 must also be necessary for identifiability.

**In Section 3 we assume, in addition, that**

C3 The component distributions $F_j$ possess derivatives of all orders in $(0, \infty)$, have right derivatives of all orders at 0 and $F_j(t)$ can be expanded in a power series about 0 for all $t$ in $[0, \infty)$ and for $j = 1, \ldots, m$.  

5
Assuming C3, we give (in Theorem 4) a sufficient condition for identifiability. Let \( r_j \) be the power of the first non-zero term in the power series expansion of \( F_j(t) \) about 0. Let \( R \) be the diagonal matrix of order \( l \) (recall that \( l \) is the number of fatal sets or the column length of \( D \)) whose \( i^{th} \) diagonal element is

\[
R_{ii} = \sum_{j=1}^{m} r_j c_{ij}.
\]

We show that \( S \) is identifiable if \( RD + pC \) has rank \( m \) for all non-negative integers \( p \) and for all possible choices of the positive integers \( r_j \). This condition may appear difficult to check since, although \( D \) and \( C \) are known from the structure of \( S \), \( R \) is not and the condition has to be checked for all possible choices of \( R \). Nonetheless, we will show that this condition is easy to verify for large classes of systems, is more general than Meilijson's and that his condition can easily be deduced from it (assuming condition C3).

At the end of Section 3 we point out some strong similarities between these conditions and those given by Campbell (1980, Chapter 3) for the system of singular differential equations

\[
A\dot{x} + Bx = f,
\]

where \( A \) and \( B \) are matrices, \( x \) and \( f \) are vector valued functions and \( f \) is known, to have a unique solution.

In Theorems 6 and 7 we show that A3 is necessary and sufficient if the component life distributions belong to certain parametric families, including the exponential, half-normal and the gamma and Weibull distributions with integer shape parameters.

Intensive computational studies have been made of some simple systems in which it is not true that \( RD + pC \) has rank \( m \) for all \( p \). The results of these studies lend strong support to the view that Condition A3 is a sufficient condition for identifiability in the class of analytic functions. We conjecture that

**Conjecture 1** If C1 and C2 hold then a necessary and sufficient condition for \( S \) to be identifiable is that the fatal incidence matrix \( D \) be of full column rank \( m \).

## 2 A Necessary Condition for Identifiability

This section begins by giving two more examples of coherent systems which serve to illuminate basic definitions and concepts. It is then shown that the conditions A2 and A3 are equivalent.

First note that the fatal sets of \( S \) and their corresponding critical sets can be derived from the collection of minimal fatal sets or, equivalently, from the minimal matrix. In fact:

(i) \( D \) is a fatal set of \( S \) if and only if the intersection of all the minimal fatal sets of \( S \) contained in \( D \) is non-empty.

(ii) The critical set corresponding to the fatal set \( D \) is precisely the intersection of all the minimal fatal sets contained in \( D \).
These two properties will be used in the proof of Proposition 2, and are among those illustrated in the following two examples.

**Example 2** Consider the system

![Diagram](image1)

The minimal fatal sets are \{1, 2\} and \{1, 3\}. There is another fatal set. This is \{1, 2, 3\}. This can be observed only if components 2 and 3 fail before 1 does. Note that this set contains both minimal fatal sets, whose intersection is non-empty and is just \{1\}. Hence the critical set corresponding to \{1, 2, 3\} is just \{1\}. Thus it is sometimes possible to determine which element caused system failure, and therefore the failure time of that element, from a knowledge of the fatal set. In most cases this cannot be done. For example, if \{1, 2\} is observed either element may have been the last to fail. Hence the critical set corresponding to this fatal set is also \{1, 2\}. In the case of minimal fatal sets, the critical set always coincides with the fatal set. Thus the minimal, fatal and critical incidence matrices of this system are respectively

\[
M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ \end{pmatrix}.
\]

Components 2 and 3 form a series system of order two. The set \{2, 3\} is therefore a module, \(S_1\), of \(S\) in parallel with component 1. Consequently, \(S\) may be written as

\[
S = s_1 \prod S_1.
\]

**Example 3** Consider the system with minimal matrix

\[
M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.
\]

(3)

This system is given diagrammatically as

![Diagram](image2)
but we prefer to view it entirely through the matrix (3). The set \{1, 2, 3\} contains the minimal fatal set \{1, 2\}, so it is clearly a cut set. It is also a fatal set since the only minimal fatal sets contained in \{1, 2, 3\} are \{1, 2\} and \{2, 3\}, whose intersection is non-empty and is just \{2\}. Thus \{1, 1, 1, 0\} is a row of \(D\) and \{0, 1, 0, 0\} is a corresponding row of \(C\). Other rows of \(D\) and \(C\) can be similarly obtained.

The fatal and critical matrices of this system are

\[
D = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{pmatrix}
\quad \text{and} \quad
C = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]

Note that 1 belongs to the fatal set \(D_7 = \{1, 3, 4\}\) and that \(D_3 = D_7 - \{1\}\) is also a fatal set. In addition, the critical sets corresponding to \(D_7\) and \(D_3\) are the same, namely \(\{3, 4\}\). Thus 1 satisfies the condition of Definition 8. The same is true of component 4. Hence \(E = \{1, 4\}\). Nowik (1990)'s matrix \(C_J\) is the matrix consisting of those columns of \(C\) corresponding to components not in \(E\). Thus \(J = \{2, 3\}\) and

\[
C_J = \begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

We now show that conditions A2 and A3 are equivalent.

The following proposition is an immediate consequence of the fact that if a component \(s\) is in parallel with the rest of the system, then the system \(S\) fails only if \(s\) fails.

**Proposition 1** Let \(S\) and \(S_1\) be two coherent systems of orders \(m\) and \(m - 1\), minimal matrices \(M\) and \(M_1\) and fatal matrices \(D\) and \(D_1\) respectively. Let the components of \(S\) be \(s_1, \ldots, s_m\) and of \(S_1\) be \(s_2, \ldots, s_m\).

Then if \(S = s_1 \uplus S_1\)

(i) \(M\) has only ones in its first column.

(ii) Columns 2 to \(m\) of \(M\) are identical to the corresponding columns of \(M_1\).

(iii) If \(E\) is a superset of a fatal set of \(S\) then \(E\) is a fatal set of \(S\).
The condition \( S = s_1 \cup S_1 \) is given diagrammatically by

\[ \text{Diagram of } S = s_1 \cup S_1 \]

**Proposition 2** If two columns of the fatal incidence matrix of a coherent system \( S \) are identical then they must both be columns of ones.

**Proof** Let \( S \) be a coherent system of \( m \) components with fatal matrix \( D \). Suppose that the \( i^{th} \) and \( n^{th} \) columns of \( D \) are identical but not columns of ones. Then there exists a fatal set and thus a minimal fatal set, \( M = \{s_n, \ldots, s_n\} \), which contains neither \( s_i \) nor \( s_n \). Moreover, no fatal set contains \( s_i \) but not \( s_n \). We show, in contradiction to this, that \( D = M \cup \{s_i\} \) is a fatal set. The set \( D \) is certainly a cut set. No minimal fatal set which is a subset of \( D \) can contain \( s_i \), for then there would be a fatal set containing \( s_i \) but not \( s_n \). Hence, every minimal fatal set in \( D \) must be a subset of \( M \). Since \( M \) is a minimal fatal set, this means that \( M \) must be the only such subset of \( D \). Hence the intersection of all the minimal fatal sets of \( D \) is non-empty. Thus \( D \) is a fatal set.

**Theorem 1** Suppose that a coherent system \( S \) has \( m \) components. Then the rank of the fatal matrix \( D \) of \( S \) is \( m \) if and only if no two columns are identical.

**Proof** Necessity is obvious. We prove sufficiency by induction.

(1) Consider \( m = 2 \). There are only two coherent systems, the series and parallel systems of order two. The fatal incidence matrices of these two systems coincide with their minimal matrices and are respectively

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

and

\((1,1)\).

Thus the theorem is true for \( m = 2 \).

(2) Suppose that the statement is true for all \( m \leq n \). Let \( S \) be a coherent system of \( n + 1 \) components such that its fatal incidence matrix has rank less than \( (n + 1) \). Suppose that no two columns of \( D \) are identical. There are two cases to consider.

(a) Suppose that \( D \) has a column of ones. We may assume this to be the first column. Then the system has the form

\[ S = s_1 \cup S_1, \]

where \( S_1 \) has \( n \) components \( \{s_2, \ldots, s_{n+1}\} \). We first show that column one must be a linear combination of the other columns and then show that this leads to a contradiction.
Let the fatal matrix of $S_1$ be $D_1$ and the submatrix of $D$ composed of columns 2 to $n + 1$ be $D_{-1}$. Note that every row of $D_1$ is also a row of $D_{-1}$. If $D_1$ has two identical columns then, by Proposition 2, they must both be columns of ones. Thus $D$ would also have two identical columns (in fact, three identical columns of ones). Hence no two columns of $D_1$ are identical. Thus, by assumption, $D_1$ has rank $n$ and the columns of $D_1$ are linearly independent. Consequently, the columns of $D_{-1}$ are linearly independent (recall that every row of $D_1$ is also a row of $D_{-1}$). Moreover, since the rank of $D$ is less than $(n + 1)$, the columns of $D$ are linearly dependent. Thus the first column of $D$ is a linear combination of the $n$ columns of $D_{-1}$,

$$1 = \sum_{j=2}^{n+1} a_j v_j,$$  \hspace{1cm} (4)\n
where $v_j$ is the $j^{th}$ column of $D$. Note also that $D_{-1}$ cannot contain a column of ones, since then $D$ would have two identical columns. Hence every column of $D_{-1}$ contains a zero. Suppose that the $r^{th}$ column of $D$ contains a zero in the $i^{th}$ row i.e. $D_{ir} = 0$. Then there exists a fatal set of $S$,

$$E = \{s_1, s_{n_1}, \ldots, s_{n_r}\},$$

not including $s_r$. Consequently, by (4),

$$1 = \sum_{j \in E} a_j = a_{n_1} + \cdots + a_{n_r}$$  \hspace{1cm} (5)\n
But $E' = E \cup \{s_r\}$ is a fatal set by Proposition 1. Thus, because of (4), we have

$$1 = \sum_{j \in E'} a_j = a_{n_1} + \cdots + a_{n_r} + a_r.$$  \hspace{1cm} (6)\n
Hence, using (5) and (6), we get that $a_r = 0$. But then, by the same reasoning applied to every column of $D_{-1}$, we have that

$$a_j = 0; \quad j = 2, \ldots, n + 1.$$\n
Thus no column of $D$ is a column of ones.

(b) Suppose that no column of $D$ is a column of ones. Then the first column contains a zero in some row. Thus there exists a minimal fatal set $M$ which does not contain $s_1$. Let

$$M = \{s_{n_1}, \ldots, s_{n_r}\}.$$

Since the rank of $D$ is less than $(n + 1)$, assume, without loss of generality, that

$$v_1 = \sum_{j=2}^{n+1} a_j v_j,$$  \hspace{1cm} (7)\n
where $v_j$ is the $j^{th}$ column of $D$. 

10
Hence,

$$0 = \sum_{s_j \in M} a_j = a_{n_1} + \cdots + a_{n_r}. \quad (8)$$

Let $t_{n_1}, \ldots, t_{n_r}$ and $t_1$ be the failure times of $s_{n_1}, \ldots, s_{n_r}$ and $s_1$ respectively. Suppose that

$$t_{n_1} < t_{n_2} < \cdots < t_{n_r} < t_1,$$

with the failure times of the other components being greater than $t_{n_r}$. Thus $s_{n_r}$ causes system failure. But there is a non-zero probability that the even:

$$t_{n_1} < t_{n_2} < \cdots < t_{n_r-1} < t_1 < t_{n_r}$$

with the other failure times being greater than $t_{n_r}$, will occur. This set

$$\{s_{n_1}, \ldots, s_{n_r-1}, s_1, s_{n_r}\}$$

also causes system failure. If it is a fatal set then, by (7), we have

$$1 = a_{n_1} + \cdots + a_{n_r}. \quad (9)$$

But this contradicts (8). Moreover, the set $\{s_{n_1}, \ldots, s_{n_r-1}\}$ cannot be a fatal set, since it is a proper subset of a minimal fatal set. Hence,

$$\{s_{n_1}, \ldots, s_{n_r-1}, s_1\}$$

must be a fatal set. Thus, because of (7),

$$1 = a_{n_1} + \cdots + a_{n_r-1}. \quad (10)$$

But (8) and (10) imply that $a_{n_r} = -1$. However, since $M$ is a minimal fatal set, the same argument may be applied to any component of $M$. Thus

$$a_{n_j} = -1 \text{ for all } s_j \in M.$$

Hence at least two columns of $D$ must be identical.

**Theorem 2** The rank of the fatal incidence matrix $D$ of a coherent system of $m$ components $S$ is less than $m$ if and only if $S$ is of the form

$$S = s_i \coprod s_j \coprod S_1,$$

where $S_1$ is a module of $S$ not containing the components $s_i$ and $s_j$.

**Proof** Sufficiency of the condition is obvious. Suppose that the rank of the fatal matrix $D$ is less than $m$. By Theorem 1, $D$ must contain at least two identical columns. By Proposition 2, these must be columns of ones. The theorem follows.

The following theorem is an immediate consequence of Theorems 1 and 2.

**Theorem 3** A necessary condition for a coherent system of $m$ components $S$ to be identifiable is that the rank of the fatal incidence matrix of $S$ be $m$.  

11
Meilijson (1981) proved that \( \text{rank}(M) = m \) is a sufficient condition for identifiability and conjectured that the condition \( \text{rank}(C) = m \) is also sufficient for identifiability. These conditions are related to each other as follows.

**Proposition 3** Suppose that a coherent system of \( m \) components has minimum, critical and fatal matrices \( M, C \) and \( D \) respectively. Then

\[
\text{rank}(M) \leq \text{rank}(C) \leq \text{rank}(D) \leq m.
\]

**Proof** Every row of \( M \) is also a row of \( C \) and \( D \). Hence \( \text{rank}(M) \leq \text{rank}(C) \) and \( \text{rank}(M) \leq \text{rank}(D) \). Thus we need only prove that \( \text{rank}(C) \leq \text{rank}(D) \). There are two cases.

(a) \( \text{rank}(D) = m \),

(b) \( \text{rank}(D) < m \).

The first case is trivial. In the second case, either \( S \) is a parallel system, in which case

\[
\text{rank}(C) = \text{rank}(D) = 1,
\]

or else

\[
S = s_1 \prod s_2 \prod \cdots \prod s_r \prod S_1,
\]

where \( r \geq 2 \) and \( S_1 \) has no columns of ones. Define the module \( S_2 \) by

\[
S_2 = S_1 \prod s_r.
\]

Let the fatal and critical matrices of \( S_2 \) be \( D_2 \) and \( C_2 \) respectively. Note that \( S_2 \) does not have two columns of ones and is therefore of full rank. Thus, since the columns 1 through \( r \) of \( D \) are identical columns of ones,

\[
\text{rank}(C) = \text{rank}(C_2) \leq \text{rank}(D_2) = \text{rank}(D).
\]

### 3 Sufficiency when Distributions are Analytic

Theorem 1 was proved using only \( C_1 \) and \( C_2 \). We now impose \( C_3 \) as well. Let the fatal and critical sets corresponding to the \( i \)-th rows of \( D \) and \( C \) be \( D_i \) and \( C_i \) respectively. Let \( D \) be the (random) fatal set observed at the time of system failure \( T \). Define the subdistribution function

\[
G_i(t) = P(T \leq t, D = D_i).
\]

The set of components may be partitioned into three subsets:

(i) The set of components, \( D_i' \), that are alive at the time of system failure;

(ii) The set of components, \( D_i - C_i \) that definitely died before time \( t \);

(iii) The set of components, \( C_i \), that may have caused system failure.

Thus

\[
G_i(t) = \int_0^t \prod_{s_j \in D_i - C_i} F_j s_j \prod_{s_j \in C_i} (1 - F_j) d\left( \prod_{s_j \in C_i} F_j \right)
\]

\[
= \int_0^t \prod_{s_j \in D_i} F_j^{s_j} (1 - F_j)^{1 - \delta_{ij}} d\left( \prod_{s_j \in C_i} F_j^{\delta_{ij}} \right),
\]

(11)
where $\delta_{ij}$ and $c_{ij}$ are indicator functions for membership of the $j$th component in the sets $D_i$ and $C_i$ respectively.

Suppose that each $F_j$ is analytic in $[0, \infty)$ with a power series expansion given by

$$F_j(t) = a_j t^{r_j} \left(1 + \sum_{p=1}^{\infty} a_{jp} t^p\right) \quad \text{for} \quad j = 1, \ldots, m.$$  

Since $F_j(0) = 0$, it follows that $r_j \geq 1$. Let the density of $G_i(t)$ be $g_i(t)$, where

$$g_i(t) = b_i t^{d_i} \left(1 + \sum_{p=1}^{\infty} b_{ip} t^p\right) \quad \text{for} \quad i = 1, \ldots, l. \quad (12)$$

From (11) we get

$$g_i(t) = \prod_{j=1}^{m} F_j(t)^{(\delta_{ij} - c_{ij})} \prod_{j=1}^{m} \left(1 - F_j(t)\right)^{(1 - \delta_{ij})} \frac{d}{dt} \left(\prod_{j=1}^{m} F_j^{c_{ij}}(t)\right) \quad (13)$$

$$= \prod_{j=1}^{m} \left(a_j t^{r_j} \left(1 + \sum_{p=1}^{\infty} a_{jp} t^p\right)^{(\delta_{ij} - c_{ij})}\right) \prod_{j=1}^{m} \left(1 - a_j t^{r_j} \left(1 + \sum_{p=1}^{\infty} a_{jp} t^p\right)^{1 - \delta_{ij}}\right) \sum_{j=1}^{m} \left(c_{ij} f_j(t) \prod_{l \neq j} F_j^{c_{il}}(t)\right), \quad (14)$$

where $f_j$ is the density of $F_j$ and has a power series representation

$$f_j(t) = a_j \left(r_j t^{(r_j-1)} + \sum_{p=1}^{\infty} a_{jp} (r_j + p) t^{(r_j+p-1)}\right).$$

Equate the coefficients of $t^n$ in (12) and (14) for all nonnegative integers $n$.

The power of the first non-zero term of $g_i(t)$ is

$$d_i = \sum_{j=1}^{m} r_j (\delta_{ij} - c_{ij}) + \sum_{j=1}^{m} c_{ij} r_j - 1.$$ 

The second term, $\sum_{j=1}^{m} c_{ij} r_j - 1$, is obtained from powers of components in the critical set. Each component contributes $c_{il} r_l$ except the density term, which contributes one less. Thus

$$d_i = \sum_{j=1}^{m} r_j \delta_{ij} - 1 \quad \text{for} \quad j = 1, \ldots, m.$$ 

This leads to the matrix equation

$$d + 1 = Dr,$$

where $r$ is the vector of $r_j$'s.

Hence, if the rank of $D$ is $m$,

$$r = (D^T D)^{-1} D^T (d + 1), \quad (15)$$
and the first powers of the $F_j$ can be found.

Again, since $\delta_{ij} a_{j0} = \delta_{ij} a^{(\delta_{ij} - c_{ij})}_{j0}$, the coefficient of the first non-zero term of $g_1(t)$ is

$$b_{i0} = \prod_{j=1}^{m} a^{(\delta_{ij} - c_{ij})}_{j0} \prod_{j=1}^{m} a^{c_{ij}}_{j0} \sum_{j=1}^{m} c_{ij} r_j.$$ 

Thus

$$\prod_{j=1}^{m} a^{\delta_{ij}}_{j0} = \frac{b_{i0}}{(\sum_{j=1}^{m} c_{ij} r_j)}.$$ 

By taking logarithms of both sides we get the equation

$$D \log a_0 = \log c,$$

(16)

where $a_0^T = (a_{10}, a_{20}, \ldots, a_{m0})$ and $c^T = (b_{10}/(\sum_{j=1}^{m} c_{ij} r_j), \ldots, b_{00}/(\sum_{j=1}^{m} c_{ij} r_j))$. Thus, if $\text{rank}(D) = m$, the values of the $a_{j0}$'s can be determined.

In general, the coefficients of the terms in $t$ of order $(\sum_{j=1}^{m} r_j \delta_{ij} + p - 1)$, or the $(p + 1)^{th}$ term, are functions of $a_0, \ldots, a_p$, where, as before, $a^T_r = (a_{1r}, a_{2r}, \ldots, a_{mr})$. If the first $p$ vectors of coefficients are known, then the coefficient of the $(p + 1)^{th}$ term is a linear combination of $a_{1p}, a_{2p}, \ldots, a_{mp}$ plus a known function of the coefficients $a_{0}, \ldots, a_{p-1}$. From (14) we get

$$g_1(t) = \left(\prod_{j=1}^{m} a^{\delta_{ij}}_{j0}\right) t^{(\sum_{j=1}^{m} r_j \delta_{ij} - 1)} \prod_{j=1}^{m} (1 + \sum_{p=1}^{\infty} a_{jp} t^p)^{(\delta_{ij} - c_{ij})}$$

$$\prod_{j=1}^{m} \left(1 - a_{j0} t^{r_j} (1 + \sum_{p=1}^{\infty} a_{jp} t^p)^{(1 - \delta_{ij})}\right)$$

$$\left\{ \left(\sum_{j=1}^{m} r_j c_{ij}\right) \prod_{j=1}^{m} (1 + \sum_{p=1}^{\infty} a_{jp} t^p)^{c_{ij}} + t \sum_{j=1}^{m} \left\{ c_{ij} \left(\sum_{p=1}^{\infty} p a_{jp} t^{p-1}\right)^{c_{ij}} \prod_{j' \neq j}^{\infty} (1 + \sum_{p=1}^{\infty} a_{jp} t^p)^{c_{ij'}} \right\} \right\}.$$ 

The coefficient of $a_{jp}$ consists of a contribution of $(\sum_{j=1}^{m} r_j c_{ij})(\delta_{ij} - c_{ij})$ from the components in $D_i - C_i$ and nothing from the components in $S - D_i$. The components in $C_i$ contribute $c_{ij} (\sum_{j=1}^{m} r_j c_{ij})$ from the left side of the expression and $pc_{ij}$ from the right side of the expression. Hence the term involving $a_{jp}$ is

$$(\sum_{j=1}^{m} r_j c_{ij})(\delta_{ij} - c_{ij}) + c_{ij} \sum_{j=1}^{m} r_j c_{ij} + pc_{ij} = (\sum_{j=1}^{m} r_j c_{ij}) \delta_{ij} + pc_{ij}.$$ 

Let $R$ be the diagonal matrix of order $l$ given by

$$R(i, i) = \sum_{j=1}^{m} c_{ij} r_j \quad \text{for} \quad i = 1, \ldots, l.$$ 

Then the coefficients of the terms in $t$ of order $\sum_{j=1}^{m} r_j \delta_{ij} + p - 1$ are of the form

$$RD + pC + h(a_0, a_1, \ldots, a_{p-1}) \quad \text{for} \quad p = 1, 2, 3, \ldots,$$

where $h$ is some known function.

Thus, if $\text{rank}(RD + pC) = m$, the coefficients $a_p$ can be determined.
Theorem 4 Let $S$ be a coherent system of $m$ components satisfying Conditions C1, C2 and C3. Suppose that the $j$th component distribution $F_j$ has a power series expansion about 0 given by

$$F_j(t) = \sum_{p=0}^{\infty} a_{j,p} t^{r_j + p} \quad \text{for} \quad j = 1, \ldots, m.$$  

Then, if $RD + pC$ has rank $m$ for all non-negative integers $p$ and for all possible sets of positive integers $r_j$, the system is identifiable.

The condition rank$(D) = m$ is not sufficient to ensure rank$(RD + pC) = m$ for all non-negative integers $p$.

Example 4 Consider the familiar Example 2.

In this system,

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$  

Suppose that $r_1 = r_2 = r_3 = 1$, as would be true if the distributions were all exponential with mean 1. Then $R$ is the diagonal matrix

$$R = \begin{pmatrix} \sum_{j=1}^{3} c_{1j} r_j & 0 & 0 \\ 0 & \sum_{j=1}^{3} c_{2j} r_j & 0 \\ 0 & 0 & \sum_{j=1}^{3} c_{3j} r_j \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$RD + pC = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + p \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + p & 2 + p & 0 \\ 2 + p & 0 & 2 + p \\ 1 + p & 1 & 0 \end{pmatrix}. $$
Hence, the determinant

\[
\begin{vmatrix}
2 + p & 2 + p & 0 \\
2 + p & 0 & 2 + p \\
1 + p & 1 & 1
\end{vmatrix} = -(2 + p)^2(p - 1).
\]

Thus, \(RD + pC\) is singular if \(p = 1\) and nonsingular otherwise. We may therefore seek to determine which classes of coherent systems satisfy the criterion of Theorem 4. The following theorems and propositions give a partial answer to this question.

The next theorem is just the main result of Meilijon (1981) restricted to the class of analytic functions. However, when the component distributions are so constrained, the proof becomes quite simple.

**Theorem 5** Let \(S\) be a coherent system of \(m\) components. Suppose that the component distributions are independent and are analytic in the region \([0, \infty)\). Then, if the minimal incidence matrix has rank \(m\), \(S\) is identifiable.

**Proof** Suppose that \(\text{rank}(M) = m\). We show that \(RD + pC\) has rank \(m\) for all non-negative integers \(p\).

Every minimal fatal set is also a fatal set with a critical set identical to itself. Thus \(M\) is a submatrix of both \(D\) and \(C\). Let \(R_M\) and \(C_M\) be the submatrices of \(R\) and \(C\) corresponding to \(M\). Then \(R_M M + p C_M\) is a submatrix of \(RD + pC\). But

\[
R_M M + p C_M = R_M M + p M = (R_M + pI)M,
\]

and \((R_M + pI)\) is a diagonal matrix whose diagonal elements are strictly positive. Hence,

\[
\text{rank}(R_M M + p C_M) = \text{rank}(M) = m.
\]

Therefore the rank of \(RD + pC\) is \(m\).

**Proposition 4** Let \(S\) be such that

\[
S = s_j \prod S_1
\]

for some component, \(s_j\), and disjoint module, \(S_1\). Then the rank of \(RD + pC = m\) for all non-negative integers, \(p\), and \(S\) is identifiable.

**Proof** Suppose that \(S_1\) has no component in series with the rest of \(S_1\) and, without loss of generality, let \(s_1\) be the component of \(S\) in series with \(S_1\). Then \(\{s_1\}\) is a fatal set. But since \(S_1\) has no component in series with the rest of \(S_1, \{s_1, s_2\}, \{s_1, s_3\}, \ldots, \{s_1, s_m\}\) must all be fatal sets. The corresponding rows of \(D\) form a submatrix, \(D_m\), of \(D\) such that

\[
D_m = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\
1 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}.
\]
The corresponding submatrices of $R$ and $C$ are $R_m$ and $C_m$, where $R_m$ is a diagonal matrix of order $m$ with the $j^{th}$ diagonal element being $r_j$, and $C_m$ is a square matrix of order $m$ with ones in the first column and zeros everywhere else. Thus

$$R_m D_m + p C_m = \begin{pmatrix}
    r_1 + p & 0 & 0 & 0 & \cdots & 0 \\
    r_2 + p & r_2 & 0 & 0 & \cdots & 0 \\
    r_3 + p & 0 & r_3 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    r_m + p & 0 & 0 & 0 & \cdots & r_m
\end{pmatrix}.$$ 

Hence, 

$$\det(R_m D_m + p C_m) = (r_1 + p)r_2 r_3 \cdots r_m,$$

which is non-zero for all non-negative $p$. Therefore, the rank of $R_m D_m + p C_m$ is $m$. Consequently, the rank of $RD + pC$ is also $m$. The general case follows easily from this.

**Proposition 5** If $S$ has two modules, $S_1$ and $S_2$, such that

$$S = S_1 \coprod S_2,$$

then $S$ is identifiable.

**Proof** Consider $S_1$ as a single object and apply Proposition 4. Then do the same for $S_2$.

Consequently, the system in Example 1 is identifiable.

The collection of all coherent systems of order $m$ may therefore be partitioned into three distinct classes:

(i) Those for which the rank of $D$ is less than $m$. We know that these have the form

$$S = s_1 \coprod s_2 \coprod s_1$$

and are not identifiable.

(ii) Those for which the rank of $RD + pC$ is $m$ for all non-negative integers $p$. These are identifiable.

(iii) Systems for which $D$ has rank $m$ but the rank of $RD + pC$ is less than $m$ for some $p$. The status of these systems is uncertain at present but the following should be noted.

**Proposition 6** If the rank of $D$ is $m$ then the rank of $RD + pC$ is less than $m$ for at most $m$ values of $p$.

**Proof** This is really very similar to an eigenvalue problem. First note that the rank of $RD + pC$ is the same as the rank of $(RD + pC)^T (RD + pC)$. The determinant of the latter matrix is a polynomial in $p$ of order $m$ and therefore has at most $m$ roots, provided that the coefficients of the polynomial are not all zero. However, since
the rank of $RD = RD + 0C$ is $m$ if the rank of $D$ is $m$, zero is not a root of that polynomial. Hence the coefficients are not all zero.

If $RD + pC$ is singular for some integer $p$ then the method that has been used leads to several candidate power series solutions. If it could be shown that at most one of these ‘solutions’ converges, then Conjecture 1 would have been shown to hold in the class of analytic functions and the problem of identifiability would have been settled for this class of functions. Intensive computational studies were made of a few of the simpler systems, such as Example 2, which lead to singular matrices for some value of $p$. The results of these studies indicate that, in general, the behaviour of these ‘solutions’ is far from straightforward. For example, more than one of these may actually converge in a finite interval (and thus be a solution in this interval), though the functions so obtained are not monotonic and thus cannot be distribution functions. However, if one insists that the power series solutions converge over the entire domain of the $G_i$ (all of $\mathbb{R}$), then the results of these computational studies lend strong support to the view that Conjecture 1 is correct.

Meilijson (1981) and Nowik (1990) chose to view the problem of identifiability in the autopsy model as one of showing that the system of non-linear integral equations (11) has a unique solution. We have viewed the problem as one of showing that the system of non-linear differential equations (13) has a unique solution. Campbell and Meyer (1979, Chapter 9) and Campbell (1980, Chapter 3) show that the system of differential equations

$$A\dot{x} + Bx = f,$$

where $A$ and $B$ are known matrices and $A$ is singular, $x$ and $f$ are vector valued functions and $f$ is known, has a unique solution if $B + \lambda A$ has full column rank for some scalar $\lambda$. This condition is very similar to that of Theorem 4. Moreover, Condition A3 can be expressed exactly in this form since $D = D + 0C$. Hence it is trivially true that Conjecture 1 is equivalent to the following statement.

**Conjecture 2** A coherent system $S$ is identifiable if and only if there exists a scalar $p$ such that $D + pC$ has full column rank.

The system of differential equations (13) cannot be put in the form of (17) but can be transformed into a similar form

$$C\dot{x} + B(x) = f,$$

where $B$ is a vector valued function of $f$, by the method used by Meilijson (1981) to obtain his equations (5) and (20) and by Nowik (1990) to obtain his equation (8). These differential equations have been studied by Campbell (1982, Chapter 6), but the results there do not tell us whether (13) has a unique solution.

### 4 Identifiability in the Parametric Case

Condition A3 was shown in Theorem 3 to be necessary for identifiability. If the class of functions is restricted to certain parametric families then one can give a complete proof that rank($D$) = $m$ is also sufficient for identifiability.
Consider first the exponential family. Suppose that
\[ F_j(t) = \begin{cases} 
1 - e^{-\beta_j t} & \text{for } t \geq 0, \\
0 & \text{otherwise.} 
\end{cases} \]

Expanding in power series about 0 gives
\[ F_j(t) = \beta_j t - \frac{\beta_j^2 t^2}{2!} + \cdots \quad \text{for } j = 1, \ldots, m. \]

Recall that if the rank of \( D \) is \( m \) then both the coefficients and powers of the first non-zero term are uniquely determined. Thus the \( \beta_j \)'s are uniquely determined and \( S \) is identifiable.

Suppose that \( F_j \) is a Weibull distribution with an integer shape parameter
\[ f_j(t) = \begin{cases} 
\frac{\beta_j}{\alpha_j} \left( \frac{t}{\alpha_j} \right)^{\beta_j-1} \exp \left\{ -\frac{t}{\alpha_j} \right\} \beta_j & \text{for } t \geq 0, \\
0 & \text{otherwise,} 
\end{cases} \]

where the \( \beta_j \) are integers.

Then, using an identical argument to that used above, we have:
(i) The parameters \( \beta_j \) are uniquely determined since the power of the first non-zero term of the power series expansion of \( F_j(t) \) is a linear function of \( \beta_j \).
(ii) The coefficients of the first non-zero terms of the power series expansions of the distribution functions are \( \beta_j/(\alpha_j)^{\beta_j} \) for \( j = 1, \ldots, m \). Since the parameters \( \beta_j \) can be determined, it follows that the parameters \( \alpha_j \) can also be determined. Hence \( S \) is identifiable.

These ideas are summarized in the following two theorems, which are stated without proof.

**Theorem 6** Suppose that the component life distributions belong to a parametric family indexed by a single parameter \( \theta \),
\[ F_j(t) = F(t; \theta_j) \quad j = 1, \ldots, m. \]

Suppose further that, for \( j = 1, \ldots, m \), the distributions \( F(t; \theta_j) \) can be expanded in a power series in \( t \) about 0 such that the first non-zero term has power \( r(\theta_j) \) and coefficient \( a(\theta_j) \), where either \( r(\theta_j) \) or \( a(\theta_j) \) is a one-to-one function from \( \mathbb{R} \) onto \( \mathbb{R} \). Then \( \text{rank}(D) = m \) is a necessary and sufficient condition for \( S \) to be identifiable.

**Theorem 7** Suppose that the distribution function of the \( j \)th component has the form \( F(t; \alpha_j, \beta_j) \) and \( F_j(t) \) can be expanded in a power series about 0. Let \( r(\alpha_j, \beta_j) \) and \( c(\alpha_j, \beta_j) \) be the power and coefficient of the first non-zero term of the power series expansion of \( F_j \). Then, if \( r(\alpha_j, \beta_j) \) and \( c(\alpha_j, \beta_j) \) jointly determine \( \alpha_j \) and \( \beta_j \), it follows that \( \text{rank}(D) = m \) is a necessary and sufficient condition for identifiability.

The distributions that satisfy the conditions of Theorem 7 include the half-normal distribution and the gamma and Weibull distributions with integer shape parameters.
References


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