Applications of the Hazard Rate Ordering in Reliability and Order Statistics

by

Philip J. Boland\(^1\), Emad El-Neweihi\(^1\) and Frank Proschan\(^2\)

University College, Dublin
Department of Statistics
Belfield, Dublin 4, Ireland

and

University of Illinois at Chicago
Department of Mathematics
Statistics and Computer Science
Chicago, Illinois 60680

and

Florida State University
Department of Statistics
Tallahassee, Florida 32306-3033

Department of Statistics
Florida State University
Tallahassee, Florida 32306-3033

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Applications of the Hazard Rate Ordering in Reliability and Order Statistics

PHILIP J. BOLAND, EMAD EL-NEWEIHI AND FRANK PROSCHAN

ABSTRACT

The hazard rate ordering is an ordering for random variables which compares lifetimes with respect to their hazard rate functions. It is an ordering which is stronger than the usual stochastic order for random variables, yet is weaker than the likelihood ratio ordering. The hazard rate ordering is particularly useful in reliability theory and survival analysis, due to the importance of the hazard rate function in these areas. In this paper earlier work on the hazard rate ordering is reviewed, and extensive new results related to coherent systems are derived. Initially we fix the form of a coherent structure and investigate the effect on the hazard rate function of the system when we switch the lifetimes of its components from the vector \((T_1, \ldots, T_n)\) to the vector \((T'_1, \ldots, T'_n)\), where the hazard rate functions of the two vectors are assumed to be comparable in some sense. Although the hazard rate ordering is closed under the formation of series systems, we see that this is not the case for parallel systems even when the system is a two component parallel system with exponentially distributed lifetimes. A positive result shows that for two component parallel systems with proportional hazards \((\lambda_1 r(t), \lambda_2 r(t))\), the more diverse \((\lambda_1, \lambda_2)\) is in the sense of majorization the stronger in the system in the hazard rate ordering. Unfortunately even this result does not extend to parallel systems with more than two components, demonstrating again the delicate nature of the hazard rate ordering.

In another direction of investigation of the hazard rate ordering, we fix the lifetimes of \(n\) components and compare the hazard rate functions of two possible systems that may be formed form these components. In particular it is shown that if \(T_{k|n}\) is the lifetime of a \(k\) out of \(n\) system, then \(T_{k|n}\) is greater than \(T_{k+1|n}\) in the hazard rate ordering for any \(k\). This has an interesting interpretation in the language of order statistics. For independent (not necessarily identically distributed) lifetimes \(T_1, \ldots, T_n\), we let \(T_{k:n}\) represent the \(k^{th}\) order statistic (in increasing order). Then it is shown that \(T_{k+1:n}\) is greater than \(T_{k:n}\) in the hazard rate ordering for all \(k = 1, \ldots, n - 1\). The result does not however extend to the stronger likelihood ratio order.
§1. Introduction.

There are several senses in which one can assert that a random variable $X$ (or equivalently its distribution function $F_X$) is "greater" than another random variable $Y$. Stochastic ordering, hazard rate ordering, and likelihood ratio ordering are among the various notions of partial ordering between random variables. A random variable $X$ is said to be stochastically larger than a random variable $Y$, written $X \geq^s Y$, if $P(X > x) \geq P(Y > x)$ for all $x$. In this paper we will be concerned with random variables which are absolutely continuous (with respect to Lebesque measure) lifetimes, and hence in particular possess density functions. Therefore we will say that the random variable $X$ is greater than the random variable $Y$ in the hazard rate ordering (written $X \geq^h Y$) if $r_X(t) = \frac{f_X(t)}{F_X(t)} \leq r_Y(t) = \frac{f_Y(t)}{F_Y(t)}$ for all $t \geq 0$. Here we use $f_X(t)$ and $F_X(t) = 1 - F_X(t)$ to denote the density function and survival function for $X$ respectively. Hence $X$ is greater than $Y$ in the hazard rate ordering if the hazard rate function of $X$ is less than that of $Y$ at any point in time. In other words given that the devices represented by lifetimes $X$ and $Y$ have survived up to an arbitrary point in time $t$, that represented by $Y$ is more likely to fail in the immediate future than that represented by $X$. One says that $X$ is larger than $Y$ in the sense of likelihood ratio (written $X \geq^l Y$) if $f_X(t)/f_Y(t)$ is a nondecreasing function of $t$ where defined.

It is well known (see for example Roes (1983)) that $X \geq Y \Rightarrow X \geq^h Y \Rightarrow X \geq^l Y$. The hazard rate ordering is of particular interest in reliability because of the importance of the hazard rate function for systems. $r_X(t) \leq r_Y(t)$ for all $t \geq 0$ is equivalent to saying that $F_X(t)/F_Y(t)$ is nondecreasing in $t$. This later property has been used (in the more general setting where densities might not exist) to define an ordering termed "uniform stochastic order in the positive direction" (see for example Keilson and Sumita (1982)). The hazard rate ordering has also been considered by Pinedo and Ross (1980), Whitt (1980) and Lynch, Mimmack and Proschan (1987), amongst others. Lynch, Mimmack and Proschan (1987) have considered some closure properties of the hazard rate ordering. Preservation properties of the hazard rate ordering under convolutions are given in Keilson and Sumita (1982) and Shanthikumar and Yao (1991). The focus of the later paper is on bivariate functional relationships for various stochastic orderings including the hazard rate ordering. Capéraà (1988) characterizes the hazard rate ordering in terms of inequalities between expectations of functions belonging to a well specified set, and uses this characterization to compare asymptotic efficiencies of rank tests in two sample problems. Recently Dykstra, Kochar and Robertson (1991) considered nonparametric maximum likelihood estimation for $k$ population problems under hazard rate order restrictions.

Another stochastic order which is useful in reliability theory is that of mean residual life. We say that $X$ is greater than $Y$ in mean residual life (written $X \geq^m Y$) if

$$E(X \mid X \geq t) \geq E(Y \mid Y \geq t)$$

for all $t \geq 0$. 

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It is not difficult to show (see Alzaid (1988)) that

\[ X^{kr} \geq Y \Rightarrow X^{mr} \geq Y, \]

although in general it is not true that \( X^{st} \geq Y \Rightarrow X^{mr} \geq Y. \)

Our purpose in this paper is to investigate closely the hazard rate ordering and in particular to what extent it is preserved under the formation of coherent systems. Interesting applications are also obtained for order statistics from heterogeneous populations. Some of our work is motivated by the papers of Lynch, Mimmack and Proschan (1987) and Chan, Proschan and Setheraman (1991). The later paper deals primarily with the likelihood ratio order.

§2. Preservation of the Hazard Rate Ordering under the Formation of Coherent Systems.

Among the basic reliability operations is that of the formation of a coherent system from \( n \) components whose lifetimes are generally assumed to be independent. It is natural to investigate the effect of this operation on hazard rate ordering. The following are two interesting lines of investigation. We fix the coherent structure and investigate the effect on the hazard rate function of the system when we switch the lifetimes of its components from the vector \((T_1, \ldots, T_n)\) to the vector \((T'_1, \ldots, T'_n)\), where the hazard rate functions of the two vectors are assumed to be comparable in some sense. Alternatively, we may fix the lifetimes of the \( n \) components and compare the hazard rate functions of two possible coherent systems \( \phi_1 \) and \( \phi_2 \) that may be constructed from the given \( n \) components. In this section we present results pertaining to the former line of investigation and in the next section we study the latter direction.

To begin with we give a useful expression for the failure rate function of a coherent system in terms of the failure rate functions of its independent components (see Esary and Proschan (1963)). We use the standard notation as in Barlow and Proschan (1981) for coherent systems in reliability theory. Let \( \phi \) be the structure function of a coherent system with reliability function \( h \). That is if \( T_i \) is the lifetime of component \( i \) with survival distribution \( \overline{F}_i(t) = P(T_i > t), i = 1, \ldots, n \), then the probability that the system lifetime \( \tau \) exceeds \( t \) (i.e. the reliability of the system at time \( t \)) is given by

\[
P(\tau > t) = h(\overline{F}_1(t), \ldots, \overline{F}_n(t)) = h(\overline{F}(t)) = h(p) \bigg|_{p=\overline{F}(t)}.
\]

Let \( r_i(t) \) be the hazard rate function of component \( i \), for \( i = 1, \ldots, n \), and \( r(t) \) the hazard rate function of the system life \( \tau \). By the chain rule for differentiation we have that

\[
r(t) = \sum_{i=1}^{n} r_i(t) \frac{\partial h(p)}{\partial p} \bigg|_{p=\overline{F}(t)} \frac{\overline{F}_i(t)}{h(\overline{F}_1(t), \ldots, \overline{F}_n(t))}.
\]

(2.1)
In particular for a series system the failure rate function has the form \( r(t) = \sum_{i=1}^{n} r_i(t) \).

Now assume that \( T_i' \) is an alternative lifetime for component \( i \) with failure rate function \( r_i'(t), i = 1, \ldots, n \). Also assume that for each \( i = 1, \ldots, n \), \( T_i \leq T_i' \), i.e. \( r_i(t) \leq r_i'(t) \) for all \( t \geq 0 \). Clearly then for a series structure the failure rate \( r(t) \) corresponding to \( T_1, \ldots, T_n \) is equal to \( \sum_{i=1}^{n} r_i(t) \leq \sum_{i=1}^{n} r_i'(t) = r'(t) \), the failure rate function of the series system corresponding to \( T_1', \ldots, T_n' \). In other words \( T_i \leq T_i' \), \( i = 1, \ldots, n \Rightarrow \min_{1 \leq i \leq n} T_i \leq \min_{1 \leq i \leq n} T_i' \), hence the hazard rate ordering is preserved under the formation of series systems.

Unfortunately the same is not true for all coherent structures (although the corresponding result is clearly true for ordinary stochastic ordering). In fact even for a parallel system of two components with exponential lifetimes the hazard rate ordering is not preserved as is shown by the following example.

**Example 2.1.** Let \( T_1, T_2 (T_1', T_2') \) be independent exponential random variables with parameters \( \lambda_1, \lambda_2 (\lambda_1', \lambda_2') \) and densities \( f_1(t) = \lambda_1 e^{-\lambda_1 t}, f_2(t) = \lambda_2 e^{-\lambda_2 t} \) \( (f_1'(t) = \lambda_1' e^{-\lambda_1' t}, f_2'(t) = \lambda_2' e^{-\lambda_2' t}) \). Assume that \( \lambda_2 = \lambda_2' \) and \( \lambda_1 < \lambda_1' \). Hence it follows that \( T_1 \leq T_1' \) and \( T_2 \leq T_2' \) (in fact \( T_2 \parallel T_2' \)). We let \( \tau, \tau' \) be the lifetime of a parallel system whose two components have lifetimes \( T_1, T_2 \) \( (T_1', T_2') \). If \( \tau(t) \) and \( \tau'(t) \) are the hazard rate functions of \( \tau \) and \( \tau' \), then clearly

\[
\tau(t) = \frac{f_1(t)F_2(t) + f_2(t)F_1(t)}{1 - F_1(t)F_2(t)} \quad \text{and} \quad \tau'(t) = \frac{f_1'(t)F_2'(t) + f_2'(t)F_1'(t)}{1 - F_1'(t)F_2'(t)}.
\]

Now basic algebra shows that \( \tau \leq \tau' \) iff

\[
\{ [\lambda_1' e^{-\lambda_1'} - \lambda_1](1 - e^{-\lambda_2 t}) + \lambda_2 e^{-\lambda_2 t} [1 - e^{-(\lambda_1'-\lambda_1) t}] + [\lambda_1 - \lambda_1' e^{-(\lambda_1'-\lambda_1) t}][1 - e^{-\lambda_2 t}]^2 \\
+ (\lambda_1' - \lambda_1) e^{-\lambda_1 t} [1 - e^{-\lambda_2 t}]^2 \} \geq 0 \quad \text{for all} \quad t \geq 0. \tag{2.2}
\]

Let \( \epsilon > 0 \) be an arbitrarily small quantity (yet to be determined) and let \( t_0 \) be large enough so that \( t \geq t_0 \Rightarrow e^{-t(\lambda_1' - \lambda_1 t)} < \frac{\lambda_2}{\lambda_1} \) and \( e^{-\lambda_1' t} < \frac{\lambda_2}{\lambda_1} \). For \( t \geq t_0 \) the left hand side of (2.2) is less than or equal to \( 2\epsilon - \lambda_1 (1 - e^{-\lambda_2 t}) + \lambda_1 (1 - e^{-\lambda_2 t})^2 + \lambda_2 e^{-\lambda_2 t} \). Now choose \( \lambda_2 \) and \( t \geq t_0 \) such that \( 1 - e^{-\lambda_2 t} = \frac{1}{2} \) and \( \lambda_2 < \epsilon \). Then for such a \( t \) the left hand side of (2.2) is less than or equal to \( 3\epsilon - \frac{\lambda_1}{2} + \frac{\lambda_1}{2} = 3\epsilon - \frac{\lambda_1}{2} < 0 \) if \( \epsilon < \frac{\lambda_1}{12} \). So based on \( \lambda_1, \lambda_1' \) one can choose \( \lambda_2, t \) such that \( \tau(t) > \tau'(t) \).

The above example shows that in general hazard rate ordering is not preserved under the formation of coherent systems. However we know that it is preserved under the formation of series systems. It is then natural to ask if there are some other special cases in which
hazard rate ordering is preserved when coherent systems are formed. One important special case is when the components of the system have identically distributed lifetimes.

Consider a $k$–out–of–$n$ system whose component lifetimes are independent and identically distributed with common distribution either that of $T_1$ or that of $T_1'$. Suppose that $T_1 \geq T_1'$. Let $\tau_{k|n}$ ($\tau'_{k|n}$) be the lifetime of the $k$–out–of–$n$ system formed from $n$ components with lifetimes distributed as $T_1(T_1')$. Lynch, Mimmack and Proshan (1987) show that $\tau_{k|n} \geq \tau'_{k|n}$. So $k$–out–of–$n$ systems whose components have identically distributed lifetimes preserve the hazard rate ordering. Clearly if one connects $m$ $k_i$–out–of–$n_i$ systems, $i = 1, \ldots, m$, in series and requires all the $n_1 + n_2 + \cdots + n_m$ components to have a common life distribution, then such a superstructure will also preserve the hazard rate ordering. This is a consequence of the fact that the hazard rate function of the superstructure is equal to the sum of the hazard rates of the $m$ $k_i$–out–of–$n_i$ systems, and each such subsystem preserves the hazard rate ordering. One might then wonder whether or not in the case of identically distributed components all coherent structures preserve the hazard rate ordering. The following example shows that this is not the case.

Example 2.2. Let $T_1, T_2, T_3$ be independent exponential random variables with common parameter $\lambda$. Let $\tau(T_1, T_2, T_3) = \max(T_1, \min(T_2, T_3)) = \tau$ be the lifetime of a coherent system constructed from these three components. We let $\tau_\lambda(t)$ denote the failure rate of the system. Clearly $\tau_\lambda(t) = \frac{\lambda e^{-\lambda t} + \lambda e^{-2\lambda t} - 3\lambda e^{-3\lambda t}}{e^{-\lambda t} + e^{-2\lambda t} - e^{-3\lambda t}} = \lambda g(p) = \lambda \left[ \frac{1 + 2p - 3p^2}{1 + p - p^2} \right]$ where we let $p = e^{-\lambda t}$.

To construct a counter example it is enough to show that

$$\frac{d}{d\lambda}(\tau_\lambda(t)) = \lambda g'(e^{-\lambda t})(-\lambda e^{-\lambda t}) + g(e^{-\lambda t}) < 0$$

for some $\lambda$ and $t$, where $g'$ is the derivative of $g$. Note that $g(p) \leq 2$ for small $p$ and $g'(p) = \frac{1 - 4p - p^2}{(1 + p - p^2)^2} \rightarrow 1$ as $p \rightarrow 0$. Hence there exists $p_0$ small enough such that $g'(p_0) \geq \frac{1}{2}$ and $g(p_0) \leq 2$. Let $\lambda, t$ be such that $e^{-\lambda t} = p_0$ and $\lambda \geq \sqrt{5/p_0}$. Then $\lambda^2 g'(p_0)p_0 \geq (5/p_0)p_0 \cdot \frac{1}{2} = 2.5 > g(p_0)$, giving the result.

In the remainder of this section we use the language of majorization and Schur-functions. We refer the reader to Marshall and Olkin (1979) for an excellent exposition on this topic.

Let $T_{\lambda}$ be an exponential random variable with parameter $\lambda$, $i = 1, 2$. We let $\tau_{\lambda, 1}$ and $\tau_{\lambda, 2}(t)$ be respectively the lifetime and the failure rate function of a parallel system constructed from two independent components whose lifetimes are $T_{\lambda, 1}, T_{\lambda, 2}$. Now if $\lambda_i \leq \lambda'_i$ for $i = 1, 2$, then clearly $\tau_{\lambda, 1, 2} \geq \tau_{\lambda'_1, \lambda'_2}$, but example 2.1 shows that it does not necessarily follow that $\tau_{\lambda, 1, 2} \geq \tau_{\lambda'_1, \lambda'_2}$. In other words $\tau_{\lambda, 1, 2}(t)$ is not necessarily a nondecreasing function of $(\lambda_1, \lambda_2)$ for each fixed $t$. However $\tau_{\lambda, 1, 2}(t)$ may be a monotone function of $(\lambda_1, \lambda_2)$.
(for each fixed \( t \)) with respect to some other partial order on \((0, \infty) \times (0, \infty)\). In particular Figure 1 (from Barlow and Proschan (1981)) strongly suggests that if \((\lambda_1, \lambda_2) \geq^m (\lambda'_1, \lambda'_2)\) then \(r_{\lambda_1, \lambda_2}(t) \leq r_{\lambda'_1, \lambda'_2}(t)\) for all \( t \). Here \( \geq^m \) stands for the partial order of majorization, and \((\lambda_1, \lambda_2) \geq^m (\lambda'_1, \lambda'_2)\) implies that the component hazard rates of \(r_{\lambda_1, \lambda_2}\) are more dispersed than those of \(r_{\lambda'_1, \lambda'_2}\). The following theorem shows that indeed this is the case even under the more general assumption of proportional hazards.

Figure 1 here

**Theorem 2.3.** Let \(r_{\lambda_1, \lambda_2}(t)\) be the hazard rate function of a parallel system of two components whose lifetimes are independent with respective hazard rate functions \(\lambda_1r(t), \lambda_2r(t)\). Then for all \( t \), \((\lambda_1, \lambda_2) \geq^m (\lambda'_1, \lambda'_2)\) implies that \(r_{\lambda_1, \lambda_2}(t) \leq r_{\lambda'_1, \lambda'_2}(t)\). In the language of majorization, this says that for each \( t \), \(r_{\lambda_1, \lambda_2}(t)\) is a Schur concave function of \((\lambda_1, \lambda_2)\).

**Proof:** Let \( t > 0 \) be fixed. Assume \( \lambda_1 + \lambda_2 = c \), and without loss of generality that \( R(t) = \int_0^t r(u)du = 1 \). We want to show that \( r_{c-\lambda, \lambda}(t) = \)

\[
g(\lambda) = \frac{(c-\lambda)e^{-(c-\lambda)} + \lambda e^{-\lambda} - ce^{-c}}{e^{-(c-\lambda)} + e^{-\lambda} - e^{-c}} = \frac{(c-\lambda)e^{\lambda} + \lambda e^{(c-\lambda)} - c}{e^{\lambda} + e^{(c-\lambda)} - 1}
\]

is increasing on \([0, \frac{c}{2}]\) as a function of \( \lambda \).

Now \( g'(\lambda) \)

\[
= \frac{[e^{\lambda}(c-\lambda-1) + e^{(c-\lambda)}(1-\lambda)][e^{\lambda} + e^{(c-\lambda)} - 1] + [-e^{\lambda} + e^{(c-\lambda)}][c-\lambda]e^{\lambda} + e^{(c-\lambda)} - c}{D^2}
\]

\[
= \frac{h(\lambda)}{D^2},
\]

where \( D \) equals \( e^{\lambda} + e^{c-\lambda} - 1 \). It suffices to show that \( h(\lambda) \geq 0 \) on \([0, \frac{c}{2}]\). Now \( h(\lambda) = -e^{2\lambda} + e^\lambda(\lambda+1) + e^c(2\lambda-4\lambda) + e^{(c-\lambda)}[\lambda-1-c] + e^{2(c-\lambda)} \). Since \( h(0) = (c+e^c-1)e^c > 0 \) and \( h(\frac{c}{2}) = 0 \), it suffices to show that \( h'(\lambda) \leq 0 \) on \([0, \frac{c}{2}]\). Now \( h'(\lambda) = -2e^{2\lambda} + e^\lambda(2+\lambda) - 4e^c + e^{(c-\lambda)}[2-\lambda+c] - 2e^{2(c-\lambda)} \) and we need only show that \( (2+\lambda)e^\lambda + [2+c-\lambda]e^{(c-\lambda)} \leq 2e^{2\lambda} + 4e^c + 2e^{2(c-\lambda)} \) for \( \lambda \in [0, \frac{c}{2}] \). But for such \( \lambda \),

\[
(2+\lambda)e^\lambda \leq 2(1+\lambda)e^\lambda \leq 2e^{2\lambda},
\]

\[
(c-\lambda)e^{(c-\lambda)} \leq e^{2(c-\lambda)} \leq 2e^{2(c-\lambda)},
\]

and \( 2e^{c-\lambda} \leq 2e^c \leq 4e^c \),

\( \)
concluding the proof.

We give two interesting consequences of this theorem in the following two corollaries.

**Corollary 2.4.** Let \( r_{\lambda_1, \lambda_2}(t) \) be the hazard rate function of a parallel system whose two independent components have respective hazard rate functions \( \lambda_1 r(t), \lambda_2 r(t) \). Let \( \overline{R}(t) = \int_0^t r(u)du \) and \( \overline{\lambda} = \frac{\lambda_1 + \lambda_2}{2} \). Then

\[
  r_{\lambda_1, \lambda_2}(t) \leq r_{\overline{\lambda}, \overline{\lambda}}(t) = \frac{2\overline{\lambda}r(t)[e^{-\overline{\lambda}R(t)} - e^{-2\overline{\lambda}R(t)}]}{2e^{-\overline{\lambda}R(t)} - e^{-2\overline{\lambda}R(t)}} = \frac{2\overline{\lambda}r(t)[e^{\overline{\lambda}R(t)} - 1]}{(2e^{\overline{\lambda}R(t)} - 1)}
\]

**Proof:** The proof follows immediately from Theorem 2.3 since

\[
  (\lambda_1, \lambda_2) \geq (\overline{\lambda}, \overline{\lambda}).
\]

**Corollary 2.5.** Suppose we want to build a parallel-series system by connecting in parallel two series subsystems of sizes \( n_1, n_2 \). We may vary \( n_1, n_2 \) (which are assumed to be positive) without changing the total number of components \( n = n_1 + n_2 \). Assume all the components have identically distributed lifetimes with common failure rate function \( r(t) \). To minimize the failure rate function of the parallel series system we must choose \( n_1 = 1, n_2 = n - 1 \) (or equivalently \( n_1 = n - 1, n_2 = 1 \)).

**Proof:** Each of the two series subsystems may now be viewed as a "super" component with failure rate functions \( n_1 r(t), n_2 r(t) \) respectively. The result now follows by Theorem 2.3 and the fact that \((0, n) \geq (n_1, n_2)\) for all choices of \( n_1, n_2 \) such that \( n_1 + n_2 = n \).

The following example shows that Theorem 2.3 is not necessarily true for \( n > 2 \).

**Example 2.6.** Let \( r_{\lambda_1, \lambda_2}(t), r_{\lambda'_1, \lambda'_2}(t) \) be the hazard rate functions of two parallel systems each consisting of two components which are exponential with parameters \( (\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2) \) respectively. Assume \( \lambda_1 > \lambda'_1 > \lambda_2 \) and \( \lambda_1 + \lambda_2 = \lambda'_1 + \lambda'_2 \), i.e. \( (\lambda_1, \lambda_2) \geq (\lambda'_1, \lambda'_2) \). Let \( \overline{F}_{\lambda_1, \lambda_2}(t), \overline{F}_{\lambda'_1, \lambda'_2}(t) \) denote the respective survival distributions of the two systems. From Theorem 2.3 we know that \( r_{\lambda_1, \lambda_2}(t) \leq r_{\lambda'_1, \lambda'_2}(t) \) for all \( t \). Also observe that \( r_{\lambda_1, \lambda_2}(t) \rightarrow \lambda_2 > 0 \) as \( t \rightarrow \infty \). It is not difficult to show (see Barlow and Proschan (1981)) that there exists \( t_0 \) depending on \( (\lambda_1, \lambda_2) \) such that \( r_{\lambda_1, \lambda_2}(t) \) increases on \([0, t_0]\) and decreases on \([t_0, +\infty)\). In particular therefore if \( t \geq t_0 \) then \( r_{\lambda_1, \lambda_2}(t) \geq \lambda_2 > 0 \). Observe also that \( \overline{F}_{\lambda'_1, \lambda'_2}(t)/\overline{F}_{\lambda_1, \lambda_2}(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Hence given \( \epsilon > 0 \), there exists a \( t_1 \) such that for \( t \geq t_1, \overline{F}_{\lambda'_1, \lambda'_2}/\overline{F}_{\lambda_1, \lambda_2}(t) \leq \epsilon \). For the rest of this proof we denote \( r_{\lambda_1, \lambda_1}(t) \) and \( \overline{F}_{\lambda_1, \lambda_2}(t) \)
(respectively $r_1 \lambda_1 \lambda_2$ and $F_1 \lambda_1 \lambda_2(t)$) by $r(t)$ and $F(t)$ ($r'(t)$ and $F'(t)$). Now consider the two parallel systems as two super components and connect to each of them in parallel a component whose exponential with parameter $\lambda < \lambda_2$, where $\lambda$ is a small value to be specified later. Let $F$ and $g$ denote the survival distribution and density of the exponential random variable with parameter $\lambda$. We now let $s(t)$ (respectively $s'(t)$) be the failure rate function of the parallel system with three independent components whose lifetimes are exponential with parameters $\lambda_1, \lambda_2, \lambda(\lambda_1', \lambda_2', \lambda)$. Clearly

$$s(t) = \frac{f(t)G(t) + g(t)F(t)}{1 - F(t)G(t)} \quad \text{and} \quad s'(t) = \frac{f'(t)G(t) + g(t)F'(t)}{1 - F'(t)G(t)},$$

where $f(t)$ and $f'(t)$ are the densities corresponding to $F$ and $F'$ respectively. Now $s(t) \leq s'(t)$ for all $t \geq 0 \Leftrightarrow (f'(t) - f(t))G(t) + g(t)(F'(t) - F(t)) + (f(t)F'(t) - f'(t)F(t))G^2(t) \geq 0$ for all $t \geq 0$

$$\Rightarrow [r'(t)F(t) - r(t)F(t)]G(t) + g(t)(1 - \frac{F'(t)}{F(t)}) + (r(t) - r'(t)F(t))G^2(t)
\quad + (r'(t) - r(t)F'(t))G^2(t) \geq 0 \quad \text{for all } t. \quad (2.3)$$

Now let $\epsilon > 0$ be arbitrarily small and $t_2 = \max(t_0, t_1)$. Then for $t \geq t_2$ the left hand side of expression (2.3) is less than or equal to

$$[r'(t)\epsilon - r(t)]G(t) + g(t) + r(t)G^2(t) + [r'(t)\epsilon - r(t)F(t)]G^2(t). \quad (2.4)$$

Since $r'(t)\epsilon - r(t)$ is bounded and $F'(t) \rightarrow 0$ as $t \rightarrow \infty$, there exists a $t_3 > t_2$ such that for $t \geq t_3$ we have the expression (2.4) is less than or equal to

$$r'(t)\epsilon - r(t)G(t) + r(t)G^2(t) + g(t) + \epsilon. \quad (2.5)$$

Now choose $\lambda < \epsilon$ and $t^* \geq t_3$ such that $e^{-\lambda t^*} = \frac{1}{2}$. Hence $r'(t^*)\epsilon - r(t^*)G(t^*) + r(t^*)G^2(t^*) + g(t^*) + \epsilon \leq r'(t^*)\epsilon - \frac{r(t^*)}{4} + 2\epsilon \leq r'(t^*)\epsilon - \frac{\lambda^2}{4} + 2\epsilon$ (since $r(t^*) \geq \lambda_2$). Observing that $r(t)$ is a bound function of $t$, $\epsilon$ can be chosen small enough (say less than $\min \{ \frac{\lambda^2}{12}, \frac{\lambda^2}{12} \}$ where $b$ is a positive upper bound for $r'(t^*)$) such that $r'(t^*)\epsilon - \frac{\lambda^2}{4} + 2\epsilon$ is negative. Hence $s(t) \leq s'(t)$ for all $t$. Consequently Theorem 2.3 does not generalize to parallel systems with more than two components.

§3. Comparisons of Hazard Rate Functions for $k$–out-of–$n$ Systems.

As mentioned in section 2, there is yet another direction in which one can study the interplay between the hazard rate ordering and the formation of coherent systems. We now fix the lifetimes of the $n$ components available to us and compare the hazard rate functions for two possible systems that may be constructed from these given $n$ components.
We carry out such comparisons when both systems considered are of the \(k\)-out-of-\(n\) type, and give interpretations for these results in terms of order statistics.

As usual \(T_1, \ldots, T_n\) will denote \(n\) independent lifetimes, and for each \(i = 1, \ldots, n\), \(r_i(t)\) and \(\bar{F}_i(t)\) will represent the failure rate function and survival function respectively of \(T_i\). We will let \(\tau_{k|n}\) denote the lifetime of a \(k\) out of \(n\) system constructed by using the \(n\) components with lifetimes \(T_1, \ldots, T_n\). Clearly \(\tau_{k|n} = T_{n-k+1:n}\) where \(T_{j:n} \leq \cdots \leq T_{n:n}\) are the order statistics of \(T_1, \ldots, T_n\). The failure rate function of \(\tau_{k|n}\) will be denoted by \(r_{k|n}(t)\), and we will use \(h_{k|n}(\bar{F}(t)) = h_{k|n}(p) \mid _{p=\bar{F}(t)} = P[\tau_{k|n} > t] \) for its survival function. The following three theorems allow comparisons to be made between \(k\) out of \(n\) systems for the hazard rate ordering.

**Theorem 3.1.** \(\tau_{k|n} \geq \tau_{k+1|n}^{hr}\) for all \(k = 1, \ldots, n - 1\).

**Proof:** Recall that

\[
r_{k|n}(t) = \sum_{i=1}^{n} \frac{r_i(t)\bar{F}_i(t)}{h_{k|n}(\bar{F}(t))} \frac{\partial}{\partial p_i} (h_{k|n}(p)) \mid_{p=\bar{F}(t)},
\]

where \(\bar{F}(t) = (\bar{F}_1(t), \ldots, \bar{F}_n(t))\). It suffices to show that for each \(i\),

\[
\frac{1}{h_{k|n}(\bar{F}(t))} \frac{\partial}{\partial p_i} (h_{k|n}(p)) \mid_{p=\bar{F}(t)} \leq \frac{1}{h_{k+1|n}(\bar{F}(t))} \frac{\partial}{\partial p_i} (h_{k+1|n}(p)) \mid_{p=\bar{F}(t)}
\]

or equivalently

\[
P(\sum_{j \neq i} X_j(t) = k - 1) / [\bar{F}_i(t)P(\sum_{j \neq i} X_j(t) \geq k - 1) + F_i(t)P(\sum_{j \neq i} X_j(t) \geq k)]
\]

\[
\leq P(\sum_{j \neq i} X_j(t) = k) / [\bar{F}_i(t)P(\sum_{j \neq i} X_j(t) \geq k) + F_i(t)P(\sum_{j \neq i} X_j(t) \geq k + 1)],
\]

(3.1)

where \(P(X_j(t) = 1) = 1 - P(X_j(t) = 0) = \bar{F}_j(t), j = 1, \ldots, n\). Note then that \(X_j(t) = 1\) if and only if \(T_j > t, j = 1, \ldots, n\). Now (3.1) reduces to showing that

\[
[P(\sum_{j \neq i} X_j(t) \geq k - 1) - P(\sum_{j \neq i} X_j(t) \geq k)] [\bar{F}_i(t)P(\sum_{j \neq i} X_j(t) \geq k) + F_i(t)P(\sum_{j \neq i} X_j(t) \geq k + 1)]
\]

\[
\leq [P(\sum_{j \neq i} X_j(t) \geq k) - P(\sum_{j \neq i} X_j(t) \geq k + 1)] [\bar{F}_i(t)P(\sum_{j \neq i} X_j(t) \geq k - 1) + F_i(t)P(\sum_{j \neq i} X_j(t) \geq k)],
\]

or by cancellation that

\[
P(\sum_{j \neq i} X_j(t) \geq k - 1)P(\sum_{j \neq i} X_j(t) \geq k + 1) \leq P^2(\sum_{j \neq i} X_j(t) \geq k).
\]

(3.2)

Karlin (1968) proved that if \(Y_\ell, \ell = 1, \ldots, m\) are independent indicator random variables with \(p_\ell = P(Y_\ell = 1)\), then \(P(\sum_{\ell} Y_\ell = k)\) is totally positive of order infinity in differences.
of $k$ (see also Esary and Proschan (1963)). Hence it follows that 3.2 is valid. Note that (3.2) is equivalent to saying that either

$$\frac{P(\sum_{j \neq i} X_j(t) \geq k - 1)}{P(\sum_{j \neq i} X_j(t) \geq k)} \quad \text{or} \quad \frac{P(\sum_{j \neq i} X_j(t) = k - 1)}{P(\sum_{j \neq i} X_j(t) \geq k)}$$

are increasing in $k$, $k = 1, \ldots, n - 1$.

Theorem 3.1 implies of course that in general (for independent absolutely continuous components)

$$r_{k+1,n}(t) \leq r_{k,n}(t) \quad \text{for all } t \geq 0.$$  

Theorem 3.1 also has an important interpretation in terms of order statistics. It says that if $T_1, \ldots, T_n$ are independent absolutely continuous random lifetimes, then

$$T_{1:n} \leq T_{2:n} \leq \cdots \leq T_{n:n}.$$  

Of course the order statistics are always ordered in this way with respect to the (weaker) usual stochastic ordering. One might expect the order statistics to be ordered in this way with respect to the (even stronger) likelihood ratio order. Surprisingly enough we will see in the following example that this is not the case in general (even for $n = 2$), although Chan, Proschan and Sethuraman [1987] proved that it is true for the special case when all of the $T_i$'s are identically distributed.

**Example 3.2.** Let $T_1, T_2$ be the lifetimes of two independent components. In this example we will see that $T_{2:2}$ is not necessarily greater than $T_{1:2}$ in the likelihood ratio ordering. It also provides an example of two random variables $X$ and $Y$ where $X \leq_Y Y$ but $X \neq_{hr} Y$.

Let $f_i, F_i = 1 - F_i$ be the density and survival distribution of $T_i$, $i = 1, 2$. The densities of $T_{2:2}$ and $T_{1:2}$ are given respectively by: $f_1(t)(1 - F_2(t)) + f_2(t)(1 - F_1(t))$ and $f_1(t)F_2(t) + f_2(t)F_1(t)$. Now $T_{2:2}$ is better in the likelihood ratio ordering than $T_{1:2} \Leftrightarrow (f_1(t)(1 - F_2(t)) + f_2(t)(1 - F_1(t)))/(f_1(t)F_2(t) + f_2(t)F_1(t))$ is increasing in $t$. Equivalently by taking the right-hand derivative and performing simple algebraic manipulations we must have

$$[f_1(t)f_2(t) - f_2(t)f_1(t)][F_2(t) - F_1(t)] + 2[f_1(t)f_2(t) + f_2(t)f_1(t)] \geq 0 \quad \text{for all } t \geq 0. \quad (3.3)$$

Clearly one can construct $f_1, f_2$ such that the left hand side of (3.3) is negative for at least one $t > 0$. Here is such a choice: let

$$f_1(t) = \begin{cases} \frac{3}{2} & 0 \leq t \leq \frac{1}{4} \\ 0 & \text{otherwise}, \end{cases}$$

and let

$$f_2(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ -\frac{1}{2}(t - \frac{1}{2}) + 1 & \frac{1}{2} \leq t \leq \frac{1}{2} + \epsilon \\ \frac{1}{2} + \epsilon & \frac{1}{2} + \epsilon \leq t \leq m, \end{cases}$$

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where \( m \) is chosen so that \( \int_0^m f_2(t) dt = 1 \), and \( \epsilon \) is an arbitrarily small quantity. Now at \( t = \frac{1}{2} \) the left hand side of (3.3) is equal to \( \frac{3}{2} - \frac{1}{34}(\frac{1}{2} - \frac{1}{4}) + 2(\frac{3}{3} + \frac{\epsilon}{4}) = -\frac{3}{16\epsilon} + \frac{1}{2} < 0 \) when \( \epsilon < \frac{1}{46} \). 

In the following theorem we shall let \( r_{k|n-1} \) denote the lifetime of a \( k \) out of \( n - 1 \) system formed from \( T_1, \ldots, T_{n-1} \).

**Theorem 3.3.** Let \( T_1, \ldots, T_n \) be such that \( r_n(t) \leq \min_{i=1,\ldots,n-1} r_i(t) \) for all \( t \geq 0 \). Then \( r_{k|n} \geq r_{k|n-1} \) for \( k = 1, \ldots, n-1 \).

**Proof:** We want to show that

\[
\tau_{k|n}(t) = \left[ \sum_{i=1}^{n} r_i(t) \bar{F}_i(t) P\left(\sum_{j \neq i} X_j(t) = k - 1\right) / P\left(\sum_{j \neq i} X_j(t) \geq k\right) \right]
\leq \left[ \sum_{i=1}^{n-1} r_i(t) \bar{F}_i(t) P\left(\sum_{j \neq i} X_j(t) = k - 1\right) / P\left(\sum_{j \neq i} X_j(t) \geq k\right) \right]
= \tau_{k|n-1}(t).
\]

Rearranging terms we see that this is equivalent to

\[
r_n(t) \leq \frac{P(\sum_{i=1}^{n} X_i(t) \geq k)}{\bar{F}_n(t) P(\sum_{i=1}^{n-1} X_i(t) = k - 1)} \left[ \sum_{i=1}^{n-1} r_i(t) \frac{\bar{F}_i(t)}{\bar{F}_n(t)} \left( \frac{P(\sum_{j \neq i} X_j(t) = k - 1)}{P(\sum_{j \neq i} X_j(t) \geq k)} - \frac{P(\sum_{j \neq i} X_j(t) = k - 1)}{P(\sum_{j \neq i} X_j(t) \geq k)} \right) \right]
= \frac{1}{P(\sum_{i=1}^{n-1} X_i(t) = k - 1)} \sum_{i=1}^{n-1} r_i(t) \frac{\bar{F}_i(t)}{\bar{F}_n(t)} \left( P(\sum_{j \neq i} X_j(t) \geq k) P(\sum_{j \neq i} X_j(t) = k - 1) - P(\sum_{j \neq i} X_j(t) \geq k) P(\sum_{j \neq i} X_j(t) = k - 1) \right)
\]

Noting that in this last expression the quantity enclosed in \{ \} reduces to

\[
\bar{F}_n(t) [P(\sum_{j \neq i} X_j(t) \geq k - 1) P(\sum_{j \neq i} X_j(t) = k - 1) - P(\sum_{j \neq i} X_j(t) \geq k) P(\sum_{j \neq i} X_j(t) = k - 2)],
\]
we need only show

\[
 r_n(t) \leq \sum_{i=1}^{n-1} r_i(t) \left[ \frac{F_i(t)}{P(\sum_{i=1}^{n-1} X_i(t) = k - 1)P(\sum_{i=1}^{n-1} X_i(t) \geq k)} \right] \times
\]

\[
 \left[ P(\sum_{i=1}^{n-1} X_i(t) \geq k - 1)P(\sum_{j \neq i}^{n-1} X_j(t) = k - 1) - P(\sum_{i=1}^{n-1} X_i(t) \geq k)P(\sum_{j \neq i}^{n-1} X_j(t) = k - 2) \right]
\]

\[
= \sum_{i=1}^{n-1} r_i(t) \{ [P(\sum_{i=1}^{n-1} X_i(t) \geq k - 1)P(X_i(t) = 1, \sum_{i=1}^{n-1} X_i(t) = k)
\]

\[- P(\sum_{i=1}^{n-1} X_i(t) \geq k)P(X_i(t) = 1, \sum_{i=1}^{n-1} X_i(t) = k - 1) ] \}
\]

/P(\sum_{i=1}^{n-1} X_i(t) = k - 1)P(\sum_{i=1}^{n-1} X_i(t) \geq k).

Now \( r_n(t) \leq \min_{i=1, \ldots, n-1} r_i(t) \), and hence this last inequality is true iff

\[
P(\sum_{i=1}^{n-1} X_i(t) \geq k - 1)(\sum_{i=1}^{n-1} P(X_i(t) = 1, \sum_{i=1}^{n-1} X_i(t) = k)
\]

\[- P(\sum_{i=1}^{n-1} X_i(t) \geq k)(\sum_{i=1}^{n-1} P(X_i(t) = 1, \sum_{i=1}^{n-1} X_i(t) = k - 1) \]

\[\geq P(\sum_{i=1}^{n-1} X_i(t) \geq k)P(\sum_{i=1}^{n-1} X_i(t) = k - 1) \]

Now using the fact that

\[
\sum_{i=1}^{n-1} P(X_i(t) = 1, \sum_{i=1}^{n-1} X_i(t) = k) = kP(\sum_{i=1}^{n-1} X_i(t) = k),
\]

we have that the above inequality is true iff

\[
kP(\sum_{i=1}^{n-1} X_i(t) \geq k - 1)P(\sum_{i=1}^{n-1} X_i(t) = k)
\]

\[\geq P(\sum_{i=1}^{n-1} X_i(t) \geq k)[P(\sum_{i=1}^{n-1} X_i(t) = k - 1) + (k - 1)P(\sum_{i=1}^{n-1} X_i(t) = k - 1)] \]

\[\Leftrightarrow P(\sum_{i=1}^{n-1} X_i(t) \geq k - 1)P(\sum_{i=1}^{n-1} X_i(t) = k) \geq P(\sum_{i=1}^{n-1} X_i(t) \geq k)P(\sum_{i=1}^{n-1} X_i(t) = k - 1) \]

\[\Leftrightarrow P(\sum_{i=1}^{n-1} X_i(t) = k)/P(\sum_{i=1}^{n-1} X_i(t) \geq k) \uparrow \text{ in } k,
\]

which we have already observed in Theorem 3.1 to be true.
Note that Theorem 3.3 implies that \( r_{n+1} \geq r_{n+1} \) (or \( r_{n+1}(t) \leq r_{n+1}(t) \) for all \( t \)) when \( r_n(t) \leq \min_{i=1, \ldots, n-1} r_i(t) \) for all \( t \). This implies that the parallel system formed from \( T_1, \ldots, T_n \) is better (in the hazard rate sense) than the parallel system formed from \( T_1, \ldots, T_{n-1} \). Of course \( r_{n+1} \geq r_{n+1} \) under no conditions on \( T_n \). However even in the case where \( n = 2 \), \( r_2 \geq r_1 \) if and only if \( r_2(t) F_1(t) \leq r_1(t) \) for all \( t \), and so we have that \( r_2 \geq r_1 \) does not hold in general. We see once again therefore that the hazard rate ordering is considerably more delicate than that of the usual stochastic order.

Let us formulate the result of Theorem 3.3 in the language of order statistics. Suppose \( T_1, \ldots, T_n \) are independent absolutely continuous lifetimes where \( T_i \geq T_j \) for \( i = 1, \ldots, n-1 \). Then for every \( k = 1, \ldots, n-1 \),

\[
T_{n-k+1:n} \geq T_{n-k:n-1}.
\]

In the following Theorem 3.4 we will use \( r_{k-1:n-1} \) to denote the lifetime of a \((k-1)\)-out-of-\((n-1)\) system constructed from the \( n-1 \) components with lifetimes \( T_1, \ldots, T_{n-1} \). This theorem is an analogue of Theorem 3.3, but in the case where \( T_n \) is the weakest component in the hazard rate ordering.

**Theorem 3.4.** Let \( T_1, \ldots, T_n \) be such that \( r_n(t) \geq \max_{i=1, \ldots, n-1} r_i(t) \) for all \( t \geq 0 \). Then

\[
r_{k-1:n-1} \geq r_{k:n}.
\]

**Proof:** Now

\[
r_{(k-1):(n-1)}(t) = \sum_{i=1}^{n-1} r_i(t) F_i(t) \frac{P(\sum_{j \neq i}^{n-1} X_j(t) = k - 2)}{P(\sum_{j=1}^{n-1} X_j(t) \geq k - 1)}
\]

\[
\leq \sum_{i=1}^{n} r_i(t) F_i(t) \frac{P(\sum_{j \neq i}^{n} X_j(t) = k - 1)}{P(\sum_{j=1}^{n} X_j(t) \geq k)}
\]

\[
= \left( \sum_{i=1}^{n-1} r_i(t) F_i(t) \left[ \frac{P(\sum_{j \neq i}^{n-1} X_j(t) = k - 2)}{P(\sum_{j=1}^{n-1} X_j(t) \geq k - 1)} - \frac{P(\sum_{j \neq i}^{n} X_j(t) = k - 1)}{P(\sum_{j=1}^{n} X_j(t) \geq k)} \right] \right)
\]

\[
\frac{P(\sum_{j=1}^{n} X_j(t) \geq k)}{F_n(t) P(\sum_{j=1}^{n-1} X_j(t) = k - 1)}
\]

\[
\leq r_n(t).
\]
Hence if \( r_n(t) \geq r_i(t) \) for all \( i = 1, \ldots, n-1 \) and all \( t \geq 0 \), it suffices to show that

\[
\sum_{i=1}^{n-1} \frac{1}{\bar{F}_n(t) P(\sum_{j=1}^{n-1} X_j(t) = k-1)} [P(\sum_{j=1}^{n} X_j(t) \geq k) P(\sum_{j \neq i}^{n-1} X_j(t) = k-2) - P(\sum_{j \neq i}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) \geq k-1)]
\leq 1.
\tag{3.4}
\]

for all \( t \geq 0 \). But for \( i = 1, \ldots, n-1 \),

\[
P(\sum_{j=1}^{n} X_j(t) \geq k) P(\sum_{j \neq i}^{n-1} X_j(t) = k-2) - P(\sum_{j \neq i}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) \geq k-1)
\]

\[
= \bar{F}_n(t) \{P(\sum_{j=1}^{n} X_j(t) \geq k-1) P(\sum_{j \neq i}^{n-1} X_j(t) = k-2) - P(\sum_{j \neq i}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) = k-2)\}
\]

\[
+ F_n(t) \{P(\sum_{j=1}^{n} X_j(t) \geq k) P(\sum_{j \neq i}^{n-1} X_j(t) = k-2) - P(\sum_{j \neq i}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) \geq k-1)\}
\]

and hence inequality (3.4) is true if

\[
\frac{1}{P(\sum_{j=1}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) \geq k-1) \bar{F}_n(t)} \times
\sum_{i=1}^{n-1} \left[P(\sum_{j=1}^{n} X_j(t) \geq k) P(X_i(t) = 1, \sum_{j=1}^{n-1} X_j(t) = k-1) - P(\sum_{j=1}^{n-1} X_j(t) \geq k-1) P(X_i(t) = 1, \sum_{j=1}^{n-1} X_j(t) = k)\right]
\]

\[
= \frac{1}{P(\sum_{i=1}^{n-1} X_j(t) = k-1) P(\sum_{i=1}^{n-1} X_j(t) \geq k-1)} \frac{F_n(t)}{\bar{F}_n(t)} \left[(k-1)P(\sum_{j=1}^{n} X_j(t) = k-1) P(\sum_{j=1}^{n-1} X_j(t) \geq k) - k P(\sum_{j=1}^{n-1} X_j(t) = k) P(\sum_{j=1}^{n-1} X_j(t) \geq k-1)\right]
\leq 1
\]

But as

\[
P(\sum_{i=1}^{n-1} X_j(t) = k) / P(\sum_{i=1}^{n-1} X_j(t) \geq k)
\]

is nondecreasing in \( k \),

\[
\frac{(k-1) P(\sum_{j=1}^{n-1} X_j(t) \geq k)}{P(\sum_{j=1}^{n-1} X_j(t) \geq k-1)} - \frac{k P(\sum_{j=1}^{n-1} X_j(t) = k)}{P(\sum_{j=1}^{n-1} X_j(t) = k-1)} \leq 0
\]

from which the result follows.
The interpretation of Theorem 3.4 in terms of order statistics from \( n \) populations is as follows. Let \( T_1, \ldots, T_n \) be independent lifetimes where \( T_i^{hr} \leq T_i \) for \( i = 1, \ldots, n - 1 \). Then

\[
T_{n-k+1:n-1}^{hr} \geq T_{n-k+1:n}^{hr}
\]

for \( k = 2, \ldots, n \).

§4. Preservation of Hazard Rate Ordering under Mixtures.

Let \( \{ X_\alpha : \alpha \in A \} \) and \( \{ Y_\beta : \beta \in B \} \) be two families of random variables. Assume \( X_\alpha^{hr} \geq Y_\beta \) for all \( \alpha \in A \) and \( \beta \in B \). Let \( F_\alpha(t) \) and \( G_\beta(t) \) denote the survival distributions of \( X_\alpha \) and \( Y_\beta \) respectively, and suppose \( u \) and \( v \) are two mixing distributions on \( A \) and \( B \) respectively. Let \( X \) and \( Y \) be two random variables whose survival distributions are \( \int_A F_\alpha(t)du(\alpha) \) and \( \int_B G_\beta(t)dv(\beta) \) respectively. The following theorem extends a result by Chan, Proschan and Sethuraman (1991) to the hazard rate ordering.

**Theorem 4.1.** \( X \geq Y \).

**Proof:** Let \( F \) and \( G \) be the survival functions of \( X \) and \( Y \) respectively. We shall prove the theorem by showing that if \( t_1 \leq t_2 \) then

\[
\frac{F(t_1)}{G(t_1)} \leq \frac{F(t_2)}{G(t_2)}.
\]

This is actually equivalent to showing that \( X \) is uniformly stochastically larger than \( Y \) in the positive direction. When the random variables in question possess densities, this is the same as hazard rate ordering. Now

\[
F(t_1)G(t_2) = (\int_A F_\alpha(t_1)du(\alpha))(\int_B G_\beta(t_2)dv(\beta))
\]

\[
= \int \int F_\alpha(t_1)G_\beta(t_2)du(\alpha)dv(\beta)
\]

\[
\leq \int \int F_\alpha(t_2)G_\beta(t_1)du(\alpha)dv(\beta)
\]

\[
= (\int_A F_\alpha(t_2)du(\alpha))(\int_B G_\beta(t_1)dv(\beta)) = F(t_2)G(t_1).
\]

Theorem 4.1.

\[\]

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References.


Barlow, R.E. and Proschan, F. "Statistical theory of reliability and life testing, probability models", To Begin With, Silver Spring, MD., 1981.


Figure 1
Applications of the Hazard Rate Ordering in Reliability and Order Statistics

Philip J. Boland, Emad El-Neweihi, and Frank Proschan

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Stochastic order, hazard rate order, likelihood ratio order, order statistics, coherent systems, k out of n systems, proportional hazard rates.

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