MODELS AND INference FOR SERIES SYSTEMS
OPERATING UNDER DIFFERENT ENVIRONMENTS

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Abstract

Let $T_i, (i = 1, \ldots, p)$, denote the failure-time variable of the $i$th component in a $p$-component series system, and let the environment on which it is operating be described by a fixed vector $\mathbf{x} = (x_1, \ldots, x_q)'$. In this paper it is postulated that $T_i$ and $\mathbf{x}$ are related via a proportional hazards model given by $\lambda_{T_i|\mathbf{x}}(t) = \lambda_{o_i}(t) \exp(\beta_i' \mathbf{x})$, $(i = 1, \ldots, p)$, where $\lambda_{T_i|\mathbf{x}}(\cdot)$ is the conditional hazard function of $T_i$ given $\mathbf{x}$, and $\lambda_{o_i}(\cdot), (i = 1, \ldots, p)$ are unknown component-specific baseline hazard functions, and $\beta_i, (i = 1, \ldots, p)$, are unknown component-specific regression coefficient vectors. The probabilistic implication of this model on the observable vector $\left(Z, \delta \equiv (\delta_1, \ldots, \delta_p)\right)$, where $Z = \min_{1 \leq i \leq p} T_i$ and $\delta_i = I(T_i = Z)$ is examined. In particular, it is proved that under mild conditions, $Z$ and $\delta$ are independent if and only if $\lambda_{o_i}(\cdot) = \rho_i \lambda_o(\cdot), (i = 1, \ldots, p)$, where $\lambda_o(\cdot)$ is some hazard function, and $\rho_1, \ldots, \rho_p$ are positive real numbers. Furthermore, given $n$ such systems operating under different environments, and under the assumption that $\lambda_{o_i}(t) = \theta_i, (i = 1, \ldots, p)$, estimation and hypotheses testing procedures are presented. The asymptotic properties of the estimators, such as consistency and asymptotic normality, are rigorously developed.

Keywords and Phrases: Competing risk; Cox model; Hazard function; Maximum likelihood; Score statistic; Series system.

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1 Introduction and summary

A system with \( p \) (\( p \in \mathbb{N} \)) components, where \( \mathbb{N} \) is the set of natural numbers, is called a series system or a simple competing risks model if its failure time coincides with the minimum of the failure times of its \( p \) components. Thus if \( T_i \) (\( i = 1, \ldots, p \)) is the failure time of the \( i \)th component, and \( Z \) is the failure time of the system, then

\[
Z = \min_{1 \leq i \leq p} T_i. \tag{1.1}
\]

It will be assumed in this paper that upon failure of a series system it is possible to determine the component that failed. For this failed component its exact failure time will be known, while for the other components it will only be known that their failure times exceeded the system failure time \( Z \). The observation on the failed component is complete, while those for the other components are censored by the system failure time. Thus, aside from being able to observe \( Z \), one will also be able to observe the \( p \times 1 \) vector \( \delta = (\delta_1, \ldots, \delta_p)' \), where

\[
\delta_i = I_{\{Z = T_i\}} \quad (i = 1, \ldots, p), \tag{1.2}
\]

with \( I_A \) denoting the indicator function of event \( A \). If \( T_i \)'s are stochastically independent and their distributions are continuous, which from hereon will be assumed, then \( \delta' 1_p = 1 \) almost surely (a.s.), where \( 1_p \) is the \( p \times 1 \) vector of 1's. We point out that in reliability settings, in contrast to medical or clinical trial settings, the independence assumption of the failure times is reasonable, since the components of coherent systems are typically not affecting the functioning of the other components. Now, suppose that such series systems operate in certain environments which can be reasonably described by a \( q \times 1 \) (\( q \in \mathbb{N} \)) vector \( x = (x_1, \ldots, x_q)' \), where the \( x_j \)'s may represent temperature, pressure, humidity, gravitational pull, type of terrain, etc.

The first goal of this paper is to introduce models for the relationship between \((Z, \delta')'\) and \( x \). Such models are of importance in determining which environmental descriptors have significant effects on the performance of the system and will be of use in predicting
the performance of the system under specific environmental conditions given data of the system's performance under varied environments. For instance, space systems are tested and the astronauts are trained on earth-based environments which purport to simulate conditions in space in which the systems are to operate, and one problem here is to predict system's performance based on these training runs on earth. A recent example was the problem that the space shuttle Endeavour's astronauts encountered in grabbing the Intelsat satellite. It turned out that their earth-based underwater training was quite different from what they encountered in space. Prediction models would be crucial in such situations, since adjustments could be made for the discrepancies in the training and operational environments.

The main feature of the model-building is that instead of directly postulating a relationship between \((Z, \delta')'\) and \(x\), we first postulate reasonable and plausible relationships between the \(T_i\)'s and \(x\). The relationships in (1.1) and (1.2) are then utilized to obtain the relationships between \((Z, \delta')'\) and \(x\). Cox [3] introduced the proportional hazards regression model which specifies that the conditional hazard function of a failure time variable \(T\), given a covariate \(x\), is

\[
\lambda_{T|x}(t \mid x) = \lambda_0(t) \exp\{\beta'x\},
\]

(1.3)

where \(\lambda_0(\cdot)\) is some baseline hazard function, and \(\beta = (\beta_1, \ldots, \beta_q)'\) is a \(q \times 1\) vector of regression coefficients. This model has been the subject of intense research in the past two decades, and it is now widely accepted as a reasonable model for failure time data. Model (1.3) is the linchpin for our model about the dependence between \((Z, \delta')'\) and \(x\). In the model that we consider, the starting postulate is that, given \(x\), the conditional hazard function of \(T_i\), \((i = 1, \ldots, p)\), is

\[
\lambda_{T_i|x}(t \mid x) = \lambda_{oi}(t) \exp\{\beta_i'x\}, \quad (i = 1, \ldots, p),
\]

(1.4)

where \(\lambda_{oi}(\cdot), (i = 1, \ldots, p),\) are baseline hazard functions, and \(\beta_i = (\beta_{i1}, \ldots, \beta_{iq})', (i = 1, \ldots, p),\) are \(q \times 1\) regression coefficient vectors describing the effect of the environment
vector \( \mathbf{x} \) on the failure time of the \( i \)th component. This model was considered by Holt [5] in the context of twin studies, but we will subject this model to a thorough theoretical treatment. Denoting by \( \Lambda_{oi}(t) = \int_0^t \lambda_{oi}(u)du \), \( (i = 1, \ldots, p) \), the baseline cumulative hazard function for the \( i \)th component, (1.4) is equivalent to

\[
\Lambda_{T_i|x}(t \mid \mathbf{x}) = \Lambda_{oi}(t) \exp(\beta_i' \mathbf{x}), \quad (i = 1, \ldots, p),
\]

where \( \Lambda_{T_i|x}(\cdot \mid \mathbf{x}) \) is the associated conditional cumulative hazard function of \( T_i \), given \( \mathbf{x} \). We examine the stochastic properties of model (1.4) in Section 2. The joint distribution of \((Z, \delta')'\), given \( \mathbf{x} \), will be derived, and necessary and sufficient conditions for the independence of \( Z \) and \( \delta \), given \( \mathbf{x} \), will be presented. Some special specifications of the baseline hazard functions will then be discussed. These specifications can be classified as parametric, semiparametric, and nonparametric in nature. Such specifications have a bearing on the statistical inferences appropriate for model (1.4).

The second goal of this paper is to develop statistical inference procedures for model (1.4). This will be done in Sections 3, 4 and 5. It will be assumed in these sections that \( n \) independent series systems, each with \( p \) (\( p \) known) components, are observed, with the \( k \)th system having operated in an environment described by the fixed vector

\[
\mathbf{x}_k = (x_{1k}, \ldots, x_{pk})', \quad (k = 1, \ldots, n).
\]

The observable entities are \((Z_k, \delta_k', \mathbf{x}_k')'\), \( (k = 1, \ldots, n) \), where

\[
Z_k = \min_{1 \leq i \leq p} T_{ik} \quad \text{and} \quad \delta_k = (\delta_{1k}, \ldots, \delta_{pk})',
\]

with \( \delta_{ik} = I_{z_k = T_{ik}} \). The basic assumption about the component failure times \( \{T_{ik}, i = 1, \ldots, p; k = 1, \ldots, n\} \) is that they satisfy (1.4) with \( T_i \) replaced by \( T_{ik} \) and \( \mathbf{x} \) replaced by \( \mathbf{x}_k \), and they are stochastically independent. The unknown parameters of the statistical model are then the baseline hazard functions \( \lambda_{oi}(\cdot) \) and the vectors of regression coefficients \( \beta_i \), \( (i = 1, \ldots, p) \). In the present paper we restrict our inference considerations to the case where the baseline hazard functions are constants, and def. the treatment
of the semiparametric and nonparametric cases to forthcoming papers. Under this restriction, procedures for estimating and testing hypotheses about the parameters based on \( (Z_k, \delta_k', x_k') \), \( k = 1, \ldots, n \), will be presented in Sections 3 and 4. The problem of estimating the system failure time will also be addressed in Section 5. In Section 3 we develop the estimators, and present a rigorous examination of their asymptotic properties; while in Section 4 we present procedures for testing the effects of the environment vectors and the equality of the baseline hazards.

2 Probabilistic properties of model

2.1 Dependence of observable vector on environment

In this section we consider the setting where a series system as described in Section 1 is operating in an environment represented by the vector \( x = (x_1, \ldots, x_q)' \), and when model (1.4) holds.

**Theorem 2.1** For each \( t \in \mathbb{R}_+ \), and for each \( d = (d_1, \ldots, d_p)' \in \{0, 1\}^p \) with \( d'1_p = 1 \),

(i) \( \mathcal{P}\{Z > t \mid x\} = \exp\left(-\sum_{i=1}^p \Lambda_o(t) \exp\{\beta'_i x\}\right) \);

(ii) \( \mathcal{P}\{\delta = d \mid x\} = \mathcal{E}\left\{ \frac{\prod_{i=1}^p [\lambda_o(Z) \exp(\beta'_i x)]^{d_i}}{\sum_{j=1}^p \lambda_o(Z) \exp(\beta'_j x)} \mid x\right\} \); and

(iii) \( \mathcal{P}\{Z > t, \delta = d \mid x\} = \mathcal{E}\left\{ I_{(Z>t)} \frac{\prod_{i=1}^p [\lambda_o(Z) \exp(\beta'_i x)]^{d_i}}{\sum_{j=1}^p \lambda_o(Z) \exp(\beta'_j x)} \mid x\right\} \).

**Proof:** The first result is immediate from the facts that \( \mathcal{P}\{Z > t \mid x\} = \prod_{i=1}^p \mathcal{P}\{T_i > t \mid x\} \) by independence of the \( T_i \)'s, \( \mathcal{P}\{T_i > t \mid x\} = \exp\{-\Lambda(T_i|x)(t \mid x)\} \), and (1.4). The second result follows from the third one by setting \( t = 0 \), so it suffices then to prove the latter. Let us denote by \((1, 0)'\) the \( p \times 1 \) vector with \( i \)-th element 1 and all others 0. Then

\[
\mathcal{P}\{Z > t, \delta = (1, 0)' \mid x\} = \mathcal{P}\{t < T_i < \min_{j \neq i} T_j \mid x\}
= \int_t^\infty \mathcal{P}\{T_i < \min_{j \neq i} T_j \mid x, T_i = w\} f_{T_i|x}(w \mid x)dw
= \int_t^\infty \exp\left(-\sum_{j \neq i} \Lambda_{o,j}(w) e^{\beta'_j x}\right) \lambda_o(w) e^{\beta'_i x} \times
\]
\[ \exp\{-\Lambda_{oi}(w)e^{\beta_i'x}\} dw = \int_t^\infty \left[ \frac{\lambda_{oi}(w)e^{\beta_i'x}}{\sum_{j=1}^{p} \lambda_{oj}(w)e^{\beta_j'x}} \right] \times \left\{ \sum_{j=1}^{p} \lambda_{oj}(w)e^{\beta_j'x} \right\} \exp\{-\sum_{j=1}^{p} \lambda_{oj}(w)e^{\beta_j'x}\} dw. \]

By noting now that the expression in the second set of brackets in the last integral is the conditional density function of \( Z \) given \( x \) by the first result of the theorem, then the proof of the third result is completed. \( \Box \)

### 2.2 Independence of system life and failure indicators

Theorem 2.1 implies that if the baseline hazard functions are proportional to each other, that is,

\[ \lambda_{oi}(t) = \rho_i \lambda_o(t), \; \forall t \in \mathbb{R}_+, \; (i = 1, \ldots, p), \]  

(2.1)

for some positive real numbers \( \rho_i, \; (i = 1, \ldots, p), \) and some hazard function \( \lambda_o(\cdot), \) then \( Z \) and \( \delta \) are stochastically independent. It is of interest to know whether (2.1) is also necessary for the independence of \( Z \) and \( \delta \). To answer this, we first prove the following lemma.

**Lemma 2.1** Let \( V \) be a random variable with density function \( f_V(\cdot) \) and support \( \mathcal{R}_V \). Let \( r : \mathbb{R} \rightarrow \mathbb{R}_+ \) be a continuous function. Then

\[ \mathcal{E}\left\{ \frac{I_{\{V>t\}}}{1 + r(V)} \right\} = \mathcal{E}\{I_{\{V>t\}}\} \mathcal{E}\left\{ \frac{1}{1 + r(V)} \right\}, \; \forall t \in \mathcal{R}_V, \]  

(2.2)

if and only if \( r(t) = c, \; \forall t \in \mathcal{R}_V, \) for some \( c \in \mathbb{R} \).

**Proof:** The sufficiency part is clearly true. To prove the necessity, note that by taking the derivative with respect to \( t \) of both sides of (2.2) we obtain the equation

\[ \frac{f_V(t)}{1 + r(t)} = f_V(t) \mathcal{E}\left\{ \frac{1}{1 + r(V)} \right\}, \; \forall t \in \mathcal{R}_V. \]
Since \( f_V(t) > 0 \) on a dense subset of \( \mathcal{R}_V \), then on this subset, \( r(t) = \{ \mathcal{E} \{ 1/[1+r(V)] \} \}^{-1} - 1 \equiv c \). By continuity of \( r(\cdot) \) it follows that \( r(t) = c, \forall t \in \mathcal{R}_V \). \( \square \)

**Theorem 2.2** If (2.1) is satisfied then \( Z \) and \( \delta \) are independent, given \( x \). Conversely, if \( Z \) and \( \delta \) are independent, given \( x \), and the baseline hazard functions \( \lambda_{oi}(\cdot) \), \( i = 1, \ldots, p \), have common support, then (2.1) holds.

**Proof:** The first result was pointed out just after the proof of Theorem 2.1. Suppose now that \( Z \) and \( \delta \) are independent given \( x \). Then for \( \forall t \in \mathbb{R}_+ \) and \( \forall d \in \{0,1\}^p \) with \( d'1_p = 1 \),

\[
\mathcal{P}\{Z > t, \delta = d \mid x\} = \mathcal{P}\{Z > t \mid x\} \mathcal{P}\{\delta = d \mid x\}.
\]

For \( d = (1_i,0')' \), it follows from Theorem 2.1 that the condition above becomes

\[
\mathcal{E} \left\{ \frac{I_{\{Z \geq t\}}}{1 + \sum_{j \neq i} r_{ji}(Z) \exp\{\gamma_{ji}'x\}} \mid x \right\} = \mathcal{E}\{I_{\{Z > t\}} \mid x\} \mathcal{E} \left\{ \frac{1}{1 + \sum_{j \neq i} r_{ji}(Z) \exp\{\gamma_{ji}'x\}} \mid x \right\}, \forall t \in \mathbb{R}_+,
\]

where \( r_{ji}(t) = \lambda_{oj}(t)/\lambda_{oi}(t) \) and \( \gamma_{ji} = \beta_j - \beta_i \). Applying Lemma 2.1, it follows that the mapping \( t \mapsto \sum_{j \neq i} r_{ji}(t) \exp\{\gamma_{ji}'x\} \) must be free of \( t \). Since \( \exp\{\gamma_{ji}'x\} > 0 \) and \( r_{ji}(t) > 0 \), then \( \sum_{j \neq i} r_{ji}(t) \exp\{\gamma_{ji}'x\} \) will be free of \( t \) if either (i) \( r_{ji}(t) = c_{ji}, \forall t \in \mathbb{R}_+ \), \( j \neq i \) for some constants \( c_{ji} \); or (ii) \( \sum_{j \neq i} r_{ji}(t) \exp\{\gamma_{ji}'x\} = C_i \) for some constants \( C_i \), \( i = 1, \ldots, p \).

Clearly, if (i) holds then the theorem is proved. On the otherhand, if (ii) holds for \( \forall i = 1, \ldots, p \), then we must have

\[
\sum_{j=1}^p \lambda_{oj}(t)e^{\beta_j'x} = (C_i + 1)\lambda_{oi}(t)e^{\beta_i'x}, (i = 1, \ldots, p), \forall t \in \mathbb{R}_+.
\]

This implies that

\[
\frac{\lambda_{oi}(t)}{\lambda_{oi}(t)} = \left( \frac{C_i + 1}{C_i + 1} \right) \exp\{(\beta_i - \beta_i)'x\}, (i = 2, \ldots, p),
\]

which is condition (2.1). \( \square \)
There are three broad classifications of specifications of the baseline hazard functions in model (1.4): fully parametric, semiparametric, and fully nonparametric. Of course, combinations are also possible, for instance, some baseline hazard functions might be parametrically specified, while the others might be nonparametric in nature. In the parametric specification, (1.4) becomes

$$\lambda_{T_i|x}(t \mid x) = \lambda_{o_i}(t; \theta_i)e^{\beta'_i x}, \ (i = 1, \ldots, p),$$

where $\theta_i \ (i = 1, \ldots, p)$, are finite dimensional parameter vectors, but which are not necessarily of the same dimension, and $\lambda_{o_i}(\cdot; \theta_i) \ (i = 1, \ldots, p)$ are hazard functions of known functional forms. By taking

$$\lambda_{o_i}(t; \theta_i) = \theta_i, \ (i = 1, \ldots, p),$$

where $\theta_i \in \mathbb{R}_+$, $(i = 1, \ldots, p)$, then we obtain baseline hazards associated with the exponential densities.

**Corollary 2.1** If the baseline hazard functions satisfy (2.4), then $Z$ and $\delta$ are independent given $x$. Furthermore, $Z$ is exponentially distributed with parameter $\sum_{i=1}^{p} \theta_i \exp\{\beta'_i x\}$, and $\delta$ is (singular) multinomially distributed with parameters 1 and

$$\frac{(\theta_1 \exp\{\beta'_1 x\}, \ldots, \theta_p \exp\{\beta'_p x\})'}{\sum_{j=1}^{p} \theta_j \exp\{\beta'_j x\}}.$$

**Proof:** These results are immediate from Theorem 2.2 and Theorem 2.1. \( \square \)

A richer parametric class of models, which includes (2.4), is to specify that each $\lambda_{o_i}(\cdot; \theta_i)$, $(i = 1, \ldots, p)$, is a Weibull hazard function, that is,

$$\lambda_{o_i}(t; \alpha_i, \theta_i) = (\alpha_i \theta_i)(\theta_i t)^{\alpha_i - 1}, \ (i = 1, \ldots, p).$$

**Corollary 2.2** If the baseline hazard functions satisfy (2.5), then $Z$ and $\delta$ are independent given $x$ if and only if $\alpha_1 = \ldots = \alpha_p$. In this situation, $Z$ is Weibull distributed
with shape parameter $\alpha$ and scale parameter $\left\{ \sum_{i=1}^{p} \theta_i^\alpha \exp\{\beta_i^j x\}\right\}^{\frac{1}{\alpha}}$, and $\delta$ is (singular) multinomially distributed with parameters $1$ and $\frac{\left(\theta_1^\alpha \exp\{\beta_1^j x\}, \ldots, \theta_p^\alpha \exp\{\beta_p^j x\}\right)}{\sum_{j=1}^{p} \theta_j^\alpha \exp\{\beta_j^j x\}}$.

**Proof:** These results are also immediate from Theorem 2.2 and Theorem 2.1. □

Of course, it is possible to have some baseline hazard functions belong to one parametric class of hazard functions, and the others belong to other classes. Alternative parametric classes of hazard functions, which are useful in reliability and survival analysis, are the linear failure rate and the Gompertz-Makeham class of hazard functions. See for instance Doksum and Yandell [4]. These classes, together with the Weibull class (2.5), can model failure times possessing the increasing failure rate (IFR) property (cf., Barlow and Proschan [1]).

On the other hand, if one could only assume that the baseline hazard functions $\lambda_{oi}(\cdotp)$, $(i = 1, \ldots, p)$, each belongs to $\mathcal{H}$, the family of all hazard functions on $\mathbb{R}_+$, then we would say that the specification is fully nonparametric. Under this situation, $Z$ and $\delta$ will not be independent, given $x$. If we impose the additional restriction (2.1) together with $\lambda_0(\cdotp) \in \mathcal{H}$, which makes the model semiparametric, then by Theorem 2.2, $Z$ and $\delta$ are independent, given $x$. Clearly, this model is not identifiable. One way to make it identifiable is to assume

$$\text{Condition (2.1), } \lambda_0(\cdotp) \in \mathcal{H}, \text{ with } \sum_{i=1}^{p} \rho_i = 1. \quad (2.6)$$

### 3 Estimation under constant baseline hazards

The set-up considered in this section is that described in the last paragraph of Section 1, in which $n$ series systems operating on different environments $x_k$, $(k = 1, \ldots, n)$, are each observed until their failure. The observable vectors are $(Z_k, \delta_k', x_k')', (k = 1, \ldots, n)$, which are defined in (1.6) and (1.5). For the $k$th system, it is assumed that (1.4) with $x$ replaced by $x_k$ and $T_i$ replaced by $T_{ik}$ holds. Recall that the $T_{ik}$'s are the component
failure times. For our notation let \((z_k, d_k', x_k')', (k = 1, \ldots, n)\), be the realizations of \((Z_k, \delta_k', x_k')', (k = 1, \ldots, n)\); and let
\[
S = \mathbf{1}_n' Z = \sum_{k=1}^{n} Z_k
\]  
(3.1)
and
\[
N = (N_1, \ldots, N_p)' = \mathbf{1}_n' \Delta = (\sum_{k=1}^{n} \delta_{1k}, \ldots, \sum_{k=1}^{n} \delta_{pk})',
\]  
(3.2)
where \(Z = (Z_1, \ldots, Z_n)'\) and \(\Delta = (\delta_1, \ldots, \delta_p)'\). The statistic \(S\) is the 'total-time-on-test' of the \(n\) systems, while \(N_i, (i = 1, \ldots, p)\), is the total number out of the \(n\) systems in which component \(i\) caused system failure. We will denote by \(s\) and \(n = (n_1, \ldots, n_p)'\) the realizations of \(S\) and \(N\), respectively.

### 3.1 Estimating environment effects and baseline hazards

In this subsection we consider the case where the baseline hazard functions are unknown positive constants, that is, model (1.4) together with the condition in (2.4). The parameter vector of the associated statistical model is therefore \((\theta', B')'\), where (with an obvious abuse of notation)
\[
(\theta', B')' \equiv (\theta_1, \beta_1', \ldots, \theta_p, \beta_p')' \in (\mathbb{R}_+ \times \mathbb{R}_+^p).
\]
Using Corollary 2.1, \(Z_k\) and \(\delta_k\) are independent, given \(x_k\), hence from Theorem 2.1, we obtain the full likelihood function of \((\theta', B')'\), given the data \((z_k, d_k')'\), to be
\[
L[(\theta', B')'] = \prod_{i=1}^{p} \left[ \theta_i^{n_i} e^{\beta_i' \sum_{k=1}^{n} x_{k,i}} \exp \left\{ -\theta_i \sum_{k=1}^{n} z_k e^{\beta_i' x_k} \right\} \right].
\]  
(3.3)
in (3.3) we see that the likelihood function factors into the product of \(L_i[(\theta_i, \beta_i')']\), \((i = 1, \ldots, p)\), which is the likelihood function associated with the parameters of the \(i\)th component, indicating that when making likelihood-based inferences about \((\theta_i, \beta_i')'\), one need only consider \(L_i[(\theta_i, \beta_i')']\). For instance, in order to obtain the maximum likelihood estimators (MLEs) \((\hat{\theta}_i, \hat{\beta}_i')\), one need only maximize each \(L_i[(\theta_i, \beta_i')']\), \((i = 1, \ldots, p)\). It
also follows from (3.3) and by invoking the factorization theorem that the set of statistics 
\( (N_i, \sum_{k=1}^n x_k' \delta_{ik}, Z')' \) is sufficient for \( (\theta_i, \beta_i')' \).

Let us now obtain the MLE of \( (\theta_i, \beta_i')' \). Since

\[
\mathcal{L}_i[(\theta_i, \beta_i')'] = \exp \left\{ -\theta_i \sum_{k=1}^n z_k \exp \{\beta_i' x_k\} \right\}
\]

whenever \( n_i = 0 \), then \( \mathcal{L}_i \) is maximized under this situation by taking \( \hat{\theta}_i = 0 \) and setting \( \hat{\beta}_i \) arbitrarily. Thus when \( n_i = 0 \), which means that component \( i \) never caused the failure of any of the \( n \) systems, there is a certain indeterminacy when using maximum likelihood estimation. However, if \( n \) is sufficiently large, and each component is stochastically relevant in the sense that each has positive probability of causing system failure, then \( n_i = 0 \) will occur with small probability. If \( n_i > 0 \) the log-likelihood function of \( (\theta_i, \beta_i')' \) is given by

\[
\ell_i[(\theta_i, \beta_i')'] = \log \mathcal{L}_i[(\theta_i, \beta_i')'] = n_i \log \theta_i + \beta_i' \sum_{k=1}^n x_k d_{ik} - \theta_i \sum_{k=1}^n z_k e^{\beta_i' x_k}.
\] (3.4)

The efficient score function associated with \( \ell_i \) is therefore

\[
U_i[(\theta_i, \beta_i')'] = \begin{bmatrix}
\frac{\partial \ell_i}{\partial \theta_i} \\
\frac{\partial \ell_i}{\partial \beta_i'}
\end{bmatrix} = \begin{bmatrix}
\frac{n_i}{\theta_i} - \sum_{k=1}^n z_k e^{\beta_i' x_k} \\
\sum_{k=1}^n x_k [d_{ik} - \theta_i z_k e^{\beta_i' x_k}]
\end{bmatrix}.
\] (3.5)

Solving the equation \( U_i[(\hat{\theta}_i, \hat{\beta}_i')'] = 0 \), we find that the MLEs of \( (\theta_i, \beta_i')' \) must satisfy the equations

\[
\sum_{k=1}^n \frac{x_k d_{ik}}{n_i} = \sum_{k=1}^n x_k \xi_k(\hat{\beta}_i),
\] (3.6)

and

\[
\hat{\theta}_i = \frac{n_i}{\sum_{k=1}^n z_k \exp \{\beta_i' x_k\}},
\] (3.7)

where, for any \( \beta \in \mathbb{R}^s \), we define

\[
\xi_k(\beta) = \frac{z_k e^{\beta' x_k}}{\sum_{j=1}^n z_j e^{\beta' x_j}}, \quad (k = 1, \ldots, n).
\] (3.8)

It is easy to check that the solutions of (3.6) and (3.7) do maximize \( \ell_i \) whenever \( n_i > 0 \). We can thus summarize the above results as follows.
Proposition 3.1 The maximum likelihood estimators \((\hat{\theta}_i, \hat{\beta}_i)\)' of \((\theta_i, \beta_i)\)' satisfy (3.6) and (3.7) when \(N_i > 0\); otherwise, they are \(\hat{\theta}_i = 0\) and \(\hat{\beta}_i\) can be taken to be 0.

An iterative procedure needs to be employed to solve \(\hat{\beta}_i\) in equation (3.6). For instance, the Newton-Raphson algorithm (cf., [6]) can be utilized. To use this algorithm, define

\[
\gamma_i(\beta) = \sum_{k=1}^{n} x_k [d_{ik}/n_i - \xi_k(\beta)], \quad (i = 1, \ldots, p),
\]

and

\[
I(\beta) = \sum_{k=1}^{n} x_k \otimes^2 \xi_k(\beta) - \left( \sum_{k=1}^{n} x_k \xi_k(\beta) \right) \otimes^2,
\]

where for any column vector \(b\), \(b \otimes^2 = bb'\). The symbol \(\otimes\) is the usual direct product for matrices or vectors. The iteration procedure is given by

\[
\hat{\beta}_i^{(m+1)} \leftarrow \hat{\beta}_i^{(m)} + \left\{ I(\hat{\beta}_i^{(m)}) \right\}^{-1} \gamma_i(\hat{\beta}_i^{(m)}), \quad m = 0, 1, 2, \ldots.
\]

We are also able to obtain from (3.3) the observed Fisher information matrix associated with \((\theta_i, \beta_i)\)'s. Recall that this is the matrix consisting of the negative of the second partial derivatives of \(\ell_i\) with respect to \((\theta_i, \beta_i)\)'s. This is routinely shown to be

\[
I_i((\theta_i, \beta_i)') = \begin{bmatrix}
\frac{n_i}{\theta_i} & \sum_{k=1}^{n} x_k' z_k e^{\beta_i} x_k \\
\sum_{k=1}^{n} x_k z_k e^{\beta_i} x_k & \theta_i \sum_{k=1}^{n} x_k \otimes^2 z_k e^{\beta_i} x_k
\end{bmatrix}.
\]

So far we have seen that the estimators and the observed information matrix of \((\theta_i, \beta_i)\)' do not directly involve the data and parameters of the other components. However, when we now look at the expected Fisher information, which is the expected value of (3.12) taken with respect to the \(Z_k\)'s and the \(\delta_k\)'s, then this matrix depends now on all the parameters. Let us first introduce the following notation.

\[
\eta_{ik} \equiv \eta_{ik}([\theta', B']') = \frac{\theta_i \exp\{\beta_i' x_k\}}{\sum_{j=1}^{p} \theta_j \exp\{\beta_j' x_k\}}, \quad (i = 1, \ldots, p; \quad k = 1, \ldots, n).
\]

Proposition 3.2 The expected Fisher information matrix associated with \(L_i\) is given by

\[
I_i((\theta', B')') = \begin{bmatrix}
\frac{1}{\theta_i} \sum_{k=1}^{n} \eta_{ik} & \frac{1}{\theta_i} \sum_{k=1}^{n} x_k' \eta_{ik} \\
\frac{1}{\theta_i} \sum_{k=1}^{n} x_k \eta_{ik} & \sum_{k=1}^{n} x_k \otimes^2 \eta_{ik}
\end{bmatrix}.
\]
Proof: This follows from (3.12) and Corollary 2.1, since from the latter we obtain that
\[ E\{N_i \mid \mathbf{x}_1, \ldots, \mathbf{x}_n\} = \sum_{k=1}^{p} \eta_{ik} \text{ and } E\{Z_k \mid \mathbf{x}_k\} = \{\sum_{j=1}^{p} \theta_j \exp(\beta_j'\mathbf{x}_k)\}^{-1}. \quad \Box \]

A natural estimator of the \( \eta_{ik} \)'s is obtained by substituting \( \hat{\theta}_i \) and \( \hat{\beta}_i \) for \( \theta_i \) and \( \beta_i \), respectively, in (3.13). The resulting estimators, which are the MLEs, are given by
\[ \hat{\eta}_{ik} = \frac{N_i \exp(\hat{\beta}_i'\mathbf{x}_k)}{\sum_{j=1}^{p} N_j \exp(\hat{\beta}_j'\mathbf{x}_k)}, \quad (i = 1, \ldots, p; \ k = 1, \ldots, n), \quad (3.15) \]
Substituting \( \hat{\theta}_i \) and \( \hat{\eta}_{ik} \) for \( \theta_i \) and \( \eta_{ik} \), respectively, in (3.14) we obtain the MLE of \( \mathcal{L}_i[(\theta', \mathbf{B}')'] \). We will denote this estimator by \( \hat{\mathcal{L}}_i \), \( (i = 1, \ldots, p) \).

3.2 Asymptotics of estimators

To study the asymptotic behavior of our estimators, we now consider the situation where the number of systems being observed becomes large, formally, \( n \to \infty \). The sequence of environment vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots \) will still be assumed fixed. We let \( \mathbf{X}^{(n)} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}, \ (n = 1, 2, \ldots) \), and \( \mathbf{W}_k = (W_{1k}, \ldots, W_{pk})', \ (k = 1, 2, \ldots) \), where
\[ W_{ik} = \delta_{ik} - Z_k \theta_i e^{\beta_i'\mathbf{x}_k} \overset{d}{=} \delta_{ik} - V_k \eta_{ik}, \quad (i = 1, \ldots, p; \ k = 1, 2, \ldots), \quad (3.16) \]
where \( V_1, V_2, \ldots \) are, conditionally on the \( \mathbf{x}_k \)'s, i.i.d. unit exponential variates and independent of the \( \delta_k \)'s, and \( \overset{d}{=} \) denotes 'equal-in-distribution'. From (3.5) we can then express the score function associated with \( \mathcal{L}_i \) as
\[ U_i[(\theta', \mathbf{B}')'] = \sum_{k=1}^{n} \left( \frac{1}{\delta_i} \right) W_{ik}, \quad (i = 1, \ldots, p). \quad (3.17) \]
Let us denote by \( U^{(n)} \), the overall score function, which is the \( p(1+q) \times 1 \) vector consisting of the \( (1+q) \times 1 \) vectors \( U_i, \ (i = 1, \ldots, p) \).

Proposition 3.3  Conditional on \( \mathbf{x}_k \), the mean vector and covariance matrix of \( \mathbf{W}_k \) are given, respectively, by the zero vector and
\[ \mathbf{Y}_k \equiv \mathbf{Y}_k[(\theta', \mathbf{B}')'] = \text{Diag}\{\eta_{1k}, \ldots, \eta_{pk}\}. \]
Consequently, $\mathcal{E}\{U^{(n)} \mid X^{(n)}\} = 0$ and

$$
\text{Cov}\{U^{(n)} \mid X^{(n)}\} = \sum_{k=1}^{n} \text{Diag}\{\Psi_{i_k}, \ldots, \Psi_{p_k}\},
$$

where $\Psi_{ik} = \left(\frac{1}{\delta_i} x_k\right)^{\otimes 2} \eta_{ik}$.

**Remark 3.1** Diag$\{a_1, \ldots, a_p\}$ denotes a $p \times p$ diagonal matrix with diagonal elements $a_1, \ldots, a_p$; while Diag$\{A_1, \ldots, A_p\}$, where $A_i, (i = 1, \ldots, p)$ are square matrices, denotes a block diagonal matrix with 'diagonal' elements $A_i, (i = 1, \ldots, p)$.

**Proof:** From Corollary 2.1 and (3.16) we obtain

$$
\mathcal{E}\{W_{ik} \mid x_k\} = \mathcal{E}\{\delta_{ik} - V_k \eta_{ik} \mid x_k\} = \eta_{ik} - \eta_{ik} = 0,
$$

and

$$
\text{Cov}\{W_{ik}, W_{lk} \mid x_k\} = \text{Cov}\{\delta_{ik}, \delta_{lk} \mid x_k\} + \eta_{ik} \eta_{lk} \text{Var}\{V_k \mid x_k\}, 
$$

where we used the independence of $V_k$ and $\delta_k$ to obtain the last equality. Since $\delta_{ik}$ is multinomial, then $\text{Cov}\{\delta_{ik}, \delta_{lk} \mid x_k\} = \eta_{ik}(1 - \eta_{ik})$ if $i = l$; $-\eta_{ik} \eta_{lk}$ if $i \neq l$. The last term in (3.18) becomes equal to $\eta_{ik} \eta_{lk}$ since $\text{Var}\{V_k \mid x_k\} = 1$, hence $\text{Cov}\{W_{ik}, W_{lk} \mid x_k\} = \eta_{ik}$ if $i = l$; 0 if $i \neq l$. Using (3.17) the results about $U^{(n)}$ then follow. \( \square \)

We are now ready to present simple conditions on the environment vectors in order to achieve asymptotic normality of the overall score function. For our notation below $\| \cdot \|$ will denote the usual Euclidean norm, and

$$
M_n = \max_{k \leq n} \| x_k \|^2, \ n = 1, 2, \ldots
$$

**Theorem 3.1** If the sequence $x_1, x_2, \ldots$ satisfies the conditions

(1) there exists nonnegative definite $(1 + q) \times (1 + q)$ matrices $\overline{\Psi}_i([\theta', B'])$, $(i = 1, \ldots, p)$, with at least one of them positive definite, such that for each $i = 1, \ldots, p$,

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \Psi_{ik}([\theta', B']) = \overline{\Psi}_i([\theta', B']);
$$
(II) $M_n \exp\{-n/\sqrt{M_n}\} \to 0$ as $n \to \infty$; and

(III) there exists a $\kappa > 0$ such that $M_n = o(n^{\kappa/(2+\kappa)})$ as $n \to \infty$;

then

$$
\frac{1}{\sqrt{n}} U^{(n)}[(\theta', B')'] \xrightarrow{d} \text{Normal}_{p(1+q)}(0, \overline{\Psi}[(\theta', B')'] \text{)} as n \to \infty,
$$

where $\overline{\Psi} = \text{Diag}\{\overline{\Psi}_1, \ldots, \overline{\Psi}_p\}$.

**Proof:** From (3.17) we see that $U^{(n)}$ is a sum of independent but (possibly) non-identically distributed random vectors. To prove the theorem, it suffices then to verify the conditions of the multivariate extension of the Lindeberg-Feller central limit theorem (CLT). (See for instance Serfling [7], Theorem B, page 30). From Proposition 3.3 the $k$th summand of $U^{(n)}$ has mean $0$ and covariance matrix $\text{Diag}\{\Psi_{1k}, \ldots, \Psi_{pk}\}$, where we have suppressed their arguments for brevity. Below, we also suppress writing the conditioning on the environment vectors. Condition (I) now guarantees that

$$
\frac{1}{n} \sum_{k=1}^{n} \text{Diag}\{\Psi_{1k}, \ldots, \Psi_{pk}\} \to \overline{\Psi} = \text{Diag}\{\overline{\Psi}_1, \ldots, \overline{\Psi}_p\} as n \to \infty,
$$

which is the first condition of the above-mentioned CLT. Next, we verify the Lindeberg condition, which in our situation becomes

$$
\forall \epsilon > 0, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{E}\left\{ \left[ \sum_{i=1}^{p} \left( \frac{1}{\theta_k^i} + \|x_k\|^2 \right) W_{ik}^2 \right] I\left\{ \sum_{i=1}^{p} \left( \frac{1}{\theta_k^i} + \|x_k\|^2 \right) W_{ik}^2 \geq \epsilon^2 n \right\} \right\} = 0.
$$

(3.19)

Using the inequality $I\{\sum_{i=1}^{p} y_i^2 \geq \epsilon\} \leq \sum_{i=1}^{p} I\{y_i^2 \geq \epsilon/p\}$, $\forall \epsilon > 0$, then to prove (3.19) it would suffice to show that $\forall i, l = 1, \ldots, p$,

$$
\forall \epsilon > 0, \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathcal{E}\left\{ \left[ \left( \frac{1}{\theta_k^i} + \|x_k\|^2 \right) W_{ik}^2 \right] I\left\{ \left( \frac{1}{\theta_k^i} + \|x_k\|^2 \right) W_{ik}^2 \geq \frac{\epsilon^2 n}{p} \right\} \right\} = 0.
$$

(3.20)

From (3.16), and using the facts that $|\delta_{ik}| \leq 1$, $|\eta_{ik}| \leq 1$, and $V_k$'s are i.i.d., it can be shown that

$$
\mathcal{E}\left\{ W_{ik}^2 I\left\{ W_{ik}^2 \geq \frac{\epsilon^2 \theta_k^i n}{p(1 + \theta_k^i \|x_k\|^2)} \right\} \right\} \leq \mathcal{E}\left\{ (1 + V_k^2) I\left\{ 1 + V_k^2 \geq \frac{\epsilon^2 \theta_k^i n}{p(1 + \theta_k^i M_n)} \right\} \right\}.
$$
Consequently, the left-hand-side expression in (3.20) becomes

\[
\frac{1}{n} \sum_{k=1}^{n} \mathcal{E} \left\{ \left[ \left( \frac{1}{\theta_i^2} + \| \mathbf{x}_k \|_2^2 \right) W_{ik}^2 \right] I \left\{ \left( \frac{1}{\theta_i^2} + \| \mathbf{x}_k \|_2^2 \right) W_{ik}^2 \geq \frac{\epsilon^2 n}{p} \right\} \right\} \leq \left( \frac{1}{\theta_i^2} + M_n \right) \mathcal{E} \left\{ (1 + V_i^2) \left[ 1 + V_i^2 \geq \frac{\epsilon^2 \theta_i^2 n}{p(1 + \theta_i^2 M_n)} \right] \right\}. \tag{3.21}
\]

Clearly, if \( M_n = O(1) \) as \( n \to \infty \), the expression in (3.21) will converge to zero as \( n \to \infty \) for all \( \epsilon > 0 \). It remains therefore to consider the case where \( M_n \to \infty \) as \( n \to \infty \), although it is clear that \( M_n = o(n) \) is necessary for (3.21) to converge to 0 as \( n \to \infty \). Consequently, the factor involving \( \theta_i \) in (3.21) need not be considered, since it is dominated by that which involves \( M_n \). Let us denote by

\[
C_n \equiv C_n(p, \theta, \epsilon) = \frac{\epsilon^2 \theta_i^2}{p(1/M_n + \theta_i^2)} - \frac{M_n}{n}
\]

Then

\[
M_n \mathcal{E} \left\{ (1 + V_i^2) \left[ 1 + V_i^2 \geq \frac{\epsilon^2 \theta_i^2 n}{p(1 + \theta_i^2 M_n)} \right] \right\} = M_n \mathcal{P} \{ V_i^2 \geq nC_n/M_n \} + M_n \mathcal{E} \{ V_i^2 \geq nC_n/M_n \}
\leq M_n \exp \left\{ -\sqrt{\frac{nC_n}{M_n}} \right\} + M_n \mathcal{E} \{ V_i^2 + \kappa \} \left[ \frac{nC_n}{M_n} \right]^{-\kappa/2} \tag{3.22}
\]

As \( M_n \to \infty \) with \( M_n = o(n) \), \( C_n = O(1) \) and is also bounded away from 0, hence by condition (II) the first term in (3.22) converges to 0, while condition (III) guarantees that the second term also goes to 0. This completes the proof of the theorem. \( \square \)

**Lemma 3.1** If \( M_n = O(n^{1-\gamma}) \) for some \( \gamma \in (0, \infty) \), then conditions (II) and (III) of Theorem 3.1 are satisfied.

**Proof:** The lemma is clearly true if \( \gamma \geq 1 \). If \( \gamma < 1 \) then condition (II) follows from the fact that \( n^{1-\gamma} \exp \{-n^{\gamma/2}\} \to 0 \) as \( n \to \infty \). Condition (III) is easily seen to be satisfied by taking \( \kappa > 2(1-\gamma)/\gamma \). \( \square \)

**Remark 3.2** Condition (I) of Theorem 3.1 is equivalent to the convergence to finite
limits, as $n \to \infty$, of the elements of $n^{-1} \sum_{k=1}^{n} \eta_{ik}$, $n^{-1} \sum_{k=1}^{n} x_{ik} \eta_{ik}$, and $n^{-1} \sum_{k=1}^{n} x_{ik} \omega_{ik}$. Consequently, the conditions of Theorem 3.1 are quite reasonable as implied by Lemma 3.1 and the preceding remark.

**Theorem 3.2** If the sequence $x_1, x_2, \ldots$ satisfies condition (I) of Theorem 3.1 and the conditions

(I) $n \exp \left\{ -\frac{n}{M_n^{3/2} \exp(\sqrt{M_n})} \right\} \to 0$ as $n \to \infty$;

(II) $M_n^3 \exp(\sqrt{M_n})/n \to 0$ as $n \to \infty$;

(III) there exists a finite number $A$ such that

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \| x_k \|^3 \exp(\| x_k \|) \leq A,$$

then

$$[(\hat{\theta}', \hat{B}')' - (\theta', B')']^{\text{pt}} \circ \text{ o as } n \to \infty.$$

**Proof:** We present the proof of this theorem in the Appendix. □

**Theorem 3.3** If the sequence $x_1, x_2, \ldots$ satisfies all the conditions of Theorem 3.2 and condition (III) of Theorem 3.1, then as $n \to \infty$,

$$\sqrt{n} \left[ (\hat{\theta}', \hat{B}')' - (\theta', B')' \right] \overset{d}{\to} \text{Normal}_{p(1+4)(\theta', B')}.$$ 

**Proof:** This follows from Theorem 2(iv) of [2], the proof of Theorem 3.2, and the fact that condition (III) of Theorem 3.1 guarantees the Lindeberg-type condition (III) of [2] as was verified in the proof of Theorem 3.1. □

**Remark 3.3** If the $x_k$'s satisfy the condition that $\sup_k \| x_k \| < \infty$, which will be typical in real applications, then conditions (II) and (III) of Theorem 3.2 and conditions (I), (II), and (III) of Theorem 3.2 are satisfied.
Remark 3.4 From Theorem 3.1, Proposition 3.3, and the definition of \( \hat{I}_i \) just after equation (3.15), an estimator of the asymptotic covariance matrix of \((\hat{\theta}',\hat{B}')'\) in Theorem 3.3 is given by \( \text{Diag}\{\hat{I}_1,\ldots,\hat{I}_p\} \).

4 Hypotheses tests under constant baseline hazards

Using the asymptotic results in the preceding subsections, hypotheses tests concerning the \( \theta_i \)'s and the \( \beta_i \)'s can be performed using efficient score statistics or Wald-type statistics. Thus to test the null hypothesis that the failure times of the components are unaffected by the environment variables, that is, to test \( H_o : \beta_i = \mathbf{0} \ (i = 1,\ldots,p) \) versus \( H_1 : \beta_i \neq \mathbf{0} \) for some \( i \), one could use the score statistic given by

\[
Q_1 = \sum_{i=1}^{p} \left\{ \sum_{k=1}^{n} x_k' (\delta_{ik} - \hat{\beta}_i Z_k) \left[ \sum_{k=1}^{n} x_k \otimes^2 \hat{\eta}_{ik} \right]^{-1} \left[ \sum_{k=1}^{n} x_k (\delta_{ik} - \hat{\beta}_i Z_k) \right] \right\}.
\]

If the conditions of Theorem 3.1 are satisfied, then under \( H_o \), \( Q_1 \) has an asymptotic chi-square distribution with degrees-of-freedom (df) equal to \( pq \). An asymptotic \( \alpha \)-level test of \( H_o \) is therefore

\[
\text{Reject } H_o \text{ whenever } Q_1 \geq \chi^2_{\alpha,pq}.
\]

A computationally simpler but asymptotically equivalent test statistic to \( Q_1 \) when \( H_o \) holds is obtained by setting \( \hat{\beta}_i = \mathbf{0} \ (i = 1,\ldots,p) \) in (4.1). From (3.13) and (3.8), this yields the statistic

\[
Q_1^* = n \sum_{i=1}^{p} \left\{ \sum_{k=1}^{n} x_k' \left( \frac{\delta_{ik} - Z_k}{N_i} \right) \left[ \sum_{k=1}^{n} x_k \otimes^2 \right]^{-1} \left[ \sum_{k=1}^{n} x_k \left( \frac{\delta_{ik} - Z_k}{N_i} \right) \right] \right\},
\]

where \( S = \sum_{k=1}^{n} Z_k \) is the total time on test of all \( n \) systems. When there is only one environment variable, and this variable is dichotomous taking values 0 or 1, then (4.2) simplifies to

\[
Q_1^* = \frac{m}{n} \sum_{i=1}^{p} \left\{ \frac{\sum_{k=1}^{n} x_k \delta_{ik} / m}{N_i / n} - \frac{\sum_{k=1}^{n} x_k Z_k / m}{S / n} \right\}^2,
\]

(4.3)
where \( m = \sum_{k=1}^{n} x_k \) is the number of systems that operated at environment \( x_k = 1 \). The other quantities above can be interpreted informatively as follows:

- \( \frac{1}{m} \sum_{k=1}^{n} x_k \delta_{ik} = \) proportion in which \( i \)th component failed among systems that operated in environment “1”; 
- \( N_i/n = \) proportion in which \( i \)th component failed among all systems; 
- \( \frac{1}{m} \sum_{k=1}^{n} x_k Z_k = \) average system life of those that operated in environment “1”; and 
- \( S/n = \) average system life of all systems.

The statistic in (4.3) is therefore comparing ratios of averages for those in environment “1” relative to all the systems. Under \( H_0 \), these ratios are expected to be near unity for each \( i = 1, \ldots, p \), hence \( Q_i^* \) in (4.3) will tend to be small.

Another testing problem of interest is that of deciding whether the baseline hazards are identical, that is, testing 

\[
H_0 : \theta_1 = \theta_2 = \ldots = \theta_p \quad \text{versus} \quad H_1 : \theta_i \neq \theta_j \text{ for some } i \neq j.
\]

Under the conditions of Theorem 3.1 and when \( H_0 \) holds, the score statistic

\[
Q_2 = \sum_{i=1}^{p} \left( \frac{N_i}{\hat{\theta}} \right) - \sum_{k=1}^{n} \frac{Z_k \exp\{\hat{\beta}_i' x_k\}}{\left( \sum_{k=1}^{n} \frac{Z_k \exp\{\hat{\beta}_i' x_k\}}{\bar{\theta}^2} \right)^{-1}}
\]  
(4.4)

is asymptotically chi-square with df = \( p - 1 \), where \( \bar{\theta} \) is the ML estimator of the common value of the \( \theta_i \)'s. This estimator is given by

\[
\bar{\theta} = \frac{n}{\sum_{i=1}^{p} \sum_{k=1}^{n} Z_k \exp\{\bar{\beta}_i' x_k\}}
\]  
(4.5)

as can be readily seen from the likelihood function in (3.3). Substituting the expression in (3.15) and (4.5) into (4.4), the test statistic \( Q_2 \) simplifies to

\[
Q_2 = \sum_{i=1}^{p} \left\{ N_i - n \left( \frac{\sum_{k=1}^{n} Z_k \exp\{\hat{\beta}_i' x_k\}}{\sum_{j=1}^{p} \sum_{k=1}^{n} Z_k \exp\{\hat{\beta}_j' x_k\}} \right) \right\}^2 \left\{ \sum_{k=1}^{n} \frac{N_i \exp\{\hat{\beta}_i' x_k\}}{\sum_{j=1}^{p} \sum_{k=1}^{n} N_j \exp\{\hat{\beta}_j' x_k\}} \right\}^{-1}.
\]
The asymptotic $\alpha$-level test for the equality of baseline hazards is therefore

\[ \text{Reject } H_0 : \theta_1 = \ldots = \theta_p \text{ whenever } Q_2 \geq \chi^2_{\alpha,p-1}. \]

A more restrictive type of hypothesis is that the baseline hazards are identical and there are no environmental effects, so symbolically, the hypotheses are

\[ H_0 : \theta_1 = \theta_2 = \ldots = \theta_p, \beta_i = 0, \forall i = 1, \ldots, p, \text{ versus } H_1 : \text{not } H_0. \]

By noting from (3.3) that the ML estimator of the common value of the $\theta_i$'s under $H_0$ is

\[ \hat{\theta} = \frac{n}{\sum_{i=1}^{p} \sum_{k=1}^{n} Z_k} = \frac{n}{pS}, \quad (4.6) \]

the appropriate efficient score statistic for testing $H_0$ is given by

\[ Q_3 = \sum_{i=1}^{p} \left\{ U_i(\hat{\theta}, 0) \right\} \left\{ I_i(\hat{\theta}, 0) \right\}^{-1} \left\{ U_i(\hat{\theta}, 0) \right\} \quad (4.7) \]

where the $U_i$'s and the $I_i$'s are given in (3.5) and (3.14), respectively. If the conditions of Theorem 3.1 are satisfied, then under $H_0$, $Q_3$ is asymptotically chi-square with df $p(q + 1) - 1$. Simplifying the expression for $Q_3$ we find it to be equal to

\[ Q_3 = \sum_{i=1}^{p} \left[ \frac{Es}{n} \left( N_i - \frac{n}{p} \right) \sum_{k=1}^{n} x_k' \left( \delta_{ik} - \frac{nZ_k}{pS} \right) \right] \left[ \frac{Es}{n} \sum_{k=1}^{n} x_k \frac{S}{p} \sum_{k=1}^{n} x_k' \frac{S}{p} \sum_{k=1}^{n} x_k \right]^{-1} \times \left[ \frac{Es}{n} \sum_{k=1}^{n} x_k' \left( \delta_{ik} - \frac{nZ_k}{pS} \right) \right]. \quad (4.8) \]

Notice that this test statistic decomposes into the sum of three terms: the first term can be considered as a statistic for detecting the equality of the baseline hazards; the second term can be viewed as a statistic for determining no environmental effects; while the third term involves a cross-product of the statistics for detecting equality of baseline hazards and no environmental effects. The asymptotic $\alpha$-level test for $H_0$ would therefore reject $H_0$ whenever $Q_3 \geq \chi^2_{\alpha,p(q+1)-1}$.

5 Estimating system reliability at a given environment

A major interest in this regression model is to estimate the system life distribution at some other environment vector $x$. Under the present "constant baseline hazard" assumption
(2.4), the system life $Z$ at environment $x$ has an exponential distribution with parameter $\sum_{i=1}^{p} \theta_i \exp(\beta_i' x)$. A natural estimator of this parameter is
\[
\sum_{i=1}^{p} \hat{\theta}_i \exp(\hat{\beta}_i' x) = \sum_{i=1}^{p} \hat{\theta}_i \exp(\hat{\beta}_i' x). \tag{5.1}
\]
If the conditions of Theorem 3.3 are satisfied, then a simple application of the $\delta$-method (cf., Serfling [7]) shows that
\[
\sum_{i=1}^{p} \hat{\theta}_i \exp(\hat{\beta}_i' x) \sim \text{AN} \left( \sum_{i=1}^{p} \theta_i \exp(\beta_i' x), \frac{1}{n} \sum_{i=1}^{p} \theta_i^2 \exp \left[ 2\beta_i' x (1 \ x') \Psi_i^{-1} \left( \begin{array}{c} 1 \\ x \end{array} \right) \right] \right). \tag{5.2}
\]
The asymptotic variance can then be estimated consistently, using Theorem 3.2, by
\[
\sum_{i=1}^{p} \hat{\theta}_i^2 \exp \left[ 2\hat{\beta}_i' x (1 \ x') \hat{I}_i^{-1} \left( \begin{array}{c} 1 \\ x \end{array} \right) \right]. \tag{5.3}
\]
A natural estimator of the reliability function $\mathcal{P}(Z > t | x)$ is given by
\[
\mathcal{P}(Z > t | x) = \exp \left\{ -t \sum_{i=1}^{p} \hat{\theta}_i \exp(\hat{\beta}_i' x) \right\}. \tag{5.4}
\]
By applying the $\delta$-method, this estimator is asymptotically normal with asymptotic mean $\mathcal{P}(Z > t | x)$ and an asymptotic variance given by
\[
\exp \left\{ -2t \sum_{i=1}^{p} \theta_i \exp(\beta_i' x) \right\} \sum_{i=1}^{p} \theta_i^2 \exp \left[ 2\beta_i' x (1 \ x') I_i^{-1} \left( \begin{array}{c} 1 \\ x \end{array} \right) \right]. \tag{5.5}
\]
which is consistently estimated by
\[
\exp \left\{ -2t \sum_{i=1}^{p} \hat{\theta}_i \exp(\hat{\beta}_i' x) \right\} \sum_{i=1}^{p} \hat{\theta}_i^2 \exp \left[ 2\hat{\beta}_i' x (1 \ x') \hat{I}_i^{-1} \left( \begin{array}{c} 1 \\ x \end{array} \right) \right].
\]

**APPENDIX: Proof of Theorem 3.2**

We prove this theorem by applying Theorem 2(i) of Bradley and Gart [2]. For $(z, d) \in \mathbb{R}_+ \times \{0, 1\}^p$ with $1_p d = 1$, let
\[
f_k(z, d | \theta, B) \equiv f(z, d | \theta, B, x_k) = \exp \left\{ -z \sum_{i=1}^{p} \theta_i e^{\beta_i' x_k} \right\} \prod_{i=1}^{p} (\theta_i e^{\beta_i' x_k})^{d_i}, \ k = 1, 2, \ldots,
\]

which is the joint density function of \((Z_k, \delta_k)\). Note that this is a density function with respect to the product measure \(\mu \times \nu\) on the product space \(\mathbb{R}_+ \times \{0, 1\}^p\), where \(\mu\) is Lebesgue measure on \(\mathbb{R}\), and \(\nu\) is counting measure on \(\{0, 1\}^p\). In the course of this proof, we shall denote by \(\mathcal{E}_k(\cdot)\), \(\text{Var}_k(\cdot)\), and \(\text{Cov}_k(\cdot)\) the expectation, variance, and covariance operators with respect to the density \(f_k(\cdot)\). We will also suppress \(\text{d}(\mu \times \nu)\) when writing integrals. The Kronecker delta function is defined by

\[
\delta_{ij} = 1 \quad \text{if} \quad i = j; \quad = 0 \quad \text{if} \quad i \neq j.
\]

For \(i, j, l \in \{1, \ldots, p\}\) and \(a, b, c \in \{1, \ldots, q\}\), it is straightforward to check that

\[
\frac{\partial \log f_k}{\partial \theta_i} = \frac{W_{ik}}{\theta_i}, \quad \text{and} \quad \frac{\partial \log f_k}{\partial \beta_{ia}} = x_{ak}W_{ik},
\]

\[
\frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} = -\delta_{ij} \frac{d_i}{\theta_i^2}, \quad \frac{\partial^2 \log f_k}{\partial \theta_i \partial \beta_{ja}} = \delta_{ij} x_{ak} x_{b \theta i e \beta_i' x_k}, \quad \text{and} \quad \frac{\partial^2 \log f_k}{\partial \beta_{ia} \partial \beta_{ja}} = \delta_{ij} x_{ak} x_{b \theta i e \beta_i' x_k},
\]

\[
\frac{\partial^3 \log f_k}{\partial \theta_i \partial \theta_j \partial \theta_l} = -\delta_{ij} \delta_{kl} \frac{d_i d_j}{\theta_i^2 \theta_j^2}, \quad \frac{\partial^3 \log f_k}{\partial \theta_i \partial \theta_j \partial \beta_{ia}} = \delta_{ij} \delta_{kl} \frac{x_{ak} x_{bk} x_{ck} x_{e \beta_i' x_k}}{\theta_i^2 \theta_j^2}, \quad \text{and} \quad \frac{\partial^3 \log f_k}{\partial \beta_{ia} \partial \theta_j \partial \beta_{ia}} = -\delta_{ij} \delta_{kl} \frac{x_{ak} x_{bk} x_{ck} x_{e \beta_i' x_k}}{\theta_i^2 \theta_j^2},
\]

where \(W_{ik}\) is defined in (3.16). Thus condition 1(i) of [2] is satisfied. Next, recall from the proof of Proposition 3.3 that \(\mathcal{E}_k(W_{ik}) = 0\) and \(\text{Cov}_k(W_{ik}, W_{jk}) = \delta_{ij} \eta_{ik}\). Consequently,

\[
\int \frac{\partial f_k}{\partial \theta_i} = \mathcal{E}_k \left( \frac{\partial \log f_k}{\partial \theta_i} \right) = \mathcal{E}_k \left( \frac{W_{ik}}{\theta_i} \right) = 0,
\]

\[
\int \frac{\partial f_k}{\partial \beta_{ia}} = \mathcal{E}_k \left( \frac{\partial \log f_k}{\partial \beta_{ia}} \right) = \mathcal{E}_k (x_{ak}W_{ik}) = 0,
\]

\[
\int \frac{\partial^2 f_k}{\partial \theta_i \partial \theta_j} = \mathcal{E}_k \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} \right) + \mathcal{E}_k \left( \frac{\partial \log f_k}{\partial \theta_i} \frac{\partial \log f_k}{\partial \theta_j} \right)
\]

\[
= \mathcal{E}_k \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} \right) + \mathcal{E}_k \left( \frac{W_{ik} W_{jk}}{\theta_i \theta_j} \right)
\]

\[
= -\delta_{ij} \frac{\eta_{ik}}{\theta_i^2} + \delta_{ij} \frac{\eta_{ik}}{\theta_i^2} = 0,
\]

\[
\int \frac{\partial^2 f_k}{\partial \theta_i \partial \beta_{ja}} = \mathcal{E}_k \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \beta_{ja}} \right) + \mathcal{E}_k \left( \frac{\partial \log f_k}{\partial \theta_i} \frac{\partial \log f_k}{\partial \beta_{ja}} \right)
\]

\[
= \mathcal{E}_k \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \beta_{ja}} \right) + \mathcal{E}_k \left( \frac{x_{ak} W_{ik} W_{jk}}{\theta_i} \right)
\]

\[
= -\delta_{ij} \frac{x_{ak} \eta_{ik}}{\theta_i} + \delta_{ij} \frac{x_{ak} \eta_{ik}}{\theta_i} = 0,
\]
and

\[
\int \frac{\partial^2 f_k}{\partial \beta_{i\alpha} \partial \beta_{j\beta}} = \mathcal{E}_k \left( \frac{\partial^2 \log f_k}{\partial \beta_{i\alpha} \partial \beta_{j\beta}} \right) + \mathcal{E}_k \left( \frac{\partial \log f_k}{\partial \beta_{i\alpha}} \frac{\partial \log f_k}{\partial \beta_{j\beta}} \right) \\
= \mathcal{E}_k \left( -\delta^{ij} x_{\alpha k} x_{\beta k} Z_k \theta_i \epsilon \beta_i \epsilon \beta_k \right) + \mathcal{E}_k \left( x_{\alpha k} x_{\beta k} W_k W_{\beta k} \right) \\
= -\delta^{ij} x_{\alpha k} x_{\beta k} \eta_{ik} + \delta^{ij} x_{\alpha k} x_{\beta k} \eta_{ik} = 0.
\]

Also, for a fixed \((\theta', B')\), let \(0 < \epsilon < 1\) such that \(\min_{1 \leq i \leq p} \theta_i - \epsilon > 0\), and define the functions

\[
H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B) = \delta_{ij} \delta_{ij} \frac{2d_i}{(\theta_i - \epsilon)^3} \\
H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B) = \delta_{ij} \delta_{ij} z \| x_k \|^2 \exp\{\beta_i x_k + \epsilon \| x_k \|\}, \\
H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B) = \delta_{ij} \delta_{ij} z \| x_k \|^2 (\theta_i + \epsilon) \exp\{\beta_i x_k + \epsilon \| x_k \|\}.
\]

Then for every \((\theta'', B'')\) such that \(\| (\theta', B') - (\theta'', B'') \| \leq \epsilon\), we have that for every \(z\) and \(d\),

\[
\left| \frac{\partial^3 \log f_k}{\partial \theta_i \partial \theta_j \partial \theta_l} \right|_{(\theta''', B''')} \leq H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B), \\
\left| \frac{\partial^3 \log f_k}{\partial \theta_i \partial \beta_{j\alpha} \partial \beta_{l\beta}} \right|_{(\theta''', B''')} \leq H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B) \\
\left| \frac{\partial^3 \log f_k}{\partial \beta_{i\alpha} \partial \beta_{j\beta} \partial \beta_{l\epsilon}} \right|_{(\theta''', B''')} \leq H^s_{k, \epsilon i \beta, \epsilon j \beta}(z, d; \theta, B).
\]

Furthermore, note that

\[
\mathcal{E}_k \{ H^s_{k, \epsilon i \beta, \epsilon j \beta}(Z, \delta; \theta, B) \} \leq \frac{2}{(\theta_i - \epsilon)^3}, \\
\mathcal{E}_k \{ H^s_{k, \epsilon i \beta, \epsilon j \beta}(Z, \delta; \theta, B) \} \leq \frac{\| x_k \|^2}{\theta_i} \exp\{\epsilon \| x_k \|\}, \\
\mathcal{E}_k \{ H^s_{k, \epsilon i \beta, \epsilon j \beta}(Z, \delta; \theta, B) \} \leq \frac{\| x_k \|^3}{\theta_i} (\theta_i + \epsilon) \exp\{\epsilon \| x_k \|\}.
\]

Each of these bounds are bounded above by

\[
L_k = C(\theta, B) \max\{\| x_k \|^3 \exp\{\epsilon \| x_k \|\}, 1\}, \ k = 1, 2, \ldots,
\]

where \(C\) is a constant depending only on \((\theta, B)\) and \(\epsilon\). We have thus established condition I(ii) of [2].
We now verify condition II(i) of [2]. We have that

\[
\sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{\partial \log f_k}{\partial \theta_i} \right| > n \right\} = \sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{W_{ik}}{\theta_i} \right| > n \right\} \\
\leq \sum_{k=1}^{n} \mathbb{P}\{1 + V_k \eta_{ik} > n \theta_i\} \leq \sum_{k=1}^{n} \mathbb{P}\{1 + V_k > n \theta_i\} \\
= \sum_{k=1}^{n} \exp\{-(n \theta_i - 1)\} = n \exp\{-(n \theta_i - 1)\} \\
\to 0 \text{ as } n \to \infty;
\]

\[
\sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{\partial \log f_k}{\partial \beta_{ia}} \right| > n \right\} = \sum_{k=1}^{n} \mathbb{P}\{x_{ik} W_{ik} > n\} \\
\leq \sum_{k=1}^{n} \mathbb{P}\{1 + V_k > n/\sqrt{M_n}\} = n \exp\{-(n/\sqrt{M_n} - 1)\} \\
\to 0 \text{ as } n \to \infty \text{ by condition (I)};
\]

\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{\partial \log f_k}{\partial \theta_i} \right)^2 I\left\{ \left| \frac{\partial \log f_k}{\partial \theta_i} \right| \leq n \right\} \right\} = \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{W_{ik}^2}{\theta_i^2} I\left\{ \left| \frac{W_{ik}}{\theta_i} \right| \leq n \right\} \right\} \\
\leq \frac{1}{(n \theta_i)^2} \sum_{k=1}^{n} \mathcal{E}_k (W_{ik}^2) \\
= \frac{1}{(n \theta_i)^2} \sum_{k=1}^{n} \eta_{ik} \leq \frac{n}{(n \theta_i)^2} \to 0 \text{ as } n \to \infty;
\]

\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{\partial \log f_k}{\partial \beta_{ia}} \right)^2 I\left\{ \left| \frac{\partial \log f_k}{\partial \beta_{ia}} \right| \leq n \right\} \right\} = \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{W_{ik}^2 x_{ak}^2}{\theta_i^2} I\{\left| x_{ak} W_{ik} \right| \leq n\} \right\} \\
\leq \frac{M_n n}{n^2} = \frac{M_n}{n} \\
\to 0 \text{ as } n \to \infty \text{ by condition (I) or (II)}.
\]

This proves condition II(i) of [2]. To prove their condition II(i), we note that

\[
\sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{\partial^2 \log f_k}{\partial \theta_i^2} \right| > n \right\} = \sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{\delta_{ik}}{\theta_i^2} \right| > n \right\} = 0 \text{ whenever } n > \theta_i^2;
\]

\[
\sum_{k=1}^{n} \mathbb{P}\left\{ \left| \frac{\partial^2 \log f_k}{\partial \theta_i \beta_{ia}} \right| > n \right\} = \sum_{k=1}^{n} \mathbb{P}\left\{ \left| Z_k x_{ak} e_{\beta_i} x_k \right| > n \right\} \\
= \sum_{k=1}^{n} \mathbb{P}\{V_k \eta_{ik} | x_{ik} | / \theta_i > n\} \leq \sum_{k=1}^{n} \mathbb{P}\{V_k > n \theta_i / \sqrt{M_n}\}.
\]
\[
\sum_{k=1}^{n} \mathcal{P}\left\{ \left| \frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} \right| > n \right\} \leq \sum_{k=1}^{n} \mathcal{P}\{ x_{ak} x_{bk} \mid Z_k \theta_i \beta_j' x_k > n \} \\
\leq \sum_{k=1}^{n} \mathcal{P}\{ V_k > n/M_n \} = n \exp(-n/M_n) \\
\rightarrow 0 \text{ by (I).}
\]

Also,
\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} \right)^2 I \left\{ \left| \frac{\partial^2 \log f_k}{\partial \theta_i \partial \theta_j} \right| \leq n \right\} \right\} \leq \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{\delta_{ik}}{\theta_i^4} \left\{ \frac{\delta_{ik}}{\theta_i^2} \leq n \right\} \right\} \\
\leq \frac{1}{(n \theta_i^4)} \rightarrow 0 \text{ as } n \rightarrow \infty,
\]

\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{\partial^2 \log f_k}{\partial \theta_i \partial \beta_j} \right)^2 I \left\{ \left| \frac{\partial^2 \log f_k}{\partial \theta_i \partial \beta_j} \right| \leq n \right\} \right\} \\
\leq \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{x_{ak} x_{bk}}{Z_k \theta_i \beta_j'} x_k \right)^2 I \left\{ \left( \frac{x_{ak} x_{bk}}{Z_k \theta_i \beta_j'} x_k \leq n \right\} \right\} \\
\leq \frac{M_n}{(n \theta_i)^2} \sum_{k=1}^{n} \mathcal{E}_k (V_k^2) \leq \frac{\text{Constant} M_n}{n \theta_i} \rightarrow 0;
\]

and
\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{\partial^2 \log f_k}{\partial \beta_i a \partial \beta_j b} \right)^2 I \left\{ \left| \frac{\partial^2 \log f_k}{\partial \beta_i a \partial \beta_j b} \right| \leq n \right\} \right\} \\
\leq \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( \frac{x_{ak} x_{bk}}{Z_k \theta_i \beta_j'} x_k \right)^2 I \left\{ \left( \frac{x_{ak} x_{bk}}{Z_k \theta_i \beta_j'} x_k \leq n \right\} \right\} \\
\leq \frac{M_n^2}{n^2} \sum_{k=1}^{n} \mathcal{E}_k (V_k^2) = \frac{\text{Constant} M_n^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ by condition (I) or (II).}
\]

With regards to the limiting information matrix, we note that
\[
\frac{1}{n} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{\partial \log f_k}{\partial \theta_i} \frac{\partial \log f_k}{\partial \theta_j} \right\} = \frac{1}{n \theta_i \theta_j} \sum_{k=1}^{n} \mathcal{E}_k (W_{ik} W_{jk}) \\
= \delta_{ij} \frac{1}{n \theta_i \theta_j} \sum_{k=1}^{n} \eta_{ik} = \delta_{ij} \left( \frac{1}{\theta_i \theta_j} \frac{1}{n} \sum_{k=1}^{n} \eta_{ik} \right) \\
\frac{1}{n} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{\partial \log f_k}{\partial \theta_i} \frac{\partial \log f_k}{\partial \beta_j a} \right\} = \frac{x_{ak}}{\theta_i n} \sum_{k=1}^{n} \mathcal{E}_k (W_{ik} W_{jk}) = \frac{\delta_{ij}}{\theta_i n} \sum_{k=1}^{n} x_{ak} \eta_{ik}, \text{ and}
\]
\[
\frac{1}{n} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \frac{\partial \log f_k}{\partial \beta_i a} \frac{\partial \log f_k}{\partial \beta_j b} \right\} = \frac{x_{ak} x_{bk}}{n} \sum_{k=1}^{n} \mathcal{E}_k (W_{ik} W_{jk}) = \frac{\delta_{ij}}{n} \sum_{k=1}^{n} x_{ak} x_{bk} \eta_{ik}.
\]
By condition (I) of Theorem 3.1, the sequences

\[
\{n^{-1} \sum_{k=1}^{n} \eta_{ik}\}, \{n^{-1} \sum_{k=1}^{n} x_{ak} \eta_{ik}\}, \text{ and } \{n^{-1} \sum_{k=1}^{n} x_{ak} x_{ik} \eta_{ik}\}
\]

all converge to finite quantities, which when formed into a \(p(1 + q) \times p(1 + q)\) matrix becomes the limiting information matrix \(\bar{\Psi}\), which by condition (I) of Theorem 3.1 is positive definite. This completes the verification of condition II(i) of [2].

Finally, it remains to show that their condition II(iii) is satisfied. We have that

\[
\sum_{k=1}^{n} \mathcal{P}\{H_{k}^{c}(Z_{k}, \delta_{k}; \theta, B) > n\} \leq \sum_{k=1}^{n} \mathcal{P}\{2\delta_{ik} > n(\theta_{i} - \epsilon)^{3}\} \to 0 \text{ as } n \to \infty;
\]

\[
\sum_{k=1}^{n} \mathcal{P}\{H_{k}^{c}(Z_{k}, \delta_{k}; \theta, B) > n\} \leq \sum_{k=1}^{n} \mathcal{P}\{Z_{k} \parallel x_{k} \parallel^{2} \exp(\beta_{i} x_{k} + \epsilon \parallel x_{k} \parallel) > n\}
\]

\[
= \sum_{k=1}^{n} \mathcal{P}\{V_{k} \parallel x_{k} \parallel^{2} \exp(\epsilon \parallel x_{k} \parallel) > n\theta_{i}\}
\]

\[
\leq \sum_{k=1}^{n} \mathcal{P}\left\{V_{k} > \frac{n\theta_{i}}{M_{n} \exp(\epsilon \sqrt{M_{n}})}\right\}
\]

\[
= n \exp(-n\theta_{i}/\{M_{n} \exp(\epsilon \sqrt{M_{n}})}) \to 0 \text{ as } n \to \infty
\]

by condition (I); and

\[
\sum_{k=1}^{n} \mathcal{P}\{H_{k}^{c}(Z_{k}, \delta_{k}; \theta, B) > n\} \leq \sum_{k=1}^{n} \mathcal{P}\{Z_{k} \parallel x_{k} \parallel^{3} (\theta_{i} + \epsilon) \exp(\beta_{i} x_{k} + \epsilon \parallel x_{k} \parallel) > n\}
\]

\[
= \sum_{k=1}^{n} \mathcal{P}\{V_{k} \parallel x_{k} \parallel^{3} (\theta_{i} + \epsilon) \exp(\epsilon \parallel x_{k} \parallel) > n\theta_{i}\}
\]

\[
\leq \sum_{k=1}^{n} \mathcal{P}\left\{V_{k} > \frac{n\theta_{i}}{M_{n}^{3/2} \exp(\epsilon \sqrt{M_{n}})(\theta_{i} + \epsilon)}\right\}
\]

\[
= n \exp(-n\theta_{i}/\{M_{n}^{3/2} \exp(\epsilon \sqrt{M_{n}})(\theta_{i} + \epsilon)\}) \to 0
\]

as \(N \to \infty\) by condition (I). It is also clear that

\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_{k}\left\{\left(\frac{2\delta_{ik}}{(\theta_{i} - \epsilon)^{3}}\right)^{2} I\left\{\frac{2\delta_{ik}}{(\theta_{i} - \epsilon)^{3}} \leq n\right\}\right\} \to 0 \text{ as } n \to \infty;
\]

\[
\frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_{k}\left\{H_{k}^{2}(Z_{k}, \delta_{k}; \theta, B)^{2} I\left\{H_{k}^{2}(Z_{k}, \delta_{k}; \theta, B) \leq n\right\}\right\}
\]

\[
\leq \frac{M_{n}^{2} \exp(2\epsilon \sqrt{M_{n}})}{(n\theta_{i})^{2}} \sum_{k=1}^{n} \mathcal{E}_{k}(V_{1}^{2})
\]
\[ M_n^2 \exp\left(2\epsilon \sqrt{M_n}\right) \frac{\mathcal{E}_k(V_i^2)}{n \theta_i^2} \]
\[ \to 0 \quad \text{as} \quad n \to \infty \quad \text{by condition (II);} \quad \text{and} \]
\[ \frac{1}{n^2} \sum_{k=1}^{n} \mathcal{E}_k \left\{ \left( H_{k, \delta, B}^i (Z_k, \delta_k; \theta, B) \right)^2 I \left\{ H_{k, \delta, B}^i (Z_k, \delta_k; \theta, B) \leq n \right\} \right\} \]
\[ \leq \frac{M_n^2 \exp(2\epsilon \sqrt{M_n})}{n} \left( \frac{\theta_i + \epsilon}{\theta_i} \right)^2 \mathcal{E}_k(V_i^2) \]
\[ \to 0 \quad \text{as} \quad n \to \infty \quad \text{by condition (II).} \]

The proof of the theorem is now completed by noting that condition (III) guarantees

\[ \frac{1}{n} \sum_{k=1}^{n} L_k = \frac{1}{n} \sum_{k=1}^{n} C(\theta, B) \max \{ \| x_k \|^3 \exp(\epsilon \| x_k \|) \}, 1 \]
\[ \leq \frac{C(\theta, B)}{n} \sum_{k=1}^{n} \max \{ \| x_k \|^3 \exp(\| x_k \|), 1 \} \]
\[ \leq C(\theta, B)[1 + A], \]

which is the last requirement of condition II(iii) of [2]. □.

References


