Brownian Motion on the Continuum Tree

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We construct Brownian motion on a continuum tree, a structure introduced as an asymptotic limit to certain families of finite trees. We approximate the Dirichlet form of Brownian motion on the continuum tree by adjoining one-dimensional Brownian excursions. We study the local times of the resulting diffusion. Using time-change methods, we find explicit expressions for certain hitting probabilities and the mean occupation density of the process.
0. Introduction:

Over the last several years, various authors have extended the theory of continuous Markov processes to inherently rough sets. Barlow and Perkins[8], Lindstrom[15], Hambly[13], and Barlow and Bass[5], [6], [7] have defined Brownian motions on various classes of fractal sets. A characteristic of all these works is that the Brownian motions in question must be defined through approximations by simpler processes, generally random walks on approximating lattices. As a consequence, it is difficult to describe functionals of the resulting Brownian motion explicitly.

In this paper, we approach constructing such Brownian motions from a different angle. Our underlying state space is a continuum tree; think of this a limit at infinity of a finite, graph-theoretical tree of segments in Euclidean space. A natural approach to constructing a Brownian motion on such a state space is to start from Brownian motion on the approximating finite trees, then let the finite trees converge to the continuum limit while suitably rescaling time. It turns out that this can be done rigorously through potential theoretic methods. The details comprise the bulk of this paper.

A standard property of graph-theoretic trees is that any two points \( x, y \) are joined by a unique path \([x, y]\), a property common to the real line. This property allows many standard calculations for linear Brownian motion to be transferred to Brownian motion in the tree setting. As a result, the hitting distribution and the occupation density for Brownian motion on the continuum tree may be described fairly explicitly. Standard results on Brownian local times also carry over directly to the tree.

Brownian motion on the continuum tree was proposed in Aldous[3]; a rigorous construction was not provided, although an intuitive argument for the existence of the process was sketched. This paper formalizes that argument carefully. Besides clarifying Aldous’ argument, we feel that the argument presented here has a couple of other points of interest.

i. Certain fractal subsets of Euclidean space have natural approximations by trees. The construction given here (with a slight modification) will define Brownian motion on such spaces. Many of these fractals do not admit easy approximating lattices.
ii. In Aldous[2], [3], [4] models for random continuum trees are developed, together with applications to family trees of branching process and random combinatorial structures. Brownian motions can be constructed on realizations of these random trees. These may be useful for studying random walk on finite or discrete random trees.

iii. In [3], section 6.3.1, Aldous remarks that local time at the root for Brownian motion on the continuum tree can be recovered from the superprocess constructed over the Ray-Knight diffusion. Other interesting relationships might exist between superprocesses and diffusions on continuum trees.

The remainder of this paper is organized as follows: In the first section, we review the properties of continuum trees that we will need and explicitly define our Brownian motions. In the second section, we construct the Dirichlet space for our process and develop some of its potential-theoretic properties. In the third section, we study additive functionals of our processes, prove that our Brownian motion has jointly continuous local times, and describe the occupation density of our process explicitly. We conclude with some comments on our Brownian motion and some directions for further work.

1. A Review of Continuum Trees:

Let $(S, \mu)$ be a continuum tree, in the sense of Aldous [4], Section 2.3. That is, $S \subseteq \ell^1$ is closed, contains 0, and for every $x, y \in S$, there is a unique path $[x, y]$ connecting $x$ to $y$, of length $\|x - y\|$.

Taking 0 as the root of the tree and $x, y \in S$, define the branch point $b = b(x, y)$ as the point in $[0, x] \cap [0, y]$ of maximal distance from 0. Say that $x \in S$ is in the skeleton of $S$ if there exists $y \in S$ such that $x \in [0, y]$. If $x$ is not in the skeleton of $S$, we say that $x$ is a leaf. Following Aldous[4], we assume

a. If $x_1, x_2, x_3 \in S$ are such that $b(x_1, x_2) = b(x_1, x_3) = b(x_2, x_3) = b$, then at least one of $x_1, x_2, x_3$ equals $b$.

b. $\mu\{ x : x \text{ is a leaf of } S \} = 1$.

c. $\mu\{ y : x \in [0, y] \} > 0$ for all $x$ in the skeleton of $S$.

We also assume
d. \( S \) is compact and the support of \( \mu \) is \( S \). (In the terminology of Aldous\cite{4}, \( (S, \mu) \)

is said to be "leaf-dense".)

e. \( S \) has finite box-counting dimension \( \alpha \).

For convenience, we assume that if \( x = (x_1, x_2, x_3, \ldots) \in S \) then \( (x_1, 0, 0, \ldots),

(0, x_2, 0, \ldots), (x_1, x_2, x_3, 0, \ldots), (x_1, \ldots, x_j, 0, 0, \ldots) \in S \) for all \( j \). (Thus, for any \( x \in S, \ [0, x] = [0, x]_s, \) as defined in Aldous\cite{2}, \cite{4}.) This assumption is not essential to the arguments that follow.

Let \( \{k_1, k_2, \ldots\} \) be the standard unit vector basis for \( \ell^1 \). Let \( s_n = [a_n, b_n] \) be

the maximal segment in the direction \( k_n \) contained in \( S \). Then, \( S_\infty = \bigcup_1^\infty s_n \). As

\( S \) is compact, \( d = \sup_{x,y \in S} d(x,y) < \infty \). Let \( m_i \) be Lebesgue measure on \( s_i \), and

define a measure \( m_0 \) on \( S_\infty \) by taking \( m_0(E) = \sum_{i=1}^\infty 2^{-i} m_i(s_i \cap E) \) for Borel sets \( E \subset \ell^1 \). As \( m_i(s_i) \leq d \) for all \( i \), \( m_0(S) < \infty \).

We call a graph-theoretic tree with exactly \( k \) leaves labeled \( 1, \ldots, k \), such that all

internal nodes have exactly two children a proper \( k \)-tree. If \( S_k = \bigcup_1^k s_j \) then by the

preceding assumptions \( S_k \) is a proper \( k \)-tree. Let \( \mu_k \) be Lebesgue measure on \( S_k \),

normalized so that \( \mu_k(S_k) = 1 \). Then, we assume

f. \( \mu_k \Rightarrow \mu \) as \( k \to \infty \).

Let \( \{X_t\} \) be an \( S \)-valued stochastic process. We make the following

**Definition 1.** \( \{X_t\} \) is an \( (S, \mu) \)-Brownian motion if

i. \( \{X_t\} \) has continuous sample paths.

ii. \( \{X_t\} \) is strong Markov

iii. \( \{X_t\} \) is symmetric with respect to the invariant measure \( \mu \).

iv. For each path \( [a, b] \subset S \) and each \( x \in [a, b] \),

\[
P^x[T_a < T_b] = \frac{d(x, b)}{d(a, b)},
\]

where \( T_z = \inf\{t : X_t = z\}, z \in S \).

v. For \( x, y \in S \), let \( m_{x,y} \) be the mean occupation measure for \( X_t \) started at \( x \) and

run until it first hits \( y \). Then,

\[
m_{x,y}(dz) = 2 d(c(z; [x, y]), y) \mu(dz),
\]
where \((c(z; [x, y]])\) is the point where \([y, x]\) and \([y, z]\) diverge.

2. Constructing the Dirichlet Space:

Our objective is to construct an \((S, \mu)\)-Brownian motion explicitly. To do this, define

\[
\mathcal{E}_i(f, g) = \frac{1}{2} \int_{s_i} \frac{\partial f}{\partial k_i} \frac{\partial g}{\partial k_i} dm_i
\]

\[
\mathcal{E}_\infty(f, g) = \sum_i \mathcal{E}_i(f, g)
\]

\[
\mathcal{D}(\mathcal{E}_\infty) = FC^\infty_0.
\]

\(\frac{\partial f}{\partial k_i}\) is a partial derivative in the direction \(k_i\), in the sense of Gâteau; \(FC^\infty_0\) is the set of functions \(f : \ell^1 \to R\) such that \(f\) depends on a finite set of coordinates \(x_1, \ldots, x_n\) and is a \(C_0^\infty\) function when restricted to these coordinates.

**Theorem 2.** \(\mathcal{E}_\infty\) is a symmetric, regular, local, Markovian, closable form.

**Proof:** \(\mathcal{E}_\infty\) is a monotonic limit of the forms \(\sum_1^k \mathcal{E}_i\). Thus, it suffices to show that \(\sum_1^k \mathcal{E}_i\) is symmetric, regular, local, Markovian, and closable. In turn, it is easy to see that \(\sum_1^k \mathcal{E}_i\) will have the desired properties if each \(\mathcal{E}_i\) does. Routine arguments show that each \(\mathcal{E}_i\) is symmetric, Markovian and local. (See, for example, Fukushima[12], ch. 2.)

To show that each \(\mathcal{E}_i\) is closable, note that \(\mathcal{E}_i(f, g) = \frac{1}{2} \int_{s_i} \frac{\partial f}{\partial k_i} \frac{\partial g}{\partial k_i} dm_i\) is a "classical Dirichlet form" in the sense of Albeverio and Röckner[1]; Theorem 3 of that paper gives necessary and sufficient conditions for the closability of such a form. Examination of \(\mathcal{E}_i\) show that it satisfies these conditions trivially. Thus, each \(\mathcal{E}_i\) is closable.

To see that \(\mathcal{E}_\infty\) is regular, observe that \(FC^\infty_0\) is an algebra of functions, whose restrictions to \(S\) separate points and vanish nowhere. Thus, \(\mathcal{D}(\mathcal{E}_\infty)\) is dense in \(C(S)\), by the Stone-Weierstrass theorem, and \(\mathcal{E}_\infty\) is regular.

**Corollary 3.**

i. Let \(\mathcal{E}\) be the closure of \(\mathcal{E}_\infty\). Then \(\mathcal{E}\) is a Dirichlet form.

ii. There exists an \(m\)-symmetric strong Markov process \(X^0\), with continuous sample paths and Dirichlet form \(\mathcal{E}\).
Proof: $\mathcal{E}$ is a Dirichlet form, by definition. Since $\mathcal{S}$ is compact, ii. follows from Fukushima[12], Theorem 6.22.

Let $\mathcal{D}(\mathcal{E})$ be the domain of $\mathcal{E}$. As $f \in \mathcal{D}(\mathcal{E})$ is bounded and $m_\infty$ is finite, $\mathcal{D}(\mathcal{E}) \subset L^2(m_\infty)$. For $\alpha > 0$ and $f, g \in \mathcal{D}(\mathcal{E})$ let $\mathcal{E}_\alpha(f, g) = \mathcal{E}(f, g) + \alpha \cdot \int fgd\mu_\infty$. It is well-known that $\mathcal{E}_\alpha$ defines an inner product which makes $\mathcal{D}(\mathcal{E})$ a Hilbert space.

Our Brownian motion will ultimately be produced by a time-change of $X^0$. To show that a suitable time change is possible, we prove

Lemma 4.

i. Let $\{f_n\} \subset \mathcal{D}_0$, and suppose $\mathcal{E}_1(f_n - f, f_n - f) \to 0$. Then $\{f_n\}$ is uniformly equicontinuous and $f$ is uniformly continuous.

ii. There exists a constant $C < \infty$ such that $\sup_{x \in \mathcal{S}} f(x) \leq C\mathcal{E}(f, f)$ for all $f$ in $\mathcal{D}$.

iii. All non-empty subsets of $\mathcal{S}$ have positive capacity.

Proof: Let $p, q \in \mathcal{S}_\infty$; without loss of generality, suppose $p, q \in \mathcal{S}_n$. Let $\gamma = \gamma_{pq}$ be the unique path in $\mathcal{S}_n$ connecting $p$ and $q$. If $f \in \mathcal{D}(\mathcal{E}_\infty)$, then by elementary calculus $f(p) - f(q) = \left| \sum_i \int_{s_i} \frac{\partial f}{\partial k_i} 1_\gamma(t)m_i(dt) \right|$. Then,

$$|f(p) - f(q)| = \left| \sum_i \int_{s_i} \frac{\partial f}{\partial k_i} 1_\gamma(t)m_i(dt) \right|$$

$$\leq \left( \sum_i \int_{s_i} \left( \frac{\partial f}{\partial k_i} \right)^2 dm_i \right)^{1/2} \left( \sum_i \int_{s_i} 1_\gamma dm_i \right)^{1/2}$$

$$= \sqrt{2}\mathcal{E}(f, f)^{1/2} \cdot d(p, q)^{1/2}$$

Suppose $\mathcal{E}_1(f_n - f, f_n - f) \to 0$. Then $f_n \to f$ in $L^2(m_\infty)$. Furthermore, since $\mathcal{E}(f_n, f_n)$ is bounded $\{f_n\}$ is uniformly equicontinuous. As $\mathcal{S}$ is compact, we can choose a subsequence $f_{n_k}$ such that $f_{n_k}$ converges uniformly to some limit $\tilde{f}$. But then $f = \tilde{f}$ ($m_\infty$–a. s.). So, we may take $f$ to be uniformly continuous.

ii. Let $\mathcal{B} = \{f \in \mathcal{D} : \mathcal{E}_1(f, f) \leq 1\}$. Suppose that for each $n$ there exists $f_n \in \mathcal{B}$ such that $\sup_{x \in \mathcal{S}} f_n(x) > n$. Then $\mathcal{E}_1(f_n/n, f_n/n) \to 0$, but $\sup_{x \in \mathcal{S}} f_n(x)/n > 1$. This contradicts i..
iii. ii. implies that all point masses \( \{ \delta_x \} \) are of finite energy integral. Measures of finite energy integral charge no polar sets, so all singletons have positive capacity. As \( \text{Cap}() \) is increasing, it follows that all non-empty sets have positive capacity.

From this we get

**Corollary 5.**

i. For every measure \( \nu \) on \( \mathcal{B}(S) \), there exists an additive functional \( A^\nu \) satisfying the relation

\[
\lim_{t \to 0} \frac{1}{t} \int_S h(x) E^x \int_0^t f(s) dA^\nu_s(x) m(dx) = \int_S h(x) f(x) \nu(dx)
\]

for any \( \gamma \)-excessive function \( h \).

ii. Let \( \tau_t = \inf\{ s > 0 : A^\nu_s > t \} \). Then \( X^0_{\tau_t} \) is a \( \nu \)-symmetric Hunt process, with Dirichlet form \( \mathcal{E} \).

**Proof:** i. is Theorem 5.1.3 in Fukushima[12]. ii. is Theorem 8.5 in Silverstein[18].

In particular,

**Corollary 6.** For every \( z \in S \) there exists a local time \( L^z_t \) for \( X^0_t \) at \( z \).

Besides local times, we are interested in two other additive functionals, together with their associated time-changes. First, suppose \( A = A^\mu \), with corresponding time-change \( \tau \). Then \( X_t = X^0(\tau_t) \) is a \( \mu \)-symmetric Hunt process with continuous sample paths. We restate this as

**Corollary 7.** \( \{ X_t \} \) satisfies conditions i., ii., and iii. of Definition 1.

Now, let \( x, y \in S, x \neq y \), and let \( c = d(x,y) \). Define \( \gamma : [0,1] \to [x,y] \) by taking \( \gamma(t) \) to be the unique point \( z \) on \( [x,y] \) such that \( d(x,z)/c = t \). Then, evidently, \( \gamma \) is continuous. Let \( \lambda = \lambda_{[x,y]} \) be the image of Lebesgue measure on \( [0,1] \) under \( \gamma \), let \( A^\lambda \) be the additive functional corresponding to \( \lambda \), and let \( \tau^\lambda_t \) be the time change induced by \( A^\lambda \). \( X^\lambda_t = X^0(\tau^\lambda_t) \) is then a \( \lambda \)-symmetric Markov process, with Dirichlet form \( \mathcal{E} \).
Proposition 8. Let \((\mathcal{F}^\lambda, \mathcal{E}^\lambda_1)\) denote the Dirichlet space associated with \(X^\lambda\). Then \((\mathcal{F}^\lambda, \mathcal{E}^\lambda_1)\) is isometric to the Dirichlet space for Brownian motion on \([0,1]\) reflecting at the endpoints.

Proof: If \(x, y \in \mathcal{S}_\infty\) then it is easy to see that there exist points \(0 = t_0 < v_1 < \ldots < v_k = 1\) such that \(\gamma\) restricted to \([v_j, v_{j+1}]\) is a \(C^\infty\) mapping. For arbitrary points \(x, y \in \mathcal{S}_k\), by letting \(x_k, y_k\) be the closest points in \(\mathcal{S}_k\) to \(x, y\) respectively, we can find an increasing sequence \(\{v_n\}_{n=-\infty}^\infty\) such that \(\lim_{n \to -\infty} v_n = 0, \lim_{n \to -\infty} v_n = 1\), and \(\gamma\) is a \(C^\infty\) mapping when restricted to \([v_n, v_{n+1}]\).

On \([0,1]\), define

\[
\mathcal{E}^*(f, g) = \frac{1}{2} \int_0^1 \frac{df}{dt} \frac{dg}{dt} dt \quad f, g \in \mathcal{D}^*
\]

\(\mathcal{D}^* = \{f: f\text{ restricted to }[v_n, v_{n+1}]\text{ is }C^\infty, n = \ldots, -1, 0, 1, \ldots; f\text{ is constant on }[v_{-n}, v_n]^c\text{ for some }n\}\)

If \(f \in \mathcal{D}_0 \cap \mathcal{F}_\lambda\), then we can define \(\tilde{f}\) on \([0,1]\) by \(\tilde{f}(t) = f(\gamma(t))\). Then, \(\tilde{f} \in \mathcal{D}^*\).

Clearly,

\[
\int_{[x,y]} f^2(z) \lambda(dz) = \int_0^1 f^2(t) dt \quad \mathcal{E}(f, f) = \frac{1}{2} \int_0^1 \left(\frac{d\tilde{f}}{dt}\right)^2 dt
\]

Thus, \(\gamma\) induces an isometry between \((\mathcal{F}^\lambda, \mathcal{E}^\lambda_1)\) and the closure of \((\mathcal{D}^*, \mathcal{E}^*_1)\). To complete the proof of the proposition, we must show \(\mathcal{E}^*\) is a closable form generating reflecting Brownian motion on \([0,1]\).

Let \(f \in C^\infty[0,1]\). Let \(\epsilon > 0\), and choose \(\delta\) sufficiently small that \(|f(x) - f(y)| < \epsilon/2\) for \(|x-y| < \delta\). Choose \(n\) sufficiently large that \(u_n, 1 - v_n < \delta\), and define

\[
f^*(t) = \begin{cases} 
  f(t) & u_n < t < v_n \\
  f(u_n) & 0 \leq t \leq u_n \\
  f(v_n) & v_n \leq t \leq 1
\end{cases}
\]

Then \(f^* \in \mathcal{D}^*\) and \(\mathcal{E}_1^*(f - f^*, f - f^*) < \epsilon\). As \(\mathcal{E}^*\) is closable, \(f \in \mathcal{D}^*\) for all \(f \in C^\infty[0,1]\), and \(\mathcal{E}^*(f, f)\) is given by the usual Dirichlet integral. But this Dirichlet form is associated with reflecting Brownian motion on \([0,1]\). (See, for example, Fukushima[12], Theorem 2.3.1, and the discussion following Theorem 2.3.2.)
As a consequence, if $T^\lambda_t$ is the transition semigroup associated with $X^\lambda$ and $f$ is a continuous, real-valued function on $S$ then $T^\lambda_t f(z) = T_t \tilde{f}(\gamma^{-1} z)$, where $T_t$ is the transition semigroup for reflecting Brownian motion on $[0,1]$. Thus, $\gamma^{-1}(X^0(\tau^\lambda_t))$ is a realization of reflecting Brownian motion on $[0,1]$. In particular, it follows that the hitting distribution and the occupation density for $X^\lambda$ for $a \in [z, w] \subset [x, y]$ are given by

$$P^a[T_z < T_w] = \frac{d(a, w)}{d(z, w)}, \quad m_a(dz) = \begin{cases} 2d(a, x)d(z, y) / d(x, y) \lambda(dz) & z \in [a, y] \\ 2d(z, x)d(a, y) / d(x, y) \lambda(dz) & z \in [a, x] \end{cases}$$

by directly applying the usual formulas for Brownian motion.

**Corollary 9.** $\{X_t\}$ satisfies iv. of Definition 1.

**Proof:** The distribution of hitting places is unaffected by changes in time scale, so iv. follows from Proposition 8.

3. **Analysis of the Mean Occupation Density:**

To show that $X_t$ also satisfies formula (2) of Definition 1, we proceed by a series of stages. We begin with the following

**Lemma 10.** Let $T_k$ be a proper $k$-tree. Let $x, y \in T_k$, let $Y_t$ be a Brownian motion on $T_k$ and let $T_y = \inf \{t : Y_t = y\}$. For any Borel measurable $f$,

$$E^x \int_0^{T_y} f(Y_t) dt = \int_{T_k} f(z) \cdot 2d(c(z; [x, y]), y) m(dy) \quad (3)$$

**Proof:** We proceed by induction on the order $k$ of $T_k$. If $T$ is a 1-tree, then $\{Y_t\}$ is a one-dimensional diffusion on natural scale. The integrand in (3) is then the Green’s function for $\{Y_t\}$ for a process absorbed at $y$ and $\infty$. Since $\{Y_t\}$ has reflecting boundaries at the ends of a branch, (3) follows by making a time change of the diffusion absorbed at $\infty$.

Now, suppose $k = 2$. Let $p$ be the density of $m$ with respect to Lebesgue measure $l$. Then the time change of Brownian motion giving $Y_t$ corresponds to the additive
functional \( f_0^t p(X_s) ds \). So (*) will follow for time-changed Brownian motion if we can establish it for Brownian motion with constant speed.

If \( m \) is constant, then an argument similar to that in Proposition 8 shows that \( Y_t \) is isomorphic to Walsh's Brownian motion on three rays, with equal probability assigned to each ray. So, for the moment, let \( x = j \), the junction of the three rays. For this case, we can write the transition operator for \( \{Y_t\} \) as,

\[
T_t f(x) = \frac{2}{3} T_t^+ \bar{f}(0) + \frac{1}{3} T_t^+ f_3(0)
\]

where \( T_t^+ \) is the transition operator for Brownian motion reflecting at 0 and \( \bar{f}(r) = (1/2)(f_1(r) + f_2(r)) \). (See Barlow, Pitman, and Yor[9] for a thorough discussion of Walsh's Brownian motion in this situation.) This is the same transition operator as for skew Brownian motion started at 0, with parameter \( \alpha = 2/3 \). Applying the speed measure for skew Brownian motion then gives

\[
E^x \int_0^{T_x} f(Y_t) dt = \frac{1}{3} \int_0^0 f_3(r) \cdot 2|r - y| dr + \frac{2}{3} \bar{f}(r) \cdot 2y \, dr = \int_T f(z) \cdot 2d(c(z; [x,y]), y)m(dz),
\]

which is what we wished to show.

For general \( x \in T \), we consider two cases. First, suppose that \( x \in b_3 \). If \( y \in [j, x] \), then the problem reduces to that of the 1-tree. So suppose \( x \in [j, y] \). But examination of the transition operator again shows that it is the same as for skew Brownian motion started at \( x \), with parameter \( \alpha = 2/3 \). Again, (*) follows from applying the speed measure and scale function for skew Brownian motion. Now, suppose \( x \in b_1 \), say. If \( S = \inf \{t : Y_t = j\} \), then applying the strong Markov property plus the result for 1-trees gives

\[
E^x \int_0^{T_x} f(Y_t) dt
\]

\[
= E^x \int_0^{T_x} f(Y_t) dt + E^j \int_0^{T_y} f(Y_t) dt
\]

\[
= \int_{b_1} f_1(z) \cdot 2d(c(z; [x,j]), y)m(dz) + \int_T f(z)2d(c(z; [j,y]), y)m(dz)
\]

\[
= \int_T f(z) \cdot 2d(c(z; [x,y]), y)m(dz),
\]
as desired. The case of \( x \in b_2 \) is similar.

Now, let \( k \geq 2 \) and suppose formula (3) holds for \( k \)-trees. Let \( T \) be a \( k + 1 \)-tree, and let \( b_1, \ldots, b_{k+1} \) denote the branches of \( T \). Without loss of generality, suppose \( x \in b_1 \) and \( y \in b_k \). Define \( T_a \) as the minimal subtree containing \( b_1, b_k \), and \( b_{k+1}, T_b \) as \( T \setminus b_{k+1} \) and \( T_c \) as \( T_a \cap T_b \). Then \( T_b \) is a proper \( k \)-tree, while \( T_a \) is a \( 2 \)-tree and \( T_c \) is either a 1- or a 2-tree.

Define

\[
A_t = \int_0^t 1_{T_a}(Y_s)ds \quad \alpha_t = \inf\{s : A_s > t\}
\]

\[
B_t = \int_0^t 1_{T_b}(Y_s)ds \quad \beta_t = \inf\{s : B_s > t\}
\]

\[
C_t = \int_0^t 1_{T_c}(Y_s)ds \quad \gamma_t = \inf\{s : C_s > t\}
\]

Then \( Y_\alpha, Y_\beta, \) and \( Y_\gamma \) are time-changed Brownian motions on \( T_a, T_b \) and \( T_c \), respectively, with corresponding speed measures \( m(\partial \cap T_a), m(\partial \cap T_b), \) and \( m(\partial \cap T_c) \).

Applying the inductive hypothesis and the results of the first three paragraphs,

\[
E^x \int_0^{T_y} f(X_s)ds
\]

\[
= E^x \int f(X_s)[1_{T_a} + 1_{T_b} - 1_{T_a} \cdot 1_{T_b}](Y_s)ds
\]

\[
= E^x \left[ \int_0^{T_y} f(X_s)dA_s + \int_0^{T_y} f(X_s)dB_s + \int_0^{T_y} f(X_s)dC_s \right]
\]

\[
= \int_{T_a} f(z) \cdot 2d(c(z; [x, y]), y)m(dy) + \int_{T_b} f(z) \cdot 2d(c(z; [x, y]), y)m(dy) \]

\[
- \int_{T_c} f(z) \cdot 2d(c(z; [x, y]), y)m(dy)
\]

\[
= \int_{T} f(z) \cdot 2d(c(z; [x, y]), y)m(dy),
\]

which is what we wanted to prove. (3) then follows for proper \( k \)-trees of any integer order by induction.

We next extend (2) to the process \( X^0 \).
Lemma 11. Let \( x, y \in S_\infty \), and let \( T_y = \inf\{t : X^0_t = y\} \). Then for any Borel measurable \( f \),

\[
E^x \int_0^{T_y} f(X^0_t)dt = \int_{S_\infty} f(z) \cdot 2d(c(z; [x, y]), y)\mu_\infty(dy)
\]

Remark: Since \( m_\infty(S \setminus S_\infty) = 0 \), we can replace the range of integration on the right-hand side with \( S \).

Proof: Recall that we can write \( S_\infty = \bigcup_{n=1}^{\infty} [0, x_n] \) for some sequence \( \{x_n\} \). Then \( S_k = \bigcup_{n=1}^{k} [0, x_n] \) is a proper \( k \)-tree. Without loss of generality, suppose \( x \in [0, x_1] \).

For \( k = 1, 2, \ldots \), let \( A^k_t = \int_0^t 1_{S_k}(X^0_s)ds \), \( \tau^k_t = \inf\{s : A^k_s > t\} \). Then \( X^0(\tau^k_t) \) is a time-changed Brownian motion on \( S_k \), with speed measure \( \mu_0(\circ \cap S_k) \). By Lemma 10, if \( f \) is a positive measurable function on \( S_\infty \),

\[
E^x \int_0^{T_y} f(X^0_s)1_{S_k}(X_s)ds = E^\tau \int_0^{T_y} f(X(\tau^k_s))ds
= \int_{S_\infty} f(z)2d(c(z; [x, y]), y)1_{S_k}(z)\mu_0(dz)
\]  \( \tag{4} \)

Let \( k \to \infty \) and apply the monotone convergence theorem to both sides of (4) to prove the lemma.

From this we have the following

Corollary 12. For \( x, y \in S \), \( E^xT_y \leq C \cdot d(x, y) \).

Proof: Recall that \( \mu_0(S) < \infty \). Let \( x, y \in S_\infty \) and apply the preceding lemma twice with \( f \equiv 1 \) to get

\[
E^xT_y = E^x \int_0^{T_y} 1 \cdot ds = \int_S 1 \cdot 2d(c(z; [x, y]), y)\mu_0(dz)
\]

\[
E^yT_x = \int_S 1 \cdot 2d(c(z; [x, y]), x)\mu_0(dz)
\]

Thus,

\[
E^xT_y + E^yT_x = \int_S 2d(x, y)\mu_0(dz) = 2d(x, y)\mu_0(S_0)
\]
For \( x \in \mathcal{S} \setminus \mathcal{S}_\infty, \ y \in \mathcal{S}_\infty \), let \( x_k \) be the nearest point to \( x \) in \( \mathcal{S}_k \). Then \( x_k \in \[x, y\] \) for all \( k \) and \( x_k \to x \) as \( k \to \infty \). Let \( T_k = T_{x_k} \). By sample path continuity, \( T_k \uparrow T_x \), \((P^y\text{-}a.s.) By the strong Markov property,

\[
E^z \int_0^{T_y} 1_{S_k}(X^0_s) ds = E^{x_k} \int_0^{T_y} 1_{S_k}(X^0_s) ds = \int_S 1_{S_k}(z) 2d(c(z; [x, y]), x) \mu_0(dz).
\]

Letting \( k \to \infty \) gives

\[
E^z \int_0^{T_y} 1_{S_k}(X^0_s) ds \to E^z \int_0^{T_y} 1_{S_\infty}(X^0_s) ds = E^z \int_0^{T_y} 1 \cdot c\mathcal{L}_S
\]

\[
\int_S 1_{S_k}(z) 2d(c(z; [x, y]), x) \mu_0(dz) \to \int_S 1_{S_\infty}(z) 2d(c(z; [x, y]), x) \mu_0(dz)
\]

\[
= \int_S 2d(c(z; [x, y]), x) \mu_0(dz).
\]

On the other hand,

\[
E^yT_x = \lim_{k \to \infty} E^yT_k = \lim_{k \to \infty} \int_S 2d(c(z; [x_k, y]), x_k) \mu_0(dz)
\]

\[
= \int_S 1 \cdot 2d(c(z; [x, y]), x) \mu_0(dz)
\]

Thus, \( E^zT_y + E^yT_x = 2d(x, y)\mu_0(\mathcal{S}_0) \), as before. A similar argument gives \( E^zT_y + E^yT_x = 2d(x, y)\mu_0(\mathcal{S}_0) \) for arbitrary \( y \) in \( \mathcal{S} \). \( \blacksquare \)

This estimate on the expected hitting times is extremely strong. In particular, it implies that the local times of \( X^0 \) are jointly continuous. We state this result as a separate theorem.

**Theorem 13.** The local times \( \{L^z_t\} \) of \( X^0 \) may be chosen to jointly continuous in \((z, t)\).

**Proof:** We prove this by adapting the argument in Blumenthal and Getoor [10], Theorem V.3.30. From Blumenthal and Getoor, Theorem V.3.28, we have the estimate

\[
P \left[ \sup_{0 \leq t \leq N} |L^a_t \cdot L^b_t| > 2\delta \right] \leq 2e^{-\delta^2 / \gamma},
\]

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where \( \delta > 0 \) is arbitrary and \( \gamma^2 = 1 - E^a e^{-T_b} \cdot E^b e^{-T_a} \). An elementary estimate gives \( E^a e^{-T_b} \geq 1 - E^a (T_b \wedge 1) \), so

\[
\gamma^2 \leq 1 - [1 - E^a (T_b \wedge 1)] \cdot [1 - E^b (T_a \wedge 1)] \\
\leq E^a (T_b \wedge 1) + E^b (T_a \wedge 1) \\
\leq E^a T_b + E^b T_a = 2 \mu_0(S) d(a, b).
\]

Choose a family of finite sets \( \{D_n\}_{n=1}^\infty \) such that \( D_n \subset D_{n+1}, \ n = 1, 2, \ldots \), and \( S \subset \cup_{x \in D_n} N(x, 2^{-n}) \). As \( S \) has finite metric entropy \( \lambda \), if \( \lambda > \alpha \), we can choose \( \{D_n\} \) so that \( |D_n| < C_2 \times 2^{n \lambda}, \ n = 1, 2, \ldots \), for some constant \( C_2 \).

Choose \( \theta > 2^{2\lambda} \). For \( n = 1, 2, \ldots \), let \( \delta_n = (\log \theta) \times \sqrt{C_0} \times n \times \frac{1}{\sqrt{3 \cdot 2^{-n}}} \). If \( d(a, b) < 3 \cdot 2^{-n} \), then

\[
P \left[ \sup_{0 \leq t \leq N} |L_t^a - L_t^b| > 2\delta_n \right] \leq 2e^N \exp (- \frac{\delta_n}{\sqrt{C_0 d(a, b)}}) \leq 2e^N e^{-n \log \theta}.
\]

Applying Boole's inequality,

\[
P \left[ \sup_{a, b \in D_n} \sup_{0 \leq t \leq N} |L_t^a - L_t^b| > 2\delta_n \right] \leq \frac{(C_1 \cdot 2^{n \lambda})^2 \cdot 2e^N}{2 \cdot e^n \log \theta} = C_1 e^N \left( \frac{2^{2\lambda}}{\theta} \right)^n.
\]

This sequence sums over \( n \), so we apply the Borel-Cantelli lemma to find \( \nu = \nu(\omega) \) such that \( \sup_{0 \leq t \leq N} |L_t^a(\omega) - L_t^b(\omega)| \leq 2\delta_n \) for all \( n \geq \nu(\omega) \), \( a, b \in D_n \), \( d(a, b) \leq 3 \cdot 2^{-n} \).

Now, let \( D = \cup_{j=0}^\infty D_n \), let \( a, b \in D \) and suppose \( d(a, b) \leq 2^{-m} \). There exist \( a_n, b_n, n = m, m+1, \ldots \) such that \( a_n, b_n \in D_n \), \( d(a, a_n), d(b, b_n) < 2^{-n} \). Clearly, \( d(a_m, b_m) < 3 \cdot 2^{-m} \) and \( d(a_n, a_{n+1}), d(b_n, b_{n+1}) < 3 \cdot 2^{-(n+1)} \), \( n = m, m+1, \ldots \).

Then,

\[
\sup_{0 \leq t \leq N} |L_t^a - L_t^b| \leq \sup_{0 \leq t \leq N} \left( |L_t^a - L_t^a_m| + |L_t^a_m - L_t^b_m| + |L_t^b_m - L_t^b| \right) \\
\leq 2 \sum_{j=m+1}^\infty (\log \theta)j \sqrt{C_0 \sqrt{3 \cdot 2^{-j}}} + (\log \theta)m \sqrt{C_0 \sqrt{3 \cdot 2^{-m}}}.
\]
provided \( m \geq \nu \). The series sums, so \( \sup_{0 \leq t \leq N} |L^a_t - L^b_t| \to 0 \) (a. s.) \( \Rightarrow b \to a \) in \( D \). It follows that \( L^a_{t \in D} \) is uniformly continuous. Thus, it can be extended to a continuous function \( L^a_t : S \to \mathbb{R} \). But \( \tilde{L}^a_t = L^a_t \) almost surely.

We have already remarked that for any measure \( \nu \) on \( S \) we can define a \( \nu \)-symmetric Hunt process with continuous sample paths as a time-change of \( X^{\sigma} \). This time change corresponds to an additive functional \( A^\nu \) with Revuz measure \( \nu \). As the local times \( \{L^a_t\} \) are jointly continuous we can express \( A^\nu \) explicitly as

\[
A^\nu_t = \int_S L^a_t \nu(dz)
\]

by applying the representation theorem for additive functionals. (See, for example, Sharpe [17], Exercise 75.2).

Recall that \( S_k = \bigcup_{n=1}^k x_n \) and \( \mu_k \) is normalized Lebesgue measure on \( S_k \). By hypothesis, \( \mu_k \Rightarrow \mu \). If we let \( A^k_t = \int_S L^a_t \mu_k(dz) \) and \( A_t = \int_S L^a_t \mu(dz) \), then \( A^k_t(\omega) \to A_t(\omega) \) for each \( t \) and each sample path \( \omega \) such that \( \{L^a_t\} \) is jointly continuous.

We now state the main result of this section.

**Theorem 14.** \( \{X_t\} \) satisfies formula (2).

**Proof:** It suffices to show (2) for positive continuous functions \( f \). Furthermore, without loss of generality we can take \( y = 0 \); henceforth, let \( T = T_0 \). For \( k = 1, 2, \ldots \), take \( A^k_t \) and \( A_t \) as defined above and let \( \tau^k_t \) and \( \tau_t \) be the corresponding time-changes. Let \( X^k_t = X^0(\tau^k_t) \). Then, by Lemma 10,

\[
E^x \left( \int_0^T f(X^k_s) ds \right) = \int_S f(z) \cdot 2d(c(z, [x, 0]), 0) \mu_k(dz)
\]

As \( k \to \infty \), the right-hand side of (6) converges to \( \int_S f(z)2d(c(z, [x, 0]), 0) \mu(dz) \)

Now consider the left-hand side of (6). For any \( t \) and any \( \omega \) such that \( X^0_t(\omega) \) is continuous in \( t \) and \( L^a_t(\omega) \) is jointly continuous in \( (z, t) \), we have

\[
\int_0^T f(X^k_s) ds = \int_0^T f(X^0(\tau^k_s)) ds = \int_0^T f(X^0_s) dA^k_s
\]

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As $k \to \infty$, we have
\[
\int_0^T f(X^0_s) dA^k_s \to \int_0^T f(X^0_s) dA_s = \int_0^T f(X_s) ds.
\]

It remains to show that $\{ \int_0^T f(X^0_s) dA^k_s \}$ converges in $L^1$. To do this, first note that $f$ is bounded on $S$, so
\[
\left| \int_0^T f(X_s) ds \right| \leq \int_0^T |f(X^0_s)| dA_s \leq \|f\|A_T
\]

We shall show that $E^z(A^k_T)^2$ is bounded. It will follow that $E^z(\int_0^T f(X^0_s) dA^k_s)^2$ is bounded and that $\{ \int_0^T f(X^0_s) dA^k_s \}$ is uniformly integrable. We prove

**Lemma 15.** For any $z \in S$, $E^z(L_T^z)^2 \leq 16 \times d^2$.

**Proof:** Let $z \in S$, and let $v = c(z; \|x, 0\|)$. If $v = 0$ then necessarily $L_T^z = 0$ a. s.

So suppose $v \in [0, z[$.

By proposition 8, $X^0_t$ restricted to $[0, z]$ is a time-change of linear Brownian motion. Let $\beta = d(0, v)$, $\zeta = d(0, z)$. Applying the strong Markov property and the representation theorem, we see that $L_T^z$ is the same as the local time for one-dimensional Brownian motion started at $\beta$ and run until it first reaches $0$.

By the Ray-Knight theorem, $\{ L^w_T : w \in [0, \zeta] \}$ evolves as a solution to the stochastic integral equation
\[
Z_w = \int_0^w (Z_u^+)^{1/2} dW_u + 2(w \wedge \beta),
\]

Applying the Ikeda-Watanabe comparison theorem (See Rogers and Williams [16], Section V.43), we can bound $Z_w$ pathwise by $\tilde{Z}_w$, where $\tilde{Z}$ satisfies the equation
\[
\tilde{Z}_w = \int_0^w (\tilde{Z}_u^+)^{1/2} dW_u + 2w.
\]

$\tilde{Z}_w$ is a BESQ$^2(0)$ process; thus, we can write $\tilde{Z}_w = (B^1_w)^2 + (B^2_w)^2$, where $B^1_w$ and $B^2_w$ are independent Brownian motions. Then
\[
E(\sup_{u \in [0, z]} L^w_T)^2 \leq E(\sup_{0 \leq w \leq \zeta} L^w_T)^2 \leq E(\sup_{0 \leq w \leq \zeta} \tilde{Z}_w^2)
\]
\[
\leq 2E(\sup_{0 \leq v \leq \zeta} (B^1_v)^4 + \sup_{0 \leq v \leq \zeta} (B^2_v)^4)
\]
\[
= \frac{16}{3} E(B^1_\zeta)^4 = \frac{16}{3} \cdot 3\zeta^2 \leq 16d^2 \]

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Returning to the proof of the main theorem, we have
\[
E^x (A_T^k)^2 = E^x \left( \int_S L^k \mu_k (dz) \right)^2 \leq E^x \int_S (L^k)^2 \mu_k (dz) \\
\leq \int_S 16d^2 \cdot \mu_k (ds) = 16d^2
\]
Thus, for \( f : S \to \mathbb{R}^+ \) continuous, \( \{\int_0^T f(X_s^k) ds\} \) is uniformly integrable. Therefore,
\[
E^x \left( \int_0^T f(X_s) ds \right) = \lim_{k \to \infty} E^x \left( \int_0^T f(X_s^k) ds \right) \\
= \lim_{k \to \infty} \int_S f(z) 2d(c(z; [x, 0], 0) \mu_k (dz) \\
= \int_S f(z) 2d(c(z; [x, 0], 0) \mu (dz)
\]
which is what we wished to prove.

4. **Concluding Remarks:**

1. In Aldous[2], Aldous proposes that Brownian motion on the continuum tree might be constructed by starting from Brownian motion on a proper \( k \)-tree and steadily adjoining excursions on edges until we have Brownian motion on the skeleton \( S_\infty \). Letting \( l(\infty, \tau, x) \) for \( x \in S_\infty \) be the local time at \( x \) when the local time at the origin reaches \( \tau \), Aldous remarks that if \( l(\infty, \tau, x) \) extends to a bounded continuous function on \( S \), then this will give rise to a process \( X' (t) \) on \( S \) which can be time-changed to give Brownian motion on \( S \).

Reviewing the flow of this paper, the Dirichlet form \( \mathcal{E} \) in section 3 is exactly the right form for Brownian excursions joined together on the branches, although working out the details is tedious. The Dirichlet form by itself, together with Lemma 4, gives a suitable Markov process on \( S \). Joint continuity of local times is used only to show that the resulting process has the “right” occupation density for Brownian motion. The Dirichlet form mechanics thus reduce the technicalities to manageable proportions, at the cost of a loss of intuition.

2. Certain fractal sets have a naturally tree–like structure, indicating that this construction might apply to them. For example, the Vicsec snowflake in the plane
can be constructed by the following procedure: In $\mathcal{R}^2$ let $I_h = \{(x, 0) : -1 \leq x \leq 1\}$, $I_v = \{(0, y) : -1 \leq y \leq 1\}$ and let $T_0 = I_h \cup I_v$. Define transformations

$S_1 : (x, y) \rightarrow \left(\frac{x}{3}, \frac{y}{3}\right)$
$S_2 : (x, y) \rightarrow \left(\frac{x}{3}, \frac{y}{3}\right) + \left(\frac{2}{3}, 0\right)$
$S_3 : (x, y) \rightarrow \left(\frac{x}{3}, \frac{y}{3}\right) - \left(\frac{2}{3}, 0\right)$
$S_4 : (x, y) \rightarrow \left(\frac{x}{3}, \frac{y}{3}\right) + \left(0, \frac{2}{3}\right)$
$S_5 : (x, y) \rightarrow \left(\frac{x}{3}, \frac{y}{3}\right) - \left(0, \frac{2}{3}\right)$

Then, $S_1, \ldots, S_5$ are contractive similitudes with fixed points $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$, respectively. For $X \subset \mathcal{R}^2$ let $S(X) = \bigcup_{i=1}^{5} S_i(X)$. Then, it is known that there is a unique compact set $\Gamma$ such that $S(\Gamma) = \Gamma$. Furthermore, for bounded sets $X$, $S^n(X)$ converges to $\Gamma$ in Hausdorff metric. (See Falconer [11]).

Take $\mu$ to be Hausdorff $\log_3 5$–dimensional measure, restricted to $\Gamma$. Then, it is not hard to verify that $(\Gamma, \mu)$ satisfies assumptions b., c., d., and e. of section 1. A result in Hutchinson[14] shows that $(\Gamma, \mu)$ also satisfies assumption f. Clearly $(\Gamma, \mu)$ fails assumption a. However, examining results in this paper shows that assumption a. is only used in the proof of Lemma 10, in showing that Brownian motion on a proper 2–tree satisfies the occupation density formula 2. If assumption a. is relaxed to allow branch points with some finite number $r$ of branches, then a slight modification of the argument in Lemma 10 will allow us to construct a process $X(t)$ satisfying conditions i.–v. of Definition 1.

If we now take

$S'_1 : (x, y) \rightarrow \left(\frac{x}{2}, \frac{y}{2}\right)$
$S'_2 : (x, y) \rightarrow \left(\frac{x}{4}, \frac{y}{4}\right) + \left(\frac{3}{4}, 0\right)$
$S'_3 : (x, y) \rightarrow \left(\frac{x}{4}, \frac{y}{4}\right) - \left(\frac{3}{4}, 0\right)$
$S'_4 : (x, y) \rightarrow \left(\frac{x}{4}, \frac{y}{4}\right) + \left(0, \frac{3}{4}\right)$
$S'_5 : (x, y) \rightarrow \left(\frac{x}{4}, \frac{y}{4}\right) - \left(0, \frac{3}{4}\right)$

and define $S'(X)$ correspondingly for $X \subset \mathcal{R}^2$, then $S'^n(T_0)$ again converges to a fractal, with dimension $\log_2 8/(\sqrt{17} - 1))$. The construction discussed in this paper, with the modification in the previous paragraph, gives a Brownian motion $X'(t)$ on
$\Gamma'$. Despite its simplicity, this falls outside the range of Brownian motions covered by Lindstrøm[15].

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5. References:


New York.


