Bayesian Statistics

Debdeep Pati
Florida State University

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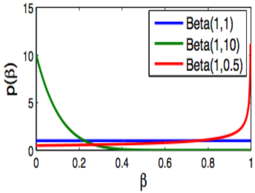
Stick Breaking construction for $G \sim DP(\cdot, \alpha, G_0)$

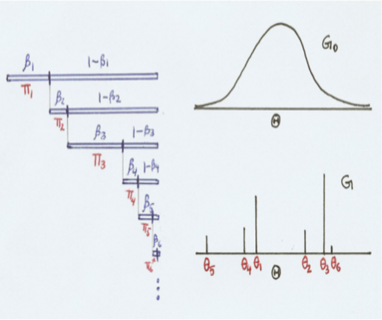
Stick-Breaking Formula

$$\pi_k = \beta_k \prod_{l=1} (1 - \beta_l), \beta_k \sim Beta(1, \alpha),$$

$$\theta_k^* \sim G_0, G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

Stick Breaking





Sketch of the proof (Sethuraman, 1994)

- Recall the posterior process

$$\left[\begin{array}{c} G \sim DP(\cdot \mid \alpha, G_0) \\ \theta \mid G \sim G \end{array}\right] \Leftrightarrow \left[\begin{array}{c} \theta \sim G_0 \\ G \mid \theta \sim DP\left(\cdot, \alpha+1, \frac{\alpha G_0 + \delta_\theta}{\alpha+1}\right) \end{array}\right]$$

- Consider a partition $(\theta, \Theta \backslash \theta)$ of Θ. We have

$$\begin{aligned} (G(\theta), G(\Theta\backslash\theta)) &\sim \mathrm{Diri}\left\{(\alpha+1)\frac{\alpha G_0 + \delta_\theta}{\alpha+1}(\theta), (\alpha+1)\frac{\alpha G_0 + \delta_\theta}{\alpha+1}(\Theta\backslash\theta)\right\} \\ &= \mathrm{Beta}(1, \alpha) \end{aligned}$$

- G has a point mass located at θ:

$$G = \beta\delta_\theta + (1-\beta)G', \quad \beta \sim \mathrm{Beta}(1, \alpha)$$

and G' is the renormalized probability measure with the point mass removed.

- What is G'?

Sketch of the proof : What is G'

- Consider a further partition of $(\theta, A_1, \ldots, A_K)$ of Θ.

$$\begin{aligned} (G(\theta), G(A_1), \ldots, G(A_K)) &= (\beta, (1-\beta)G'(A_1), \ldots, (1-\beta)G'(A_K)) \\ &\sim \mathrm{Diri}(1, \alpha G_0(A_1), \ldots, \alpha G_0(A_K)) \end{aligned}$$

- Renomalizing

$$\begin{aligned} (G'(A_1), \ldots, G'(A_K)) \mid \theta &= \mathrm{Diri}(\alpha G_0(A_1), \ldots, \alpha G_0(A_K)) \\ G' &\sim DP(\cdot, \alpha, G_0) \end{aligned}$$

Sketch of the proof

$$
\begin{aligned}
G &\sim DP(\cdot, \alpha, G_0) \\
G &= \beta_1 \delta_{\theta_1^*} + (1 - \beta_1) G_1 \\
G &= \beta_1 \delta_{\theta_1^*} + (1 - \beta_1)(\beta_2 \delta_{\theta_2^*} + (1 - \beta_2) G_2) \\
&\vdots \\
G &= \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}
\end{aligned}
$$

where $\pi_k = \beta \prod_{l=1}(1 - \beta_l), \beta_k \sim Beta(1, \alpha), \theta_k^* \sim G_0$.

Finite Mixture Models

- Finite mixture models are useful in a wide variety of settings, including density estimation, clustering, classification, etc
- Focus initially on problem in which $f(y)$,for $y \in \mathbb{R}$,is an unknown density function
- Finite mixture of Gaussians provides a flexible choice,

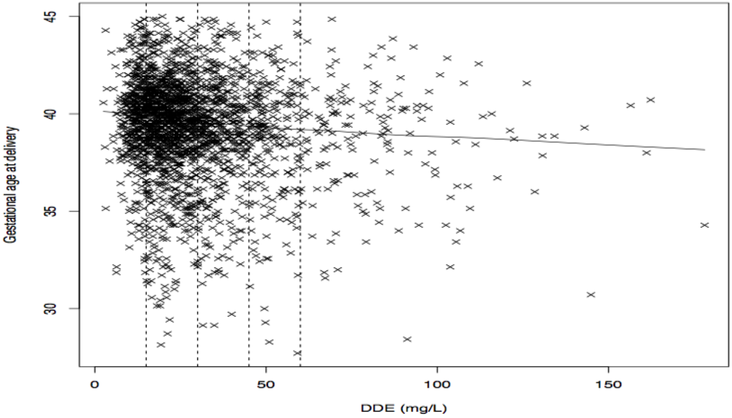
$$f(y) = \sum_{h=1}^{k} \pi_h N(y; \mu_h, \tau_h^{-1})$$

- It is well known that a mixture of normals can approximate any smooth density

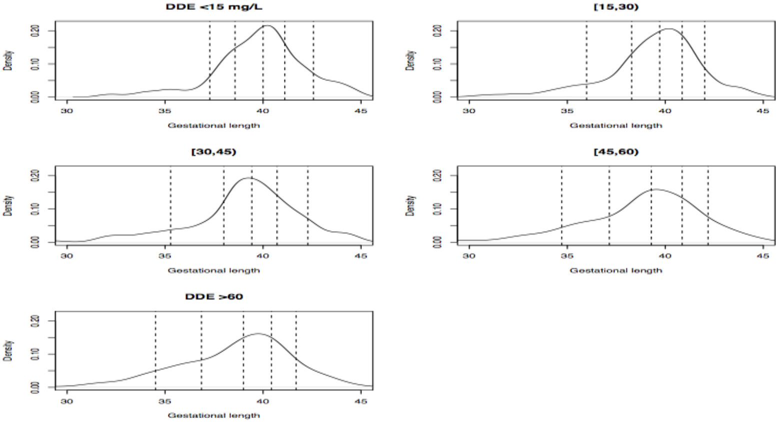
Application: Modeling length of gestation

- Preterm birth is a major public health problem leading to substantial mortality & short and long-term morbidity
- Preterm birth is typically defined as a delivery occurring prior to 37 weeks of completed gestation
- This cutoff is somewhat arbitrary & the shorter the length of gestation, the more adverse the associated health effects
- Appealing to model the distribution of gestational age at delivery as unknown & then allow predictors to impact this distribution

Gestational Length vs. DDE(mg/L)



Gestational Length Densities within DDE Categories



Comments on Gestational Length Data

- Data are non-Gaussian with a left skew
- Not straightforward to transform the data to approximate normality
- A different transformation would be needed within each DDE category
- First question: how to characterize gestational age at delivery distribution without considering predictors?

Mixture Models

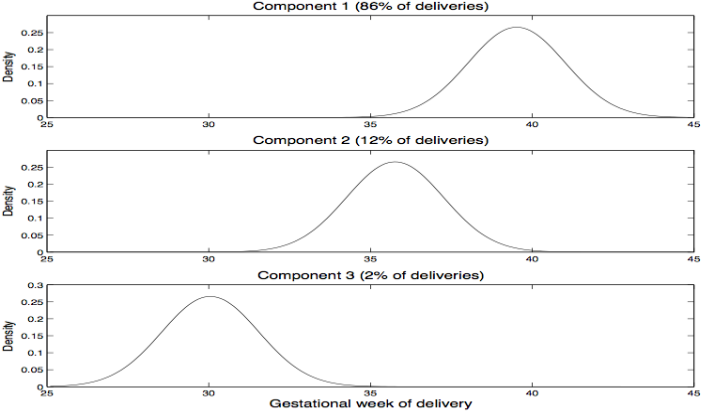
- Initially ignoring DDE
- Letting y_i = gestational age at delivery for woman i,

$$f(y_i) = \int \mathrm{N}(y_i; \mu, \sigma^2) dG(\mu, \sigma^2),$$

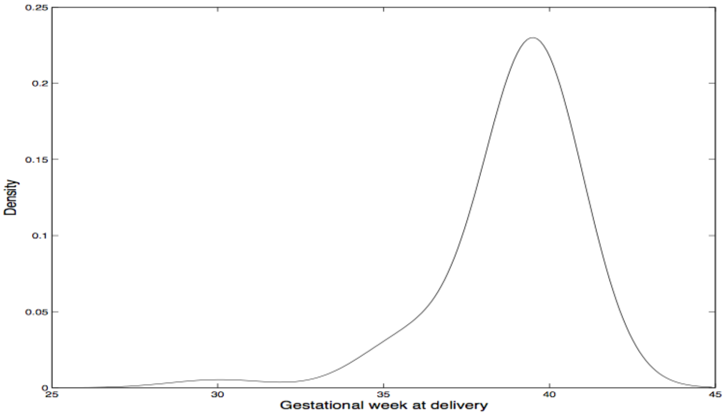
  where G = mixture distribution for $\theta = (\mu, \sigma^2)$
- Mixtures of normals can approximate any smooth density
- Finite location mixture with k components one possibility:

$$f(y) = \sum_{h=1}^{k} \pi_h N(y; \mu_h, \tau_h^{-1})$$

Mixture components for gestational age at delivery



Mixture-based density of gestational age at delivery



Some comments on mixture models

- k = 3 component mixture provides a good fit to gestational age at delivery data.
- Can be fit easily using the EM algorithm for maximum likelihood or Gibbs sampling for Bayesian inference.
- Ill focus on the Gibbs sampling approach here

Finite mixture model

- The finite mixture of normals can be equivalently expressed as

$$y_i \sim N(\mu_{S_i}; \tau_{S_i}^{-1}), S_i \sim \sum_{h=1}^{k} \pi_h \delta_h$$

δ_h = probability measure concentrated at the integer h, $S_i \in \{1, 2, \ldots, k\}$ indexes the mixture component for subject i, $i = 1, \ldots, n$.

- A prior on $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ and $(\mu_h, \tau_h), h = 1, \ldots, k$ is given by

$$\begin{aligned} \pi &\sim \mathrm{Dir}(a_1, a_2, \ldots, a_k) \\ (\mu_h, \tau_h) &\sim \mathrm{N}(\mu_h; \mu_0, \kappa\tau_h^{-1})\mathrm{Ga}(\tau_h; a_\tau; b_\tau), h = 1, \ldots, k \end{aligned}$$

Posterior Computation in finite mixture models

- Update S_i from its multinomial conditional posterior with

$$Pr(S_i = h|-) = \frac{\pi_h \mathrm{N}(y_i; \mu_h, \tau_h^{-1})}{\sum_{l=1}^{k} \pi_h \mathrm{N}(y_i; \mu_h, \tau_h^{-1})}, h = 1, \ldots, k$$

- Let $n_h = \#\{S_i = h, i = 1, \ldots, n\}$ and $\bar{y}_h = \frac{1}{n_h} \sum_{i:S_i=h} y_i$ and update (μ_h, τ_h^{-1}) from its conditional posterior

$$(\mu_h, \tau_h^{-1}|-) = \mathrm{N}(\mu_h, \hat{\mu}_h, \hat{\kappa}_h \tau_h^{-1}) \mathrm{Ga}(\tau_h, \hat{a}_{\tau_h}, \hat{b}_{\tau_h})$$

  where
  $\hat{\kappa}_h = (\kappa^{-1} + n_h)^{-1}, \hat{\mu}_h = \hat{\kappa}(\kappa^{-1}\mu_0 + n_h \bar{y}_h), \hat{a}_{\tau_h} = a_\tau + \frac{n_h}{2}$,

$$\hat{b}_{\tau_h} = b_\tau + \frac{1}{2} \left\{ \sum_{i:S_i=h} (y_i - \bar{y}_h)^2 + \frac{n_h}{1 + \kappa n_h} (\bar{y}_h - \mu_0)^2 \right\}$$

- Update π as $(\pi|-) = \mathrm{Dir}(a_1 + n_1, \ldots, a_k + n_k)$.

Some comments

- Gibbs sampler is trivial to implement
- Discarding a burn-in,monitor $f(y) = \sum_{h=1}^{k} \pi_h N(y; \mu_h, \tau_h^{-1})$ for a large number of iterations & a dense grid of y values
- Bayes estimate of $f(y)$ under squared error loss averages the samples
- Can also obtain 95% pointwise intervals for unknown density

Choosing the Dirichlet hyperparameters

- The choice of hyperparameters in the mixture model can have an important impact
- Focus initially on the choice of $(a_1, \ldots, a_k)'$ in the Dirichlet prior
- A common choice is $a_1 = \cdots = a_k = 1$,which seems "non-informative".
- However, this is actually a poor choice in many cases, as it favors assigning roughly equal weights to the different components
- Ideally, we could choose k as an upper bound and choose hyperparameters, which favor a small number of components with relatively large weights

Finite Approximation to Dirichlet Process

- Ishwaran and Zarepour (2002), "Dirichlet prior sieves in finite normal mixtures," Statistica Sinica, 12, 941-963) propose a finite approximation to the DP
- In particular, they propose letting
- $\pi \sim \text{Dir}(\alpha/k, \ldots, \alpha/k)$ iid
- Assuming also that $\theta_h = (\mu_h, \tau_h) \sim P_0$, they show that

$$\lim_{k\to\infty} \sum_{h=1}^{k} \pi_h \delta_{\theta_h} \to DP(\alpha P_0)$$

- In addition, the posterior for the density is L_1 consistent if $\log k/n \to 0$.

Implications

- We can implement a finite mixture model analysis with a carefully chosen prior & sufficiently large k to obtain an accurate approximation to the DP mixture (DPM) model:
- $f(y) = \int K(y;\theta)dP(\theta), P \sim DP(\alpha P_0)$
- Here, $K(y;\theta)$ is a kernel parameterized by θs - e.g., $K(y;\theta) = N(y;\mu,\tau^{-1})$ with $\theta = (\mu,\tau^{-1})$ for normal mixtures
- P is now an unknown mixing measure
- Hence, we no longer use the DP as a prior directly for the distribution of the data but instead use it for the mixture distribution

Dirichlet Process Mixtures - comments

- The discreteness of the Dirichlet process is not a problem when it is used for a mixture distribution instead of directly for the data distribution
- In fact, in this setting the discreteness is appealing in leading to a simple representation of the mixture distribution that leads to clustering of the observations as a side effect
- Focusing on the finite approximation, $P \sim DP_k(\alpha, P_0)$, let

$$f(y) = \int N(y; \mu, \tau) dP(\mu, \tau)$$

- Note that induces a prior on f.