Problem 4(a)

The cumulative density function of Z_n can be calculated as:

$$\begin{aligned} F_{Z_n}(t) &= P(Z_n < t) \\ &= P(\max\{X_1, X_2, ..., X_n < t\log(n)\}) \\ &= [P(X_1 < t\log(n))]^n \\ &= (1 - n^{-t})^n, \end{aligned}$$

where t is positive.

Therefore, the density function $f_{Z_n}(t)$ can be calculated as:

$$\begin{aligned} f_{Z_n}(t) &= \frac{d}{dt}F_{Z_n}(t) \\ &= n(1 - n^{-t})^{n-1}\frac{d}{dt}(1 - n^{-t}) \\ &= n(1 - n^{-t})^{n-1}(-n^{-t})\log(n) \\ &= n\log(n)(1 - n^{-t})^{n-1}n^{-t}, \end{aligned}$$

with the support $t > 0$.

To calculate $E(Z_n)$, noticing the fact that

$$E(X) = \int_0^\infty \{1 - F_X(x)\}dx,$$

which holds for any continuous, nonnegative random variable X, where $F_X(x)$ is the cdf of X. (See P78, Ex 2.14(a) of CB)

Therefore, since Z_n is continuous and nonnegative,

$$\begin{aligned} E(Z_n) &= \int_0^\infty \big\{1 - F_{Z_n}(x)\big\}dx \\ &= \int_0^\infty \big\{1 - (1 - n^{-x})^n\big\}dx. \end{aligned}$$

For $x > 0$, let $y = 1 - n^{-x}$, $0 < y < 1$. Therefore, $x = -\frac{\log(1-y)}{\log n}$,

$$E(Z_n) = \frac{1}{\log n}\int_0^1 (1 - y^n)\frac{1}{1-y}dy.$$

Since $0 < y < 1$,

$$\frac{1 - y^n}{1 - y} = \sum_{i=0}^{n-1} y^i.$$

Therefore,

$$\begin{aligned}
E(Z_n) &= \frac{1}{\log n}\int_0^1 \frac{(1-y^n)}{1-y}dy = \frac{1}{\log n}\int_0^1 \sum_{i=0}^{n-1} y^i dy \\
&= \frac{1}{\log n}\sum_{i=0}^{n-1}\int_0^1 y^i dy = \frac{1}{\log n}\sum_{i=0}^{n-1}\frac{1}{i+1} \\
&= \frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}.
\end{aligned}$$