

## 1 Martingales

$(\Omega, \mathcal{B}, P)$  is a probability space.

**Definition 1.** (Filtration) A filtration  $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$  is a collection of increasing sub- $\sigma$ -fields such that for  $m \leq n$ , we have  $\mathcal{F}_m \subset \mathcal{F}_n$ .

**Definition 2.** (Adaptation) A sequence of random variables  $X = \{X_n\}$  is adapted if for all  $n$ ,  $X_n$  is  $\mathcal{F}_n$  measurable.

**Definition 3.** (Martingales) An adapted pair  $(X_n, \mathcal{F}_n)$  is called sub / super / martingale if

1.  $X_n \in L_1(P)$  for all  $n$
2.  $EX_{n+1} | \mathcal{F}_n \geq / \leq / = X_n$  a.e.  $[P]$ .

**Remark 1.** a.  $X_n$  is n.n.g,  $L_2$  sub-martingale, then  $X_n^2$  is  $L_1$  sub-martingale.

b.  $X_n$  martingale,  $\phi$  is a convex function. Then  $\phi(X_n)$  is a sub-martingale.

c.  $X_n$  is a sub-martingale,  $\phi$  is convex, non-decreasing (say  $|x|$ ), then  $\phi(X_n)$  is a sub-martingale.

**Example:**

1.  $\{\xi_n\}$  i.i.d mean 0,  $S_n = \sum_{i=1}^n \xi_i$ ,  $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n) = \sigma(S_0, \dots, S_n)$ , then  $(S_n, \mathcal{F}_n)$  is a martingale. (For a sequence of random variables  $\{X_n\}$ ,  $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$  is called a natural filtration. )

**Theorem 1.**  $(X_n, \mathcal{F}_n)$  is a martingale.  $\phi : \mathbb{R} \mapsto \mathbb{R}$  is convex and  $\phi(X_n) \in L_1$  for all  $n$ . Then  $(\phi(X_n), \mathcal{F}_n)$  is a sub-martingale.

*Proof.* (Use Jensen's Inequality). □

**Properties:**

1.  $\{X_n\}$  is a martingale. Then  $|X_n|$  is a sub-martingale.
2.  $\{X_n\}$  is a  $L_p$ -martingale for  $p > 1$ . Then  $\{|X_n|^p\}$  is a sub-martingale.

3.  $\{X_n\}$  is a sub-martingale. Then  $\{X_n^+\}$  is a sub-martingale.
4.  $\{X_n\}$  is a non-negative  $L_p$  submartingale,  $p > 1$ . Then  $\{X_n^p\}$  is a sub-martingale.

**Example:**

1. Suppose  $(X_n, \mathcal{F}_n)$  is an adapted  $L_1$  sequence and  $E(X_{n+1} | \mathcal{F}_n) = a_n X_n$ . How to convert it to a martingale?

Let  $Z_n = c_n X_n$  be a martingale. Then  $E(Z_{n+1} | \mathcal{F}_n) = c_{n+1} a_n X_n$  which should be equal to  $c_n X_n$ . Hence we should have  $c_{n+1}/c_n = a_n^{-1}$  for all  $n$ . Taking  $c_0 = 1$ , we have  $c_n = \prod_{i=1}^n a_i^{-1}$ . Hence  $X_n / \prod_{i=1}^n a_i^{-1}$  is a martingale. (Discussed in class. The case of  $E(X_{n+1} | \mathcal{F}_n) = a_n X_n + b_n$  left as an exercise).

2. Suppose  $([0, 1], \mathcal{B}, P)$  be a probability space and  $Q$  be another probability measure such that  $Q \ll P$ . Let

$$\mathcal{F}_0 = \{\phi, [0, 1]\}$$

$$\mathcal{F}_n = \sigma(\{[r2^{-n}, (r+1)2^{-n}] : 0 \leq r \leq 2^n - 1\})$$

called a dyadic filtration. Clearly  $\sigma(\cup \mathcal{F}_n) = \mathcal{B}$  and  $X_n(\omega) = Q(A)/P(A)$  where  $A$  is an  $\mathcal{F}_n$ -atom containing  $\omega$ . Clearly,  $X_n$  is  $\mathcal{F}_n$ -measurable since  $X_n$  is constant on  $\mathcal{F}_n$  atoms. It is easy to check that  $(X_n, \mathcal{F}_n)$  is a martingale. (Discussed in class).

### 1.1 Doob's maximal inequalities

**Theorem 2.** Let  $(X_j, \mathcal{F}_j)_{0 \leq j \leq n}$  is a sub-martingale and  $\lambda \in \mathbb{R}$ . Then

1.  $\lambda P[\max_{0 \leq j \leq n} X_j > \lambda] \leq \int_{\max X_j > \lambda} X_n dP \leq E|X_n|$ .
2.  $\lambda P[\min_{0 \leq j \leq n} X_j \geq \lambda] \leq \int_{\min X_j \leq \lambda} X_n dP - E(X_n - X_0) \geq E(X_0) - E|X_n|$ .

*Proof.* Proof of a.) Let

$$A_0 = [X_0 > \lambda] \in \mathcal{F}_0$$

$$A_1 = [X_0 \leq \lambda, X_1 > \lambda] \in \mathcal{F}_1$$

$$\vdots$$

$$A_n = [X_0 \leq \lambda, \dots, X_{n-1} \leq \lambda, X_n > \lambda] \in \mathcal{F}_n.$$

Then  $[\max X_j > \lambda] = \cup_{j=0}^n A_j$  and

$$\lambda P[\max X_j > \lambda] = \sum_{j=0}^n \lambda P(A_j) \leq \sum_{j=0}^n \int_{A_j} \lambda dP \leq \sum_{j=0}^n \int_{A_j} X_j dP = \sum_{j=0}^n \int_{A_j} X_n dP = \int_{\max X_j > \lambda} X_n dP,$$

where the penultimate inequality follows from the fact that since  $E(X_n | \mathcal{F}_j) \geq X_j \equiv \int_A E(X_n | \mathcal{F}_j) dP \geq \int_A X_j dP$  for all  $A \in \mathcal{F}_j$ . Hence  $\int_A X_n dP \geq \int_A X_j dP$  for all  $A \in \mathcal{F}_j$ .  
Proof of b.)

$$\begin{aligned} A_0 &= [X_0 \leq \lambda] \in \mathcal{F}_0 \\ A_1 &= [X_0 > \lambda, X_1 \leq \lambda] \in \mathcal{F}_1 \\ &\vdots \\ A_n &= [X_0 > \lambda, \dots, X_{n-1} > \lambda, X_n \leq \lambda] \in \mathcal{F}_n \\ A_{n+1} &= [X_0 > \lambda, \dots, X_n > \lambda] \in \mathcal{F}_n. \end{aligned}$$

Then

$$\begin{aligned} EX_0 &= \int X_0 dP = \sum_{j=0}^{n+1} \int_{A_j} X_0 dP \\ &= \int_{A_0} X_0 dP + \int_{A_0^c} X_0 dP \\ &\leq \lambda P(A_0) + \int_{A_0^c} X_1 dP \\ &\leq \lambda P(A_0) + \int_{A_1} X_1 dP + \int_{\{A_0 \cup A_1\}^c} X_1 dP \\ &\leq \lambda P(A_0 \cup A_1) + \int_{\{A_0 \cup A_1\}^c} X_2 dP \\ &\vdots \\ &\leq \lambda P(\min X_j \leq \lambda) + \int_{A_{n+1}} X_n dP \\ &= \lambda P(\min X_j \leq \lambda) + \int X_n dP - \int_{A_{n+1}^c} X_n dP. \end{aligned}$$

This implies  $\lambda P[\min_{0 \leq j \leq n} X_j \leq \lambda] \geq \int_{\min X_j \leq \lambda} X_n dP - E(X_n - X_0) \geq E(X_0) - E|X_n|$ .

□

## 2 Convergence of Martingales

Goal: What conditions do we need so that an  $L_1$  bounded martingale converges in  $L_1$ ?

**Definition 4.** A subset  $S \subset L_1$  is called uniformly integrable if given  $\epsilon > 0$ , there exists  $c > 0$  such that  $\sup_{f \in S} \int_{|f| > c} |f| dP < \epsilon$ .

**Definition 5.** A subset  $S \subset L_1$  is called  $L_1$  bounded if  $\sup_{f \in S} E|f| < \infty$ .

Examples:

1. Any finite  $L_1$ -subset is uniformly integrable.
2. If  $S$  is dominated by an  $L_1$  function, then  $S$  is uniformly integrable.
3. Any uniformly integrable  $L_1$  subset is  $L_1$  bounded. (Converse is not true in general: construct an  $L_1$  bounded subset which is not uniformly integrable (**Discussed in class**)).

**Theorem 3.**  $S \subset L_1$  is uniformly integrable subset iff:

1.  $S$  is  $L_1$  bounded.
2. For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $P(A) < \delta$ , then  $\sup_S \int_A |f| < \epsilon$ .

*Proof.* if part: Suppose  $\sup_{f \in S} E|f| = M < \infty$ . Fix  $\epsilon > 0$ . Choose  $\delta$  from 2. Observe that  $\sup_{f \in S} P(|f| > c) \leq \sup_{f \in S} E|f|/c \leq M/c$ . Choose  $c = M/\delta$  with  $A = \{|f| > c\}$  to get the result.

only if part: Fix  $\epsilon > 0$ . Choose  $c$  such that  $\sup_{f \in S} \int_{|f|>c} |f| < \epsilon/2$ . Then

$$\int_A |f| dP \leq \int_{A \cap |f| \leq c} |f| + \int_{|f|>c} |f| < cP(A) + \epsilon/2.$$

Choose  $\delta = \epsilon/(2c)$  to get the result. □

**Theorem 4.**  $f \in L_1$ . Let  $S = \{E^C f := E(f | \mathcal{C}), \mathcal{C} \text{ a sub-}\sigma\text{-field of } \mathcal{F}\}$ . Then  $S$  is uniformly integrable.

*Proof.* Observe that

$$\int_{|E^C f|>c} |E^C f| \leq \int_{|E^C f|>c} E^C |f| = \int_{|E^C f|>c} |f|.$$

So 1. is satisfied. To verify 2., note that

$$\sup_c P(|E^C f| > c) \leq \sup_c \frac{E|f|}{c} = \frac{E|f|}{c}.$$

Choose  $c$  large enough such that  $\sup_c P(|E^C f| > c)$  is sufficiently small to have  $\int_{|E^C f|>c} |f| < \epsilon/2$ . Observe that for any set  $A$  with  $P(A) < \epsilon/(2c)$ ,

$$\int_A |f| \leq \int_{A \cap |E^C f| \leq c} |f| + \int_{|E^C f|>c} |f| < cP(A) + \epsilon/2 < \epsilon.$$

□

**Theorem 5.**  $f_n \in L_1, f_n \xrightarrow{a.s.} f$ . Then  $\{f_n\}$  is uniformly integrable iff  $f \in L_1$  and  $f_n \xrightarrow{L_1} f$ .

*Proof.* only if part: Fix  $\epsilon > 0$ . By Fatou's Lemma and  $L_1$  boundedness of  $\{f_n\}$ ,  $f \in L_1$ . Next we show that  $f_n \xrightarrow{L_1} f$ . To that end, observe that

$$E|f_n - f| = E|f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c}| + E|f_n 1_{|f_n| > c}| + E|f 1_{|f| > c}|.$$

Choose  $c$  such that

$$P(|f| = c) = 0, E|f_n 1_{|f_n| > c}| < \epsilon/3, E|f 1_{|f| > c}| < \epsilon/2$$

for all  $n$ . Since  $\{\omega : |f_n(\omega)| > c, |f_n(\omega)| \rightarrow |f(\omega)| = c\} \subset \{|f| = c\}$ , we have

$$f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c} \rightarrow 0$$

Choose  $N$  large enough such that for all  $n \geq N$ ,  $E|f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c}| < \epsilon/3$  by DCT. almost surely. Hence for large enough  $n$ ,  $E|f_n - f| \leq \epsilon/3$ .

if part: Assume  $f_n \xrightarrow{L_1} f$ . Then  $\{f_n\}$  is  $L_1$  bounded and  $f \in L_1$ . Fix  $\epsilon > 0$ . Then

$$\int_A |f_n| \leq \int_A |f_n - f| + \int_A |f|.$$

Choose  $N$  such that for all  $n \geq N$ ,  $\int |f_n - f| < \epsilon/2$  and choose  $\delta_0$  with  $P(A) < \delta_0$  implies  $\int |f| < \epsilon/2$ . Then for  $n \geq N$ ,  $P(A) \leq \delta_0$  implies  $\int_A |f_n| < \epsilon$ , implying  $\{f_n\}$  uniformly integrable. □

**Theorem 6.**  $f_n \geq 0, f_n \xrightarrow{a.s.} f, f_n, f$  in  $L_1$ . Then  $\{f_n\}$  is uniformly integrable if and only if  $Ef_n \rightarrow Ef$ .

*Proof.* The only if part follows from Theorem 5.

if part: It is enough to show that  $f_n \xrightarrow{L_1} f$  (This follows from Scheff's Lemma). Note that

$$E|f_n - f| = Ef_n + Ef - 2E \min\{f_n, f\}.$$

Since  $Ef_n \rightarrow Ef$  by assumption and  $E \min\{f_n, f\} \rightarrow Ef$  by DCT, implying  $f_n \xrightarrow{L_1} f$ . □

**Theorem 7.**  $(X_n, \mathcal{F}_n)_{n \geq 0}$  is a martingale. Let  $\mathcal{F}_\infty = \sigma(\cup_{n \geq 0} \mathcal{F}_n)$  and  $\sup E|X_n| = K$ . ( $E|X_n| \uparrow K$  since  $|X_n|$  is a submartingale). Then the following are equivalent.

- i.  $K < \infty, X_n \xrightarrow{L_1} X_\infty$ .
- ii.  $(X_n, \mathcal{F}_n)_{0 \leq n \leq \infty}$  is a martingale.

iii.  $K < \infty$ ,  $E|X_\infty| = K$ .

iv.  $\{X_n\}_{n \geq 0}$  is uniformly integrable.

*Proof.* (Complete verification is left as an exercise).

(i)  $\implies$  (iv): Since  $K < \infty$ ,  $X_n \rightarrow X_\infty$  a.e. (Since  $L_1$  bounded martingale converges almost surely, using Doob's upcrossing inequality). Since  $X_n \xrightarrow{L_1} X_\infty$ , from Theorem 5,  $\{X_n\}_{n \geq 0}$  is uniformly integrable.

(iv)  $\implies$  (i): Since  $\{X_n\}_{n \geq 0}$  is uniformly integrable,  $\{X_n\}_{n \geq 0}$  is  $L_1$  bounded implying  $K < \infty$ . Since  $X_n \xrightarrow{a.s.} X_\infty$ ,  $X_n \xrightarrow{L_1} X_\infty$  by Theorem 5. Hence (i)  $\Leftrightarrow$  (iv).

(ii)  $\implies$  (iv): This follows directly from Theorem 4.

(iv)  $\implies$  (ii): We will infact show (iv), (i)  $\implies$  (ii).  $X_\infty$  is the almost sure limit of  $X_n$  and hence  $\mathcal{F}_n$  measurable for all  $n < \infty$  and hence  $\mathcal{F}_\infty$  measurable. Also (i) implies  $X_\infty \in L_1$ . It is enough to check that  $E(X_\infty | \mathcal{F}_n) = X_n$  a.e. We already know that for  $m > n$ ,  $X_n = E(X_m | \mathcal{F}_n)$  a.s. It is enough to show that  $E(X_m - X_\infty | \mathcal{F}_n) \rightarrow 0$  a.e. as  $m \rightarrow \infty$ . We know that  $X_m \xrightarrow{L_1} X_\infty$ . Then

$$E|E[X_m - X_\infty | \mathcal{F}_n]| \leq EE[|X_m - X_\infty| | \mathcal{F}_n] = E[|X_m - X_\infty|] \rightarrow 0$$

as  $m \rightarrow \infty$ , implying  $E|X_n - E[X_\infty | \mathcal{F}_n]| = 0$  implying  $X_n = EX_\infty | \mathcal{F}_n$  a.e. Hence we have shown that (i), (ii) and (iv) are equivalent.

(iii)  $\implies$  (iv): Since  $|X_n|$  is a sub-martingale,  $E|X_n| \uparrow K$  implying  $E|X_n| \uparrow E|X_\infty|$ . By Scheffe's theorem  $|X_n| \xrightarrow{L_1} X_\infty$ . By Theorem 6  $\{|X_n|\}$  and hence  $\{X_n\}$  is uniformly integrable. Now we will show that (i) and (iv) implies (iii). Since  $X_n \xrightarrow{L_1} X_\infty$ , by Scheffe's Lemma and Fatou's Lemma  $E|X_n| \rightarrow E|X_\infty| < \infty$ . Also  $E|X_n| \uparrow K$ . Hence  $K < \infty$  and  $E|X_\infty| = K$ .  $\square$

**Theorem 8.**  $(X_n, \mathcal{F}_n)_{0 \leq n \leq N}$  is a non-negative martingale. Let  $p > 1$ . Then

$$\left\| \max_{0 \leq n \leq N} X_n \right\|_p \leq \frac{p}{p-1} \|X_N\|_p$$

*Proof.* The proof follows from the following Lemma 1 and Doob's maximal inequality.  $\square$

**Lemma 1.**  $U, V$  non-negative random variables. Let  $\lambda > 0$ ,  $P(U > \lambda) \leq (1/\lambda) \int_{U > \lambda} V dP$ . Then, for  $p > 1$ ,

$$\|U\|_p \leq \frac{p}{p-1} \|V\|_p.$$

*Proof.*

$$\begin{aligned}
EU^p &= \int_0^\infty p\lambda^{p-1}P(U > \lambda)d\lambda \\
&= \int_0^\infty p\lambda^{p-2} \int_{U>\lambda} V dP d\lambda \\
&= \int_0^\infty p\lambda^{p-2} \int_\Omega V(\omega)1_{U>\lambda}(\omega)dP(\omega)d\lambda \\
&= \int_\Omega V(\omega) \int_0^\Omega p\lambda^{p-2}d\lambda dP(\omega) \\
&= \frac{p}{p-1} \int_\Omega V(\omega)U^{p-1}dP = \frac{p}{p-1}E(VU^{p-1}) \\
&\leq \frac{p}{p-1}\{E(V^p)\}^{1/p}\{EU^{(p-1)q}\}^{1/q} \\
&= \frac{p}{p-1}\{E(V^p)\}^{1/p}\{EU^p\}^{1/q}
\end{aligned}$$

implying  $\|U\|_p \leq \frac{p}{p-1} \|V\|_p$  if  $U \in L_p$ . Otherwise, work with  $\min\{U, n\}$ .  $\square$

**Corollary 1.**  $(X_n, \mathcal{F}_n)_{n \geq 0}$  non-negative  $L_p$ -bounded sub-martingale. Then  $X^* = \sup X_n \in L_p$  (True for  $L_p$  bounded martingale with  $X^* = \sup |X_n|$ ).

*Proof.* Let  $X_N^* = \max_{0 \leq n \leq N} X_n$ ,  $X_N^* \uparrow X^*$ . By MCT,  $E(X_N^*)^p \uparrow E(X^*)^p$ . Since  $\{X_n\}$  are  $L_p$  bounded,

$$E(X_N^*)^p \leq \left(\frac{p}{p-1}\right)^p E|X_N|^p < M,$$

implying  $X^* \in L_p$ .  $\square$

**Theorem 9.** Let  $p > 1$ .  $\{X_n\}$  is  $L_p$ -bounded martingale or a non-negative sub-martingale. Then

1.  $\{X_n\}$  is uniformly integrable.

2.  $X_n \xrightarrow{L_q} X_\infty$ .

*Proof.* Proof is left as an Exercise.  $\square$

**Remark 2.** Counter example to show that one cannot get rid of the non-negativity in case of sub-martingale. Let  $([0, 1), \mathcal{B}, \lambda)$  be the measure space and  $\mathcal{F}_n$  is a dyadic filtration. Let  $X_n = -2^{n/2}1_{[0, 2^{-n})} \rightarrow 0$  a.s. Check that  $\{X_n, \mathcal{F}_n\}$  is a sub-martingale. Clearly  $X_n \not\xrightarrow{L_2} 0$  as  $\|X_n\|_2 = 1$ . Complete verification is left as an Exercise.

### 3 Example: Two color urn model

Suppose  $(X_n, \mathcal{F}_n)$  adapted  $L_1$ -sequence.  $EX_{n+1} | \mathcal{F}_n = a_n X_n + b_n$   $a_n \neq 0$ . Find the associated martingale. Start with  $(W_0, B_0)$  balls with  $0 \leq W_0, B_0$  and  $W_0 + B_0 = 1$ . At  $n$  th stage,  $W_{n-1}$  white,  $B_{n-1}$  black balls are available with  $W_{n-1} + B_{n-1} = n$ . Draw a ball at random and  $R_{2 \times 2}$  is a stochastic matrix. If you see a white ball, add  $R_{11}$  white and  $R_{12}$  black balls. If you see black ball, add according to second row.  $X_{n+1}$  is a vector which is  $(1, 0)'$  if white is drawn in  $n$ th stage,  $(0, 1)'$  if black.  $Z_n = (W_n, B_n)$ . Since  $Z_n$  is bounded for each fixed  $n$ ,  $Z_n \in L_1$ . Observe that

$$Z_{n+1} = Z_n + X'_{n+1} R.$$

Find the associated martingale. (Discussed in class).

### 4 Stopping Time

$\mathcal{F} = (\mathcal{F}_n)_{n \geq 0}$  is a filtration.  $\tau : \Omega \mapsto \{0, 1, 2, \dots, \infty\}$  is a measurable random time.  $\sigma$ -field on  $\{0, 1, \dots, \infty\}$  is  $\mathcal{P}(\{0, 1, \dots, \infty\})$ . A random time is called a stopping time if  $[\tau = n] \in \mathcal{F}_n$  for all  $n \in \{0, 1, 2, \dots, \infty\}$ .

Example: Time at which one starts smoking is a stopping time. However, the time at which one stops smoking is not a stopping time.

$[\tau = n]$  is equivalent to  $[\tau \leq n] \in \mathcal{F}_n$  for all  $n \in \{0, 1, 2, \dots, \infty\}$ . This is easy to see for discrete time. For continuous time  $[\tau \leq t] \in \mathcal{F}_t$  for all  $t \geq 0$ .  $[\tau = t] = [\tau \leq t] - \cup_n [\tau \leq t - 1/n : n \in \mathbb{N}]$ .

In the discrete time case  $[\tau < n] \in \mathcal{F}_n$ ,  $[\tau \leq n - 1] \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ .

**Theorem 10.** For discrete parameter spaces  $[\tau = n] \in \mathcal{F}_n \Leftrightarrow [\tau \leq n] \in \mathcal{F}_n \Leftrightarrow [\tau < n] \in \mathcal{F}_n$ .

**Definition 6.**  $(X_n, \mathcal{F}_n)$  adapted sequence.  $\tau$  is a random time,  $[\tau < \infty] = \Omega$ . Stopping random variable  $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ .

**Theorem 11.**  $X_\tau$  is  $\mathcal{B}$  measurable.

*Proof.* For  $B \in \mathcal{B}(\mathbb{R})$ ,  $X_\tau^{-1}(B) = \cup_{n=0}^{\infty} [X_\tau \in B] \cap [\tau = n] = \cup_{n=0}^{\infty} [X_n \in B] \cap [\tau = n]$ .  $\square$

**Definition 7.** (Stopping  $\sigma$ -field)  $\mathcal{F}_\tau = \{A \in \mathcal{B} : A \cap [\tau = n] \in \mathcal{F}_n \text{ for all } n\}$

Clearly  $X_\tau$  is  $\mathcal{F}_\tau$  measurable.



## 4.1 Properties

1.  $\tau \equiv \sigma \Rightarrow \mathcal{F}_\tau = \mathcal{F}_\sigma$
2.  $\tau, \sigma$  are stop times, then  $[\tau < \sigma], [\tau > \sigma], [\tau = \sigma]$  are all  $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$ -measurable. ( $[\tau < \sigma] \cap [\tau = n] = [\sigma > n] \cap [\tau = n] \in \mathcal{F}_n$ )
3.  $\tau$  is  $\mathcal{F}_\tau$  measurable. Then  $[\tau = k] \cap [\tau = n] \in \mathcal{F}_n$  for  $k \leq n$ .
4.  $\tau \leq \sigma$ , then  $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$ .
5.  $(X_n, \mathcal{F}_n)_{0 \leq n \leq N}$  is a (sub)-martingale.  $\tau \leq \sigma$  is a stopping time.  $(X_\tau, X_\sigma)$  is a (sub)-martingale corresponding to  $(\mathcal{F}_\tau, \mathcal{F}_\sigma)$ .

## 4.2 Doob's Upcrossing Inequality

$\{X_n\}_{0 \leq n \leq N}, a < b$ . Define

$$\begin{aligned}
 \tau_0 &= 0 \\
 \tau_1 &= \inf\{n : X_n \leq a\} \\
 &\vdots \\
 \tau_{2k+1} &= \inf\{n \geq \tau_{2k} : X_n \leq a\} \\
 \tau_{2k+2} &= \inf\{n \geq \tau_{2k+1} : X_n \geq b\}
 \end{aligned}$$

with the convention that  $\inf\{\emptyset\} = N$ . Define

$$\begin{aligned}
 U(\{X_n\}_{0 \leq n \leq N}; [a, b]) &= \sup\{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\} \\
 U(\{X_n\}_{n \geq 0}; [a, b]) &= \uparrow \lim_N \{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\}.
 \end{aligned}$$

**Lemma 2.**  $\tau_i$ s are stop times.

*Proof.*  $\tau_0, \tau_1$  are stop times. Assume  $\tau_{2k}$  is a stop time. Then

$$\begin{aligned}
 \{\tau_{2k+1} = i\} &= \cup_{j=0}^{i-1} \{\tau_{2k+1} = i, \tau_{2k} = j\} \\
 \{\tau_{2k+1} = i, \tau_{2k} = j\} &= \{\tau_{2k} = j\} \cap \{X_{j+1} > a\} \cap \cdots \cap \{X_{i-1} > a\} \cap \{X_i \leq a\} \in \mathcal{F}_i
 \end{aligned}$$

□

For an adapted sequence,  $\{\tau_k\}$  forms a stopping time, which implies  $\sup\{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\}$  is measurable implying  $U(\{X_n\}_{n \geq 0}; [a, b])$  is measurable.

**Theorem 12.** (*Doob's Upcrossing Inequality*).  $\{X_n, \mathcal{F}_n\}$  is a submartingale and  $a < b$ . Then

$$\begin{aligned} EU(\{X_n\}_{0 \leq n \leq N}; [a, b]) &\leq \frac{E[(X_N - a)^+] - E[(X_0 - a)^+]}{b - a} \\ &\leq \frac{E|X_N| + |a|}{b - a}. \end{aligned}$$

**Corollary 2.** An  $L_1$ -bounded martingale converges almost surely.

## 5 Problem

A branching process is defined as follows: We start with one member, namely the population size is  $Z_0 = 1$ . Let  $\xi_i^n, i = 1, 2, \dots, n$  denote the number of children of  $i$ th individual in the  $n$ th generation. Assume  $\xi_i^n$  are independent with common mean  $\mu > 0$ . Let  $Z_n$  denote the population size in the  $n$ th generation. Let  $p_k = P(\xi_i^n = k), k \geq 0, \mu = E(\xi_i^n)$ .  $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^n$ . It is easy to see that  $X_n = Z_n/\mu^n$  is a martingale.

1. **Does  $\{X_n\}$  has a limit?**

Since  $X_n$  is non-negative,  $\{X_n\}$  converges almost everywhere to an integrable random variable  $X$ .

2. **If  $\mu < 1, Z_n = 0$  almost surely for large  $n$ .**

$Z_n = X_n \mu^n$ . Since  $X_n \xrightarrow{a.s.} X$  and  $\mu^n \rightarrow 0, Z_n \xrightarrow{a.s.} 0$ . There exists a  $P$ -null set  $N$  such that for all  $\omega \notin N, Z_n(\omega) \rightarrow 0$ . Given  $\epsilon = 1/2$ , there exists  $N_0(\omega)$  such that  $Z_n(\omega) < \epsilon$  for all  $n \geq N_0(\omega)$  implying  $Z_n(\omega) = 0$  for all  $n \geq N_0(\omega)$ . This implies  $Z_n$  converges to 0 with probability 1 for all large  $n$ .

3. **If  $\mu < 1, X_n = 0$  almost surely for large  $n$ .**

$\sum_{n=1}^{\infty} P(X_n > 0) = \sum_{n=1}^{\infty} P(Z_n > 0) = \sum_{n=1}^{\infty} P(Z_n \geq 1) \leq \sum_{n=1}^{\infty} E(Z_n) = \sum_{n=1}^{\infty} \mu^n < \infty$ . Hence by Borel Cantelli Lemma  $P(\limsup A_n) = 0$  which means  $P(\bigcap_{n=1}^{\infty} \cup_{k \geq n} A_k) = 0$  implying  $P(A_n \text{ occurs infinitely often}) = 0$  implying  $P(X_n = 0 \text{ for all large } n) = 1$ . Hence  $X_0 = 0$  eventually with probability 1.

4. **If  $\mu = 1$  and  $P(\xi_1^1 > 1) > 0$ , then  $Z_n \rightarrow 0$  a.s.**

$Z_n$  non-negative martingale, hence  $Z_n \rightarrow Z_{\infty}$  a.e and  $Z_{\infty} \in L_1$ . It is enough to show that  $P(Z_{\infty} = k) = 0$  for all  $k \geq 1$ , or in other words  $P(Z_n = k \text{ eventually}) = 0$ . Observe that  $P(Z_n = k \text{ eventually}) \leq P(\text{one of } \xi_1^n, \dots, \xi_k^n \leq 1, \text{ eventually})$ . It is enough to show that  $P(\xi_1^n, \dots, \xi_k^n > 1, \text{ infinitely often}) = 1$ . To that end, note that  $\sum_{n=1}^{\infty} P(\xi_i^n > 1)^k = \infty$ . The result follows from second Borel Cantelli Lemma.