January 12, 2017 Debdeep Pati

1 Martingales

$(\Omega, \mathcal{B}, P)$ is a probability space.

Definition 1. (Filtration) A filtration $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ is a collection of increasing sub-σ-fields such that for $m \leq n$, we have $\mathcal{F}_m \subset \mathcal{F}_n$.

Definition 2. (Adaptation) A sequence of random variables $X = \{X_n\}$ is adapted if for all n, X_n is $\mathcal{F}_n$ measurable.

Definition 3. (Martingales) An adapted pair $(X_n, \mathcal{F}_n)$ is called sub / super / martingale if

1. $X_n \in L_1(P)$ for all n
2. $EX_{n+1} \mid \mathcal{F}_n \geq / \leq / = X_n$ a.e. $[P]$.

Remark 1. a. X_n is n.n.g, L_2 sub-martingale, then X_n^2 is L_1 sub-martingale.

b. X_n martingale, ϕ is a convex function. Then if $\phi(X_n)$ is L_1, then $\phi(X_n)$ is a sub-martingale.

c. X_n is a sub-martingale, ϕ is convex, non-decreasing (say $|x|$), then $\phi(X_n)$ is a sub-martingale.

Example:

1. $\{\xi_n\}$ i.i.d mean 0, $S_n = \sum_{i=1}^n \xi_i, \mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n) = \sigma(S_0, \ldots, S_n)$, then $(S_n, \mathcal{F}_n)$ is a martingale. (For a sequence of random variables $\{X_n\}$, $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ is called a natural filtration.)

Theorem 1. $(X_n, \mathcal{F}_n)$ is a martingale. $\phi : \mathbb{R} \mapsto \mathbb{R}$ is convex and $\phi(X_n) \in L_1$ for all n. Then $(\phi(X_n), \mathcal{F}_n)$ is a sub-martingale.

Proof. (Use Jensen's Inequality). □

Properties:

1. $\{X_n\}$ is a martingale. Then $|X_n|$ is a sub-martingale.

2. $\{X_n\}$ is a L_p-martingale for $p > 1$. Then $\{|X_n|^p\}$ is a sub-martingale.

3. $\{X_n\}$ is a sub-martingale. Then $\{X_n^+\}$ is a sub-martingale.

4. $\{X_n\}$ is a non-negative L_p submartingale, $p > 1$. Then $\{X_n^p\}$ is a sub-martingale.

Example:

1. Suppose $(X_n, \mathcal{F}_n)$ is an adapted L_1 sequence and $E(X_{n+1} \mid \mathcal{F}_n) = a_n X_n$. How to convert it to a martingale?
   Let $Z_n = c_n X_n$ be a martingale. Then $E(Z_{n+1} \mid \mathcal{F}_n) = c_{n+1} a_n X_n$ which should be equal to $c_n X_n$. Hence we should have $c_{n+1}/c_n = a_n^{-1}$ for all n. Taking $c_0 = 1$, we have $c_n = \prod_{i=1}^n a_i^{-1}$. Hence $X_n / \prod_{i=1}^n a_i$ is a martingale. (Discussed in class. The case of $E(X_{n+1} \mid \mathcal{F}_n) = a_n X_n + b_n$ left as an exercise).

2. Suppose $([0,1), \mathcal{B}, P)$ be a probability space and Q be another probability measure such that $Q \ll P$. Let

$$\begin{aligned} \mathcal{F}_0 &= \{\phi, [0,1)\} \\ \mathcal{F}_n &= \sigma(\{[r2^{-n}, (r+1)2^{-n}) : 0 \le r \le 2^n - 1\}) \end{aligned}$$

   called a dyadic filtration. Clearly $\sigma(\cup \mathcal{F}_n) = \mathcal{B}$ and $X_n(\omega) = Q(A)/P(A)$ where A is an $\mathcal{F}_n$-atom containing ω. Clearly, X_n is $\mathcal{F}_n$-measurable since X_n is constant on $\mathcal{F}_n$ atoms. It is easy to check that $(X_n, \mathcal{F}_n)$ is a martingale. (Discussed in class).

1.1 Doob's maximal inequalities

Theorem 2. Let $(X_j, \mathcal{F}_j)_{0 \le j \le n}$ is a sub-martingale and $\lambda \in \mathbb{R}$. Then

1. $\lambda P[\max_{0 \le j \le n} X_j > \lambda] \le \int_{\max X_j > \lambda} X_n dP \le E|X_n|$.

2. $\lambda P[\min_{0 \le j \le n} X_j \ge \lambda] \le \int_{\min X_j \le \lambda} X_n dP - E(X_n - X_0) \ge E(X_0) - E|X_n|$.

Proof. Proof of a.) Let

$$\begin{aligned} A_0 &= [X_0 > \lambda] \in \mathcal{F}_0 \\ A_1 &= [X_0 \le \lambda, X_1 > \lambda] \in \mathcal{F}_1 \\ &\vdots \\ A_n &= [X_0 \le \lambda, \ldots, X_{n-1} \le \lambda, X_n > \lambda] \in \mathcal{F}_n. \end{aligned}$$

Then $[\max X_j > \lambda] = \cup_{j=0}^{n} A_j$ and

$$\lambda P[\max X_j > \lambda] = \sum_{j=0}^{n} \lambda P(A_j) \le \sum_{j=0}^{n} \int_{A_j} \lambda dP \le \sum_{j=0}^{n} \int_{A_j} X_j dP = \sum_{j=0}^{n} \int_{A_j} X_n dP = \int_{\max X_j > \lambda} X_n dP,$$

where the penultimate inequality follows from the fact that since $E(X_n \mid \mathcal{F}_j) \ge X_j \equiv \int_A E(X_n \mid \mathcal{F}_j) dP \ge \int_A X_j dP$ for all $A \in \mathcal{F}_j$. Hence $\int_A X_n dP \ge \int_A X_j dP$ for all $A \in \mathcal{F}_j$.

Proof of b.)

$$\begin{aligned} A_0 &= [X_0 \le \lambda] \in \mathcal{F}_0 \\ A_1 &= [X_0 > \lambda, X_1 \le \lambda] \in \mathcal{F}_1 \\ &\vdots \\ A_n &= [X_0 > \lambda, \ldots, X_{n-1} > \lambda, X_n \le \lambda] \in \mathcal{F}_n \\ A_{n+1} &= [X_0 > \lambda, \ldots, X_n > \lambda] \in \mathcal{F}_n. \end{aligned}$$

Then

$$\begin{aligned} EX_0 = \int X_0 dP &= \sum_{j=0}^{n+1} \int_{A_j} X_0 dP \\ &= \int_{A_0} X_0 dP + \int_{A_0^c} X_0 dP \\ &\le \lambda P(A_0) + \int_{A_0^c} X_1 dP \\ &\le \lambda P(A_0) + \int_{A_1} X_1 dP + \int_{\{A_0 \cup A_1\}^c} X_1 dP \\ &\le \lambda P(A_0 \cup A_1) + \int_{\{A_0 \cup A_1\}^c} X_2 dP \\ &\vdots \\ &\le \lambda P(\min X_j \le \lambda) + \int_{A_{n+1}} X_n dP \\ &= \lambda P(\min X_j \le \lambda) + \int X_n dP - \int_{A_{n+1}^c} X_n dP. \end{aligned}$$

This implies $\lambda P[\min_{0 \le j \le n} X_j \le \lambda] \ge \int_{\min X_j \le \lambda} X_n dP - E(X_n - X_0) \ge E(X_0) - E|X_n|$.

□

2 Convergence of Martingales

Goal: What conditions do we need so that an L_1 bounded martingale converges in L_1?

Definition 4. *A subset $S \subset L_1$ is called uniformly integrable if given $\epsilon > 0$, there exists $c > 0$ such that $\sup_{f \in S} \int_{|f|>c} |f| \, dP < \epsilon$.*

Definition 5. *A subset $S \subset L_1$ is called L_1 bounded if $\sup_{f \in S} E|f| < \infty$.*

Examples:

1. Any finite L_1-subset is uniformly integrable.
2. If S is dominated by an L_1 function, then S is uniformly integrable.
3. Any uniformly integrable L_1 subset is L_1 bounded. (Converse is not true in general: construct an L_1 bounded subset which is not uniformly integrable (Discussed in class).

Theorem 3. *$S \subset L_1$ is uniformly integrable subset iff:*

1. *S is L_1 bounded.*
2. *For all $\epsilon > 0$, there exists $\delta > 0$ such that $P(A) < \delta$, then $\sup_S \int_A |f| < \epsilon$.*

Proof. if part: Suppose $\sup_{f \in S} E|f| = M < \infty$. Fix $\epsilon > 0$. Choose δ from 2. Observe that $\sup_{f \in S} P(|f| > c) \leq \sup_{f \in S} E|f|/c \leq M/c$. Choose $c = M/\delta$ with $A = \{|f| > c\}$ to get the result.
only if part: Fix $\epsilon > 0$. Choose c such that $\sup_{f \in S} \int_{|f|>c} |f| < \epsilon/2$. Then

$$\int_A |f| \, dP \leq \int_{A \cap |f| \leq c} |f| + \int_{|f|>c} |f| < cP(A) + \epsilon/2.$$

Choose $\delta = \epsilon/(2c)$ to get the result. □

Theorem 4. *$f \in L_1$. Let $S = \{E^{\mathcal{C}} f := E(f \mid \mathcal{C}), \quad \mathcal{C}\, a \text{ sub-}\sigma\text{-field of}\, \mathcal{F}\}$. Then S is uniformly integrable.*

Proof. Observe that

$$\int_{|E^{\mathcal{C}} f|>c} \left|E^{\mathcal{C}} f\right| \leq \int_{|E^{\mathcal{C}} f|>c} E^{\mathcal{C}} |f| = \int_{|E^{\mathcal{C}} f|>c} |f| \,.$$

So 1. is satisfied. To verify 2., note that

$$\sup_{\mathcal{C}} P(\left|E^{\mathcal{C}} f\right| > c) \leq \sup_{\mathcal{C}} \frac{E|f|}{c} = \frac{E|f|}{c}.$$

Choose c large enough such that $\sup_{\mathcal{C}} P(\left|E^{\mathcal{C}} f\right| > c)$ is sufficiently small to have $\int_{|E^{\mathcal{C}} f|>c} |f| < \epsilon/2$. Observe that for any set A with $P(A) < \epsilon/(2c)$,

$$\int_A \left|E^{\mathcal{C}} f\right| \leq \int_{A \cap |E^{\mathcal{C}} f| \leq c} \left|E^{\mathcal{C}} f\right| + \int_{|E^{\mathcal{C}} f|>c} |f| < cP(A) + \epsilon/2 < \epsilon.$$

□

Theorem 5. *$f_n \in L_1, f_n \stackrel{a.s.}{\to} f$. Then $\{f_n\}$ is uniformly integrable iff $f \in L_1$ and $f_n \stackrel{L_1}{\to} f$.*

Proof. only if part: Fix $\epsilon > 0$. By Fatou's Lemma and L_1 boundedness of $\{f_n\}$, $f \in L_1$. Next we show that $f_n \stackrel{L_1}{\to} f$. To that end, observe that

$$E|f_n - f| = E\left|f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c}\right| + E\left|f_n 1_{|f_n|>c}\right| + E\left|f 1_{|f|>c}\right|.$$

Choose c such that

$$P(|f| = c) = 0, E\left|f_n 1_{|f_n|>c}\right| < \epsilon/3, E\left|f 1_{|f|>c}\right| < \epsilon/2$$

for all n. Since $\{\omega : |f_n(\omega)| > c, |f_n(\omega)| \to |f(\omega)| = c\} \subset \{|f| = c\}$, we have

$$f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c} \to 0$$

Choose N large enough such that for all $n \geq N$, $E\left|f_n 1_{|f_n| \leq c} - f 1_{|f| \leq c}\right| < \epsilon/3$ by DCT. almost surely. Hence for large enough n, $E|f_n - f| \leq \epsilon/3$.
if part: Assume $f_n \stackrel{L_1}{\to} f$. Then $\{f_n\}$ is L_1 bounded and $f \in L_1$. Fix $\epsilon > 0$. Then

$$\int_A |f_n| \leq \int_A |f_n - f| + \int_A |f|.$$

Choose N such that for all $n \geq N$, $\int |f_n - f| < \epsilon/2$ and choose δ_0 with $P(A) < \delta_0$ implies $\int |f| < \epsilon/2$. Then for $n \geq N$, $P(A) \leq \delta_0$ implies $\int_A |f_n| < \epsilon$, implying $\{f_n\}$ uniformly integrable.

□

Theorem 6. *$f_n \geq 0$, $f_n \stackrel{a.s.}{\to} f$, f_n, f in L_1. Then $\{f_n\}$ is uniformly integrable if and only if $Ef_n \to Ef$.*

Proof. The only if part follows from Theorem 5.
if part: It is enough to show that $f_n \stackrel{L_1}{\to} f$ (This follows from Scheff's Lemma). Note that

$$E|f_n - f| = Ef_n + Ef - 2E\min\{f_n, f\}.$$

Since $Ef_n \to Ef$ by assumption and $E\min\{f_n, f\} \to Ef$ by DCT, implying $f_n \stackrel{L_1}{\to} f$. □

Theorem 7. *$(X_n, \mathcal{F}_n)_{n\geq 0}$ is a martingale. Let $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$ and $\sup E|X_n| = K$. ($E|X_n| \uparrow K$ since $|X_n|$ is a submartingale). Then the following are equivalent.*

i. $K < \infty$, $X_n \stackrel{L_1}{\to} X_\infty$.

ii. $(X_n, \mathcal{F}_n)_{0\leq n\leq\infty}$ *is a martingale.*

iii. $K < \infty$, $E|X_\infty| = K$.

iv. $\{X_n\}_{n\geq 0}$ *is uniformly integrable.*

Proof. (Complete verification is left as an exercise).
(i) $\implies$ (iv): Since $K < \infty$, $X_n \to X_\infty$ a.e. (Since L_1 bounded martingale converges almost surely, using Doobs's upcrossing inequality). Since $X_n \stackrel{L_1}{\to} X_\infty$, from Theorem 5, $\{X_n\}_{n\geq 0}$ is uniformly integrable.
(iv) $\implies$ (i): Since $\{X_n\}_{n\geq 0}$ is uniformly integrable, $\{X_n\}_{n\geq 0}$ is L_1 bounded implying $K < \infty$. Since $X_n \stackrel{a.s}{\to} X_\infty$, $X_n \stackrel{L_1}{\to} X_\infty$ by Theorem 5. Hence (i) $\Leftrightarrow$ (iv).
(ii) $\implies$ (iv): This follows directly from Theorem 4.
(iv) $\implies$ (ii): We will infact show (iv), (i) $\implies$ (ii). X_∞ is the almost sure limit of X_n and hence $\mathcal{F}_n$ measurable for all $n < \infty$ and hence $\mathcal{F}_\infty$ measurable. Also (i) implies $X_\infty \in L_1$. It is enough to check that $E(X_\infty \mid \mathcal{F}_n) = X_n$ a.e. We already know that for $m > n$, $X_n = E(X_m \mid \mathcal{F}_n)$ a.s. It is enough to show that $E(X_m - X_\infty \mid \mathcal{F}_n) \to 0$ a.e. as $m \to \infty$. We know that $X_m \stackrel{L_1}{\to} X_\infty$. Then

$$E|E[X_m - X_\infty \mid \mathcal{F}_n]| \leq EE[|X_m - X_\infty| \mid \mathcal{F}_n] = E[|X_m - X_\infty|] \to 0$$

as $m \to \infty$, implying $E|X_n - E[X_\infty \mid \mathcal{F}_n]| = 0$ implying $X_n = EX_\infty \mid \mathcal{F}_n$ a.e. Hence we have shown that (i), (ii) and (iv) are equivalent.
(iii) $\implies$ (iv): Since $|X_n|$ is a sub-martingale, $E|X_n| \uparrow K$ implying $E|X_n| \uparrow E|X_\infty|$. By Scheffe's theorem $|X_n| \stackrel{L_1}{\to} X_\infty$. By Theorem 6 $\{|X_n|\}$ and hence $\{X_n\}$ is uniformly integrable. Now we will show that (i) and (iv) implies (iii). Since $X_n \stackrel{L_1}{\to} X_\infty$, by Scheffe's Lemma and Fatou's Lemma $E|X_n| \to E|X_\infty| < \infty$. Also $E|X_n| \uparrow K$. Hence $K < \infty$ and $E|X_\infty| = K$. □

Theorem 8. *$(X_n, \mathcal{F}_n)_{0\leq n\leq N}$ is a non-negative martingale. Let $p > 1$. Then*

$$\left\| \max_{0\leq n\leq N} X_n \right\|_p \leq \frac{p}{p-1} \|X_N\|_p$$

Proof. The proof follows from the following Lemma 1 and Doob's maximal inequality. □

Lemma 1. *U, V non-negative random variables. Let $\lambda > 0$, $P(U > \lambda) \le (1/\lambda) \int_{U>\lambda} V dP$. Then, for $p > 1$,*

$$\|U\|_p \le \frac{p}{p-1} \|V\|_p .$$

Proof.

$$\begin{aligned}
EU^p &= \int_0^\infty p\lambda^{p-1} P(U > \lambda) d\lambda \\
&= \int_0^\infty p\lambda^{p-2} \int_{U>\lambda} V dP d\lambda \\
&= \int_0^\infty p\lambda^{p-2} \int_\Omega V(\omega) 1_{U>\lambda}(\omega) dP(\omega) d\lambda \\
&= \int_\Omega V(\omega) \int_0^\Omega p\lambda^{p-2} d\lambda dP(\omega) \\
&= \frac{p}{p-1} \int_\Omega V(\omega) U^{p-1} dP = \frac{p}{p-1} E(VU^{p-1}) \\
&\le \frac{p}{p-1} \{E(V^p)\}^{1/p} \{EU^{(p-1)q}\}^{1/q} \\
&= \frac{p}{p-1} \{E(V^p)\}^{1/p} \{EU^p\}^{1/q}
\end{aligned}$$

implying $\|U\|_p \le \frac{p}{p-1} \|V\|_p$ if $U \in L_p$. Otherwise, work with $\min\{U, n\}$. □

Corollary 1. *$(X_n, \mathcal{F}_n)_{n \ge 0}$ non-negative L_p-bounded sub-martingale. Then $X^* = \sup X_n \in L_p$ (True for L_p bounded martingale with $X^* = \sup |X_n|$).*

Proof. Let $X_N^* = \max_{0 \le n \le N} X_n$, $X_N^* \uparrow X^*$. By MCT, $E(X_N^*)^p \uparrow E(X^*)^p$. Since $\{X_n\}$ are L_p bounded,

$$E(X_N^*)^p \le \left(\frac{p}{p-1}\right)^p E |X_N|^p < M,$$

implying $X^* \in L_p$. □

Theorem 9. *Let $p > 1$. $\{X_n\}$ is L_p-bounded martingale or a non-negative sub-martingale. Then*

1. *$\{X_n\}$ is uniformly integrable.*
2. *$X_n \xrightarrow{L_p} X_\infty$.*

Proof. Proof is left as an Exercise. □

Remark 2. *Counter example to show that one cannot get rid of the non-negativity in case of sub-martingale. Let* $([0,1), \mathcal{B}, \lambda)$ *be the measure space and* $\mathcal{F}_n$ *is a dyadic filtration. Let* $X_n = -2^{n/2} 1_{[0,2^{-n})} \to 0$ *a.s. Check that* $\{X_n, \mathcal{F}_n\}$ *is a sub-martingale. Clearly* $X_n \overset{L_2}{\not\to} 0$ *as* $\|X_n\|_2 = 1$*. Complete verification is left as an Exercise.*

3 Example: Two color urn model

Suppose $(X_n, \mathcal{F}_n)$ adapted L_1-sequence. $EX_{n+1} \mid \mathcal{F}_n = a_n X_n + b_n$ $a_n \neq 0$. Find the associated martingale. Start with (W_0, B_0) balls with $0 \leq W_0, B_0$ and $W_0 + B_0 = 1$. At n th stage, W_{n-1} white, B_{n-1} black balls are available with $W_{n-1} + B_{n-1} = n$. Draw a ball at random and $R_{2\times 2}$ is a stochastic matrix. If you see a white ball, add R_{11} white and R_{12} black balls. If you see black ball, add according to second row. X_{n+1} is a vector which is $(1,0)'$ if white is drawn in nth stage, $(0,1)'$ if black. $Z_n = (W_n, B_n)$. Since Z_n is bounded for each fixed n, $Z_n \in L_1$. Observe that

$$Z_{n+1} = Z_n + X'_{n+1} R.$$

Find the associated martingale. (Discussed in class).

4 Stopping Time

$\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ is a filtration. $\tau : \Omega \mapsto \{0, 1, 2, \ldots, \infty\}$ is a measurable random time. σ-field on $\{0, 1, \ldots, \infty\}$ is $\mathcal{P}(\{0, 1, \ldots, \infty\})$. A random time is called a stopping time if $[\tau = n] \in \mathcal{F}_n$ for all $n \in \{0, 1, 2, \ldots, \infty\}$.
Example: Time at which one starts smoking is a stopping time. However, the time at which one stops smoking is not a stopping time.
$[\tau = n]$ is equivalent to $[\tau \leq n] \in \mathcal{F}_n$ for all $n \in \{0, 1, 2, \ldots, \infty\}$. This is easy to see for discrete dime. For continuous time $[\tau \leq t] \in \mathcal{F}_t$ for all $t \geq 0$. $[\tau = t] = [\tau \leq t] - \cup_n [\tau \leq t - 1/n : n \in \mathbb{N}]$.

In the discrete time case $[\tau < n] \in \mathcal{F}_n$, $[\tau \leq n-1] \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$.

Theorem 10. *For discrete parameter spaces* $[\tau = n] \in \mathcal{F}_n \Leftrightarrow [\tau \leq n] \in \mathcal{F}_n \Leftrightarrow [\tau < n] \in \mathcal{F}_n$.

Definition 6. $(X_n, \mathcal{F}_n)$ *adapted sequence.* τ *is a random time,* $[\tau < \infty] = \Omega$*. Stopping random variable* $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$.

Theorem 11. X_τ *is* $\mathcal{B}$ *measurable.*

Proof. For $B \in \mathcal{B}(\mathbb{R})$, $X_\tau^{-1}(B) = \cup_{n=0}^{\infty}[X_\tau \in B] \cap [\tau = n] = \cup_{n=0}^{\infty}[X_n \in B] \cap [\tau = n]$. □

Definition 7. *(Stopping σ-field)* $\mathcal{F}_\tau = \{A \in \mathcal{B} : A \cap [\tau = n] \in \mathcal{F}_n \quad \textit{for all} \quad n\}$

Clearly X_τ is $\mathcal{F}_\tau$ measurable.

4.1 Properties

1. $\tau \equiv \sigma \Rightarrow \mathcal{F}_\tau = \mathcal{F}_\sigma$
2. τ, σ are stop times, then $[\tau < \sigma], [\tau > \sigma], [\tau = \sigma]$ are all $\mathcal{F}_\tau \cap \mathcal{F}_\sigma$-measurable. ($[\tau < \sigma] \cap [\tau = n] = [\sigma > n] \cap [\tau = n] \in \mathcal{F}_n$)
3. τ is $\mathcal{F}_\tau$ measurable. Then $[\tau = k] \cap [\tau = n] \in \mathcal{F}_n$ for $k \leq n$.
4. $\tau \leq \sigma$, then $\mathcal{F}_\tau \subset \mathcal{F}_\sigma$.
5. $(X_n, \mathcal{F}_n)_{0 \leq n \leq N}$ is a (sub)-martingale. $\tau \leq \sigma$ is a stopping time. (X_τ, X_σ) is a (sub)-martingale corresponding to $(\mathcal{F}_\tau, \mathcal{F}_\sigma)$.

4.2 Doob's Upcrossing Inequality

$\{X_n\}_{0 \leq n \leq N}, a < b$. Define

$$\begin{aligned}
\tau_0 &= 0 \\
\tau_1 &= \inf\{n : X_n \leq a\} \\
&\vdots \\
\tau_{2k+1} &= \inf\{n \geq \tau_{2k} : X_n \leq a\} \\
\tau_{2k+2} &= \inf\{n \geq \tau_{2k+1} : X_n \geq b\}
\end{aligned}$$

with the convention that $\inf\{\phi\} = N$. Define

$$\begin{aligned}
\mathrm{U}(\{X_n\}_{0 \leq n \leq N}; [a, b]) &= \sup\{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\} \\
\mathrm{U}(\{X_n\}_{n \geq 0}; [a, b]) &= \uparrow \lim_N \{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\}.
\end{aligned}$$

Lemma 2. *τ_is are stop times.*

Proof. τ_0, τ_1 are stop times. Assume τ_{2k} is a stop time. Then

$$\begin{aligned}
\{\tau_{2k+1} = i\} &= \cup_{j=0}^{i-1}\{\tau_{2k+1} = i, \tau_{2k} = j\} \\
\{\tau_{2k+1} = i, \tau_{2k} = j\} &= \{\tau_{2k} = j\} \cap \{X_{j+1} > a\} \cap \cdots \cap \{X_{i-1} > a\} \cap \{X_i \leq a\} \in \mathcal{F}_i
\end{aligned}$$

□

For an adapted sequence, $\{\tau_k\}$ forms a stopping time, which implies $\sup\{l : X_{\tau_{2l-1}} \leq a < b \leq X_{\tau_{2l}}\}$ is measurable implying $\mathrm{U}(\{X_n\}_{n\geq 0}; [a,b])$ is measurable.

Theorem 12. *(Doob's Upcrossing Inequality). $\{X_n, \mathcal{F}_n\}$ is a submartingale and $a < b$. Then*

$$\begin{aligned} E\,U(\{X_n\}_{0\leq n\leq N}; [a,b]) &\leq \frac{E[(X_N - a)^+] - E[(X_0 - a)^+]}{b-a} \\ &\leq \frac{E\,|X_N| + |a|}{b-a}. \end{aligned}$$

Corollary 2. *An L_1-bounded martingale converges almost surely.*

5 Problem

A branching process is defined as follows: We start with one member, namely the population size is $Z_0 = 1$. Let $\xi_i^n, i = 1, 2, \ldots, n$ denote the number of children of ith individual in the nth generation. Assume ξ_i^n are independent with common mean $\mu > 0$. Let Z_n denote the population size in the nth generation. Let $p_k = P(\xi_i^n = k), k \geq 0, \mu = E(\xi_i^n)$. $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_i^n$. It is easy to see that $X_n = Z_n/\mu^n$ is a martingale.

1. Does $\{X_n\}$ has a limit?
   Since X_n is non-negative, $\{X_n\}$ converges almost everywhere to an integrable random variable X.

2. If $\mu < 1$, $Z_n = 0$ almost surely for large n.
   $Z_n = X_n\mu^n$. Since $X_n \overset{a.s.}{\to} X$ and $\mu^n \to 0$, $Z_n \overset{a.s.}{\to} 0$. There exists a P-null set N such that for all $\omega \notin N$, $Z_n(\omega) \to 0$. Given $\epsilon = 1/2$, there exists $N_0(\omega)$ such that $Z_n(\omega) < \epsilon$ for all $n \geq N_0(\omega)$ implying $Z_n(\omega) = 0$ for all $n \geq N_0(\omega)$. This implies Z_n converges to 0 with probability 1 for all large n.

3. If $\mu < 1$, $X_n = 0$ almost surely for large n.
   $\sum_{n=1}^{\infty} P(X_n > 0) = \sum_{n=1}^{\infty} P(Z_n > 0) = \sum_{n=1}^{\infty} P(Z_n \geq 1) \leq \sum_{n=1}^{\infty} E(Z_n) = \sum_{n=1}^{\infty} \mu^n < \infty$. Hence by Borel Cantelli Lemma $P(\limsup A_n) = 0$ which means $P(\cap_{n=1}^{\infty} \cup_{k\geq n} A_k) = 0$ implying $P(A_n \,\text{occurs infinitely often}) = 0$ implying $P(X_n = 0 \,\text{for all large}\, n) = 1$. Hence $X_0 = 0$ eventually with probability 1.

4. If $\mu = 1$ and $P(\xi_1^1 > 1) > 0$, then $Z_n \to 0$ a.s.
   Z_n non-negative martingale, hence $Z_n \to Z_\infty$ a.e and $Z_\infty \in L_1$. It is enough to show that $P(Z_\infty = k) = 0$ for all $k \geq 1$, or in other words $P(Z_n = k \,\text{eventually}) = 0$. Observe that $P(Z_n = k \,\text{eventually}) \leq P(\text{one of}\, \xi_1^n, \ldots, \xi_k^n \leq 1, \text{eventually})$. It is enough to show that $P(\xi_1^n, \ldots, \xi_k^n > 1, \text{infinitely often}) = 1$. To that end, note that $\sum_{n=1}^{\infty} P(\xi_i^n > 1)^k = \infty$. The result follows from second Borel Cantelli Lemma.