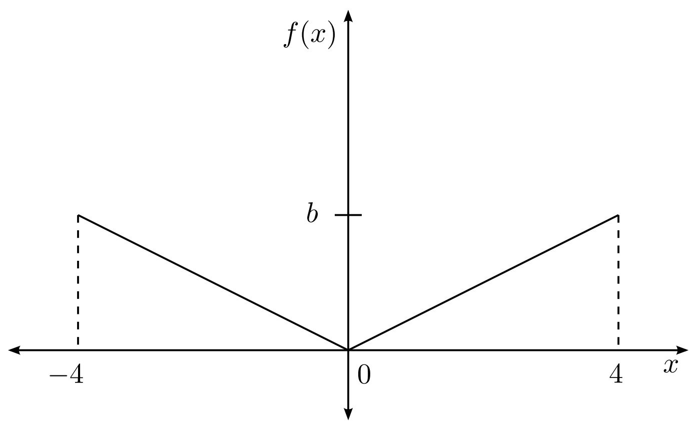
Homework 10(Due on November 28)

November 29, 2012 STA 4442/5440

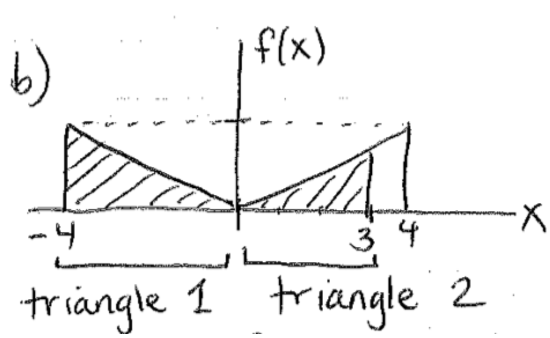
1. The preceding figure is the probability density curve of the random variable X.



(a) Find b so that $f(x)$ is a probability density function.

(b) What is $P(-4 \leq X \leq 3)$?

(c) What is $P(X = 1)$?



Solution:

(a) Total area = 1. Hence area of each of the two triangles (in the positive and the negative half) is 1/2. We have $(1/2) \times 4 \times b = 1/2$. Hence $b = 1/4$.

(b) shaded area = $P(-4 \leq X \leq 3)$ = area of triangle 1 + area of triangle 2 $= 1/2 + (1/2) \times 3 \times (3/16)$ noting that the equation of the line is $y = (1/16)x$.

(c) Probability that a continuous random variable takes on a single value is 0.

2. A survey report states that 65% of women over 30 visit their doctors for a physical exam at least once in two years. 6 adult women are randomly selected and asked whether or not they have had a physical exam in the past 2 years. Let X represent the number of women (out of the 6 asked) who have had a physical exam in the past 2 years.

   (a) What is the expected value of X?

   (b) What is the probability that fewer than 3 of the women asked have had a physical exam in the past 2 years?

   (c) What is the probability that exactly 5 of the women asked have NOT had a physical exam in the past 2 years?

   Solution: X = the number who have had exam. We are conducting 6 Bernoulli trials implies that $X \sim \text{Bin}(6, 0.65)$.

   (a) $E(X) = np = 6 \times 0.65 = 3.9$

   (b) $P(X < 3 = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
   $= \binom{6}{0}(0.65)^0(0.35)^6 + \binom{6}{1}(0.65)^1(0.35)^5 + \binom{6}{2}(0.65)^2(0.35)^4 = 0.0018 + 0.0204 + 0.095 = 0.1172$

   (c) P(exactly 5 have not had exam) = P(exactly 1 has had exam) = P(X=1) =0.0204.

3. The bonding strength of a drop of plastic glue is normally distributed with mean 100 pounds and standard deviation 8 pounds.

   (a) What is the probability that the bonding strength is between 90 and 110 pounds?

   (b) A broken plastic strip is repaired with a drop of this glue and then subjected to a test load of 98 pounds. What is the probability that the bonding will fail?

   (c) The manufacturer of the glue wishes to know the value v for which 98.5% of bonding strengths are less than v. What is the value of v?

   Solution: Let X = bonding strength (maximum amount of weight that can be supported). Hence $X \sim N(100, 8)$. Denoting by Z the standard normal,

(a) $P(90 < X < 100) = P\left(\frac{90-100}{8} < Z < \frac{110-100}{8}\right) = P(X < 1.25) - P(Z < -1.25) = 0.8944 - 0.1056 = 0.7888.$

(b) P(bonding will fail) = P(X ¡ 98) = $P\left(Z < \frac{98-100}{8}\right) = P(Z < -0.25)0.4013.$

(c) $P(Z \leq z) = 0.985$ implies $z = 2.17$. Destandardizing, we obtain $\frac{v-100}{8} = 2.17$ whence $v = 117.36$.

4. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, then one-half of the value that appears on the die. Determine her expected winnings. Solution: Let X be the amount won. Then

$$E[X] = (1/12)(2 + 4 + 6 + 8 + 10 + 12 + 1/2 + 1 + 3/2 + 2 + 5/2 + 3) = 4.375$$

5. Because 12% of the reservation holders are no-shows, a U. S. airline sells 448 tickets for a flight that can accomodate 400 passengers. For the following questions, use the normal approximation to the binomial distribution. Be sure to apply continuity correction. (Because we are using a continuous r.v. to approximate the integer-valued binomial r.v., continuity correction is necessary and is given by $P(X = i) \to P(X \in (i - 0.5, i + 0.5))$, $P(X \geq i) \to P(X \geq i - 0.5)$, $P(X > i) = P(X \geq i + 1) \to P(X \geq i + 1 - 0.5)$, $P(X \leq i) = P(X \leq i + 0.5)$).

(a) Find the approximate probability that one or more reservation holders will not be accommodated on the flight.

(b) Find the approximate probability of having fewer than 380 passengers on the flight.

Solution: success = person shows up. Let X = number of people who show up. $X \sim Bin(448, 0.88)$

(a) P(one or more people have no seat) = P(more that 400 people show up) $= P(X > 400)$. Now $n = 448, p = 0.88, np = 394.24, \sqrt{np(1-p)} = 6.878$. Closed interval: $P(X > 400) = P(X \geq 401)$. Apply continuity correction by subtracting 1/2 from the left end point. Hence $P(X \geq 401) = P(X \geq 400.5)$. Hence

$$\begin{aligned} P(X \geq 400.5) &= P\left(Z \geq \frac{400.5 - 394.24}{6.878}\right) \\ &= P(Z \geq 0.91) \\ &= 1 - P(Z \leq 0.91) \\ &= 1 - 0.8186 = 0.1814. \end{aligned}$$

(b) Want $P(X < 380) = P(X \leq 379)$. By applying continuity correction, add 1/2 to the right end point. $P(X \leq 379) = P(X \leq 379.5)$. Hence

$$\begin{aligned} P(X \leq 379.5) &= P\Big(Z \leq \frac{379.5 - 394.24}{6.878}\Big) \\ &= P(Z \leq -2.14) \\ &= 0.0162 \end{aligned}$$

6. A casino offers a game of chance for a single player in whicha fair coin is tossed at each stage. The pot starts at 1 dollar and is doubled every time a head appears. The first time a tail appears, the game ends and the player wins whatever is in the pot. Thus the player wins 1 dollar if a tail appears on the first toss, 2 dollars if a head appears on the first toss and a tail on the second, 4 dollars if a head appears on the first two tosses and a tail on the third, 8 dollars if a head appears on the first three tosses and a tail on the fourth, and so on. In short, the player wins 2^{k-1}dollars if the coin is tossed k times until the first tail appears.

   (a) What is the expected payout?

   (b) What is the expected payout if the coin is biased with success probability 1/3?

   Solution:

   (a) Expected payout $= \sum_{k=1}^{\infty} 2^{k-1} \frac{1}{2^{k-1}} \frac{1}{2} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$.

   (b) Expected payout $= \sum_{k=1}^{\infty} 2^{k-1} \frac{1}{3^{k-1}} \frac{2}{3} = \frac{2}{3} \times 3 = 2$

7. Let X and Y be independent geometric random variables with the same parameter p. Find the value of

$$P(X = i \mid X + Y = n).$$

   Argue the expression obtained above without doing any computation. Solution:

   (a)

$$P(X + Y = n) = \sum_{k=1}^{n-1} p(1-p)^{k-1} p(1-p)^{n-k-1} = (n-1)p^2(1-p)^{n-2}$$

   Hence

$$P(X = i \mid Y = n - i) = \frac{P(X = i, Y = n - i)}{P(X + Y = n)} = \frac{p(1-p)^{i-1} p(1-p)^{n-i-1}}{(n-1)p^2(1-p)^{n-2}} = 1/(n-1)$$

   (b) Clearly $(X+Y) = n$ implies that the last trial is a success and there is a success in any trial from trial 1 to trial n-1. Hence the probability is $1/(n-1)$.