Conditional probability - Class 5

September 10, 2013 Debdeep Pati

1 Conditional probability

1. Toss two dice, suppose each of the possible 36 outcomes are equally likely. If we observed that the first die is a 3, what is the probability that the sum of the two dice equals to 8? Given the first die is 3, the sample space can be reduced to (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) and the outcomes still equally likely. So the desired probability is 1/6.

2. If $P(F) > 0$, $P(E \mid F) = P(E \cap F)/P(F)$

3. A small business with 120 workers and 2 employee benefits, A and B.

	A	A^c	
B	64	34	98
B^c	7	15	22
	71	49	120

   Calculate the probability that an employee chosen at random has the retirement benefit (A) given that she takes the health-care benefit (B). Restrict our attention to the first row: 64/98

4. A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$, for all $0 \le x \le 1$. Given that the student is still working after 0.75 hours, what is the conditional probability that the full hour is used?

   Suppose the student finishes the exam at time T. Let $L_x = T < x$. We know $P(L_x) = x/2$, $x \in (0,1)$. Now the event that the student is still working at time 0.75 is the complement of the event $L_{0.75}$, so the desired probability is obtained from

$$\begin{aligned} P(\{T > 1\} \mid \{T > 0.75\}) &= \frac{P(\{T > 0.75\} \cap \{T > 1\})}{P(\{T > 0.75\})} = \frac{P(\{T > 1\})}{P(\{T > 0.75\})} \\ &= \frac{1 - P(T \le 1)}{1 - P(T \le 0.75)} = \frac{1 - 1/2}{1 - 3/8} = 0.8 \end{aligned}$$

5. If the sample space S is finite, and all outcomes are equally likely, it is often convenient to compute $P(E \mid F)$ by using F as the sample space as

$$P(E \mid F) = |E \cap F|/|F|$$

where $|A|$ = in the number of outcomes in A. Count $|E \cap F|$ in the reduced sample space F. (Note that $E \cap F$ is an event in F since $E \cap F \subset F$.)

6. A coin is flipped twice. Assuming that all four points in the sample space are equally likely, what is the conditional probability that both flips land on heads given that a) the first flip lands on heads (1/2) b) at least one flip lands on heads (1/3)

7. An urn contains 10 white balls, 5 yellow balls, and 10 black balls. Choose a ball at random. Suppose we know that it is not black. What is the prob. that it is yellow? $P(Y \mid B^c) = P(Y \cap B^c)/P(B^c) = (5/25)/(15/25)$ Alternative: Consider the reduced sample space B^c or $Y \cup W$. Then $P(Y \mid B^c)$ = P(choose a yellow out of white and yellow) = 5/15 = 1/3

8. The multiplication rule for conditional probability

$$P(E \cap F) = P(E \mid F)P(F)$$
$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2)$$
$$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n) = P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_1 \cap E_2) \ldots P(E_n \mid E_1 \cap E_2 \cap \ldots \cap E_{n-1})$$

Celine is undecided as to whether to take a French course or a chemistry course. She estimated that her probability of receiving an A grade would be 0.5 in a French course and 2/3 in a chemistry course. If she decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry? (What are the events? A: receiving an A grade; C: taking chemistry; F: taking French. $P(A \mid F) = 1/2, P(A \mid C) = 2/3, P(C) = P(F) = 1/2. P(C \cap A)$?
$P(A \cap C) = P(A \mid C)P(C) = (2/3)(1/2) = 1/3$)

9. A useful formula for calculating probabilities (Total probability formula)

$$P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c)$$

In general if $\cup_{i=1}^{n} F_i = S$, then

$$P(E) = \sum_{i=1}^{n} P(E \mid F_i)P(F_i)$$

Example: What is the probability that Celine get an A from either French or chemistry?

$$P(A) = P(A \cap C) + P(A \cap F) = P(C)P(A|C) + P(F)P(A|F) = (1/2)(2/3) + (1/2)(1/2) = 7/12$$

Example: Suppose that you are dealt two cards sequentially from a standard deck. Let A be the event that the 1st card is an ace. Let B represent the event that the second card is an ace. Questions:

(a) What is the probability that the first card is an ace? 1/13 (or $P(A \cap B) + P(A \cap B^c) = (4/52).(3/51) + (4/52).(48/51)$)

(b) What is the probability that the second card is an ace given that the first card is not an ace?(4/51)

(c) What is the probability that the second card is an ace given that the first card is an ace? (3/51)

(d) What is the probability that the second card is an ace?

10. Odds of an event A. We can express the change in the probability of a hypothesis when new evidence is introduced in a compact form using change in the odds of the hypothesis. The odds of an event A is defined by $P(A)/P(A^c) = P(A)/[1 - P(A)]$. The odds of an event A tells how much more likely it is that the event A occurs than it is that it does not occur.

11. Change of probability with a new evidence Hypothesis H with probability $P(H)$. $P(H \mid E) = P(E \mid H)P(H)/P(E), P(H^c \mid E) = P(E \mid H^c)P(H^c)/P(E)$.

12. Bayes formula

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$$

Example: An insurance company believes that people are divided into two classes, those who are accident prone and those who are not. The company's stats show that an accident-prone person will have an accident at some time within a fixed period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

A_1 denote the event that the policy holder will have accident within a year of purchasing the policy and let A denote the event that the policy holder is accident prone. Hence,

$$\begin{aligned} P(A_1) &= P(A_1 \mid A)P(A) + P(A_1 \mid A^c)P(A^c) \\ &= (0.4).(0.3) + (0.2).(0.7) = 0.26 \end{aligned}$$

Suppose the new policy holder had an accident within 1 year of making a policy. What is the probability that she is accident prone?

$P(A \mid A_1) = P(A_1 \mid A)P(A)/P(A_1) = (0.4).(0.3)/0.26 = 6/13$.