Conditional probability 2 - Class 6

September 12, 2013 Debdeep Pati

1. Change of probability with a new evidence Hypothesis H with probability $P(H)$. $P(H \mid E) = P(E \mid H)P(H)/P(E), P(H^c \mid E) = P(E \mid H^c)P(H^c)/P(E)$.

2. Bayes formula

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$$

   Example: An insurance company believes that people are divided into two classes, those who are accident prone and those who are not. The company's stats show that an accident-prone person will have an accident at some time within a fixed period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

   A_1 denote the event that the policy holder will have accident within a year of purchasing the policy and let A denote the event that the policy holder is accident prone. Hence,

$$\begin{aligned} P(A_1) &= P(A_1 \mid A)P(A) + P(A_1 \mid A^c)P(A^c) \\ &= (0.4).(0.3) + (0.2).(0.7) = 0.26 \end{aligned}$$

   Suppose the new policy holder had an accident within 1 year of making a policy. What is the probability that she is accident prone?

   $P(A \mid A_1) = P(A_1 \mid A)P(A)/P(A_1) = (0.4).(0.3)/0.26 = 6/13$.

3. Suppose that we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

   Let RR: all red card BB: all black card RB: red-black card. R: upturned side of the chosen card is red

$$\begin{aligned} P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\ &= \frac{P(R \mid RB)P(RB)}{P(R \mid RB)P(RB) + P(R \mid BB)P(BB) + P(R \mid RR)P(RR)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(1/2)(1/3)}{(1/2)(1/3) + 0.(1/3) + 1.(1/3)} \\
&= 1/3
\end{aligned}
$$

1 Independent Events

Multiple events and their relationships: Two events, E and F, are independent if knowing that one had occurred gave us no information about whether the other had or had not occurred. E and F are independent iff

$$P(E \mid F) = P(E) \equiv P(F|E) = P(F) \equiv P(E \cap F) = P(E)P(F)$$

1. If $P(E \cap F) \neq P(E)P(F)$, then E is dependent on F.

2. Verify independence by computation

$$P(E|F) = P(E)\, if\, P(F) \neq 0 \equiv P(F|E) = P(F)\, if\, P(E) \neq 0$$

3. Property: If E does not depend on the occurrence or non occurrence of F, then E and F are independent.
   Proof: If $P(E \mid F) = P(E \mid F^c)$, then $P(E) = P(E \mid F)P(F) + P(E \mid F^c)P(F^c) = P(E \mid F)$.

4. Example:
   A card is selected at random from a deck of 52.
   E : the selected card is an ace
   F : the selected card is a spade
   Then E and F are independent
   As a matter of fact, $P(E \cap F) = 1/52, P(E) = 4/52, P(F) = 13/52$

5. Example:
   Tossing 2 dice, $S = \{(i, j) : i, j = 1, 2, \ldots, 6\}$ $E_1 = \{i + j = 6\}, E_2 = \{i + j = 7\}$ $F = \{i = 4\}$ (the first die equals 4) Then E_1 and F are dependent while E_2 and F are independent. The chance of getting a total of 6 depends on the outcome of the first dice, while the chance of getting a total of 7 does not!

   $P(E_1 \cap F) = 1/36, P(E_2 \cap F) = 1/36, P(E_1) = 5/36, P(E_2) = 1/6, P(F) = 1/6$

6. Result: If E and F are independent, then so are E and F^c

7. Definiton. Three events E and F and G are said to be independent if $P(E \cap F) = P(E)P(F)$, $P(E \cap G) = P(E)P(G)$, $P(F \cap G) = P(F)P(G)$, $P(E \cap F \cap G) = P(E)P(F)P(G)$ More than pairwise independence! In fact pairwise independence does not imply (joint) independence (for ≥ 3 events)

8. Example: Tossing two dice $S = \{(i, j) : i, j = 1, 2, \ldots, 6\}$ Let $E = \{(i, j) : i + j = 7\}, F = \{(4, j) : 1 \leq j \leq 6\}$, $G = \{(i, 3) : 1 \leq i \leq 6\}$ E and F are independent E and G are independent F and G are independent E, F and G are not independent!

9. If three events A, B, C are pairwise independent (any two are independent). Can we say that knowing the two events does not tell us anything about the third? For example $P(C \mid A \cap B) = P(C)$? A fair coin is tossed twice. Let A denote the event of heads on the first toss, B denote the event of heads on the second toss, and C the event that exactly one head is thrown. A and B are clearly independent, and $P(A) = P(B) = P(C) = .5, P(C \mid A) = .5$, so A and C are independent. But $P(C \mid A \cap B) = 0, P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C)$.

10. Independence between Events: If E, F, and G are independent, then E will be independent of any event formed from F and G. $(F \cup G, F \cap G, F^c)$

    We can extend the denition of independence to more than 3 events. In some problems, we make the independence assumption to simplify the calculation

11. Example: A parallel system with n components functions if current flows from A to B. For such a system below, if component i, independent of other components, functions with prob $p_i, i = 1, 2, \ldots, n$, what is the probability that the system functions?
    Solution:
    $A_i = \{\text{component } i \text{ functions}\}$, $E = \cup_i A_i$, A_i are independent $P(E) = 1 - P(E^c) = 1 - P(\cap_{i=1}^n A_i^c) = 1 - \prod_{i=1}^n (1 - p_i)$.

12. An infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that

    (a) at least 1 success occurs in the first n trials

    (b) exactly k successes occur in the first n trials;

    (c) all trials result in successes?