Counting - Class 1

August 27, 2013 Debdeep Pati

1 General Principles

1.1 Multiplication principle

Let's say I spin a regular six sided die. There are 6 possible outcomes. When you spin 2 dice, how many different ways are there for the outcome to show up? The multiplication principle is a overwhelming idea that when you do two things one after the other in order to accomplish a complete task, then if you want to count how many way are there to do the complete task - it is the number of ways of doing the first one times the number of ways of doing the second.

How many ways can all the students in this class get in line? There are 44 ways I can choose the first person in line. Now I have to choose the second person. There are 43 ways to do that. So there are $44 \times 43 \times \cdots 1$ many ways. We now introduce $n!$ as the number of ways of lining up n people: There are n choices for the first spot, $(n-1)$ choices for the second, $(n-2)$ choices for the third, and so on giving $n(n-1)(n-2)\ldots(3)(2)(1) = n!$ in all.

n! actually grows similar to n^n. For example to form all the possible queues of 20 people if we could form a new queue every second, it actually takes billions of years. This type of exponential growth obviously makes it very important to avoid factorial numbers of steps in computer algorithms, etc.

1.2 Addition principle

Again will go by example. How many ways to spin a odd sum? Break up the problem into disjoint possibilities (answer is 36/2=18) don't always trust your intuition.

1.3 Complement principle

If you can't count what you want to count - count the opposite.

1. How many are not doubles? (36-6 = 30)

1.4 Counting double

What do I mean double here? - it means multiple counting in a controlled way. Its a really really good technique. How many ways I can choose two of you to go and complain about the leak down there? I can pick anybody for the first person .... I have counted every single person in a pair. So the right number of ways is $(36 \times 35)/2$. An instance when double counting makes life a lot easier. Suppose I wanna choose a president, a vice president, a senator and a treasurer from you guys. Ordering matters here. Introduce the permutation notation here. Suppose we just want to select a committee of size 4. Introduce the combination notation here.

2 Permutations

Often we are interested in the different orders of some objects. For example, people may want to rank some products, A, B, C and their order can be ABC, ACB, BAC, BCA, CAB, CBA, totally 6 possible ranking orders. Each arrangement is called a permutation. There are totally 6 possible permutations. Can we come out with a formula to compute the total number of permutations? Suppose we have n objects. $n(n-1)(n-2)3.2.1 = n!$

The number of different permutations (ordering is important) of r objects that could be formed from a total of n objects.

$$n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

We define n_{P_r} for $r \leq n$ by

$$n_{P_r} := \frac{n!}{(n-r)!}$$

3 Combinations

The number of different groups of r objects that could be formed from a total of n objects.

How many different groups of 3 could be selected from the 5 items A, B, C, D, and E? 5*4*3/(3*2*1) = 10

In general, the number of different groups of r items that could be formed from a set of n items (ordering is not important) is

$$\frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

We define $\binom{n}{r}$ for $r \leq n$ by

$$\binom{n}{r} := \frac{n!}{(n-r)!r!}$$

defined by the number of possible combination of n objects taken r at a time.

4 Examples of permutations

1. A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible? ($3 \times 4 \times 5 \times 2 = 120$)

2. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? ($26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$)

3. How about if repetition among letters and numbers were prohibited? ($6 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$)

4. A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score. a) how many different rankings are possible? 10! b) If the men are ranked among themselves and the women among themselves, how many different rankings are possible? $(6!)(4!) = (720)(24)$

5. Ms Jones has 10 books that she is going to put on her bookshelf. Of there, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangement are possible? $4!4!3!2!1! = 6921$

4.1 Number of permutations of a set of n objects when certain of them are indistinguishable from each other.

How many different letter arrangements can be formed using the letters P E P P E R?(6!/(3!2!)). In general, the same reasoning as that used in Ex 3d shows that there are $n!/(n_1!n_2!...n_r!)$ different permutations of n objects, of which n_1 are alike, n_2 are alike, $\ldots n_r$ are alike.

A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they places, how many outcomes are possible? $10!/(4!3!2!1!) = 12{,}600$