1 Examples of combination

1. A committee of 3 is to be formed from a group of 20 people. How many different committees are possible? ($\binom{20}{3} = \frac{20\times19\times18}{3\times2\times1}$)
2. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together? ($\binom{5}{2}\binom{7}{3} = 350, (\binom{7}{3} - \binom{2}{2}\binom{5}{1})\binom{5}{2} = 300$)
3. Consider a set of n antennas of which m are defective and n-m are functional and assume that all of the defectives and all of the functional are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive? ($\binom{n-m+1}{m}$)
4. An useful combinatorial identity is

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, 1 \le r \le n$$

1.1 Binomial coefficient

The expression $\binom{n}{r}$ is often referred to as the binomial coefficient as it appears in the Binomial theorem as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof of Binomial theorem: Consider $(x_1 + y_1)(x_2 + y_2) \ldots (x_n + y_n)$. Its expansion consists of the sum of 2^n terms, each term being the product of n factors. Each of the $2n$ terms in the sum will contain as a factor either x_i or y_i for each $i = 1, 2, n$. How many of the 2^n terms in the sum will have k of x_i and $(n-k)$ of y_i? A term in the sum has the generic formula $z_1 \cdots z_n$ where z can be either x or y. As each term consisting of k of x_i and $(n-k)$ of y_i corresponds to a choice of a group of k from the n values $z_1, z_2, \ldots, z_n$, there are $\binom{n}{k}$ such terms. Thus, letting $x_i = x, y_i = y, i = 1, , n$ we see that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

1.2 Examples

1. How many subsets are there of a set consisting of n elements? ($\sum_{k=0}^{n} \binom{n}{k} = 2^n$). Mention other argument to get this.

1.3 Multinomial coefficients

A set of n distinct items is to be divided into $r$distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where summation of them equals to n. How many divisions are possible?

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots n_{r-1}}{n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}$$

1.4 Notations

If $n_1 + \ldots + n_r = n$, define

$$\binom{n}{n_1, n_2, \ldots, n_r} := \frac{n!}{n_1!n_2!\ldots n_r!}$$

representing the number of ways of diving n distinct items is to be divided into $r$distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

1.5 Examples

1. A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible? ($\frac{10!}{5!2!3!} = 2520$)

2. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible? ($\frac{10!}{5!5!2} = 126$)

3. In the first round of a knockout tournament involving $n = 2^m$ players, the n players are divided into $n/2$ pairs, with each of these pairs then playing a game. The losers of the games are eliminated while the winners go on to the next round, where the process is repeated until only a single player remains. Suppose we have a knockout tournament of 8 players. (a) How many possible outcomes are there for the initial round? (b) How many outcomes of the tournament are possible, where an outcome gives complete information for all rounds?

   a) The number of ways of dividing the 8 players into a first pair, a second pair, a third pair and a fourth pair is $\binom{8}{2,2,2,2} = \frac{8!}{2^4}$. The number of possible pairings when there is no ordering of the player is $\frac{8!}{4!2^4}$. There are 2 possible choices from each pair to select a winner so the number of possible ways is $\frac{8!}{4!2^4} \times 2^4 = \frac{8!}{4!}$. An alternative is to select the winner in $\binom{8}{4}$ ways and pair up with the loser in $\binom{8}{4} \times 4!$ ways.

   b) $\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8!$

2 Bose-Einstein statistics

Problem numbers 31 (in Homework 1) from the Sheldon and Ross book (8th edition) are direct applications of Bose Einstein Statistics - see also Section 1.6 of Sheldon and Ross book.

In quantum statistics, Bose-Einstein statistics (or more colloquially BE statistics) is one of two possible ways in which a collection of indistinguishable particles may occupy a set of available discrete energy states. The aggregation of particles in the same state, which is characteristic of particles obeying Bose-Einstein statistics, accounts for the cohesive streaming of laser light and the frictionless creeping of superfluid helium. The theory of this behaviour was developed (1924-25) by Albert Einstein and Satyendra Nath Bose, who recognized that a collection of identical and indistinguishable particles can be distributed in this way.

A much simpler way to think of Bose-Einstein distribution function is to consider that n particles are denoted by identical balls and r shells are marked by $r-1$ line partitions. It is clear that the permutations of these n balls and $r-1$ partitions will give different ways of arranging bosons in different energy levels. Total number of ways is $\binom{n+r-1}{r-1}$.

2.1 Reformulation of BE statistics

A reformulation is to consider the number of ways n indistinguishable balls distributed in r distinguishable urns. This is same as the number of non-negative integer-valued vectors $(x_1, x_2, \ldots, x_n)$ such that

$$x_1 + x_2 + \ldots + x_n = n$$

To that end let us compute the number of positive integral solutions of the above equation. You want a partition of $1, 2, \ldots, n$. Imagine n objects with $n-1$ spaces in between and you want to put $(r-1)$ items in those $n-1$ spaces. Clearly, there are $\binom{n-1}{r-1}$ ways. To obtain the number of non-negative solutions, note that the number of is same as the positive integral solutions of

$$y_1 + \ldots + y_r = n + r$$

(seen by letting $y_i = x_i + 1, i = 1, \ldots, r$).

Then from the preceding discussion by replacing n by $(n+r)$, the number of non-negative integer-valued vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + x_2 + \ldots + x_n = n$ is precisely

$$\binom{n-r+1}{r-1}$$