Random Variables - Class 15

October 15, 2013 Debdeep Pati

1 Random variables

1.1 Examples

1. Let X be a nonnegative integer-valued r.v.Then $E(X) = \sum_{i=1}^{\infty} P(X \geq i)$.
   Proof:

$$\begin{aligned} \sum_{i=1}^{\infty} P(X \geq i) &= \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(X = k) \\ \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(X = k) &= \sum_{k=1}^{\infty} k P(X = k) = E(X) \end{aligned}$$

2. An urn contain N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that

   (a) exactly n draws are needed;

   (b) at least k draws are needed?

   (If we let X denote the number of draws needed to select a black ball, then X is a geometric random variable with parameter $\frac{M}{(M+N)}$).

2 Continuous Random Variables

Discrete random variables are those whose set of possible values is either finite or countably infinite. Continuous random variables are those whose set of possible values is uncountable, etc lifetime of a light bulb, amount of precipitation in a year.

2.1 Probability density function

1. If there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers

$$P(X \in B) = \int_B f(x)dx,$$

then the function f is called the probability density function of the random variable X.

2. All prob statements about a continuous r.v.X can be answered in terms of its density function f.

3. $\int_{-\infty}^{\infty} f(x)dx = 1$.

4. The probability of a certain interval $[a, b]$ in $P(a \le X \le b) = \int_a^b f(x)dx$.

5. $P(X = a) = \int_{x=a}^{a} f(x)dx = 0$.

6. $P(X < a) = P(X \le a) = F(a) = \int_{-\infty}^{a} f(x)dx$.

7. In addition, since $F(x) = \int_{-\infty}^{x} f(t)dt$, so that $F'(x) = f(x)$.

8. Probability density function can be considered as the measure of how likely that the random variable will be near a. Meaning of Density: When ϵ is small

$$P(a - \epsilon/2 \le X \le a + \epsilon/2) = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx = \epsilon f(a).$$

   So $f(a) = \lim_{\epsilon \to 0} \frac{P(|X-a|<\epsilon/2)}{\epsilon}$.

9. Find the probability density function for the random variable Y with cdf

$$F(y) = \begin{cases} 0, y < 0 \\ 1 - e^{-y^2}, y \ge 0 \end{cases}$$

2.2 Examples

1. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2), 0 < x < 2 \\ 0 \; o.w. \end{cases}$$

   What is the value of $C(3/8)$? and what is the value of $P(X > 1) = 1/2$

2. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100}, x \ge 0 \\ 0 \; o.w. \end{cases}$$

What is the probability that (a) a computer will function between 50 and 150 hours before breaking down $(e^{-1/2} - e^{-3/2}) \approx 0.384$
(b) it will function less than 100 hours? $(1 - e^{-1}) \approx 0.633$

3. The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} 0, x \le 100 \\ 100/x^2, x > 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events $E_i, i = 1, 2, 3, 4, 5$, that the i-th such tube will have to be replaced within this time, are independent. $(P(E_i) = \int_0^{150} f(x)dx = 100 \int_{x=100}^{150} 1/x^2 dx = 1/3.$ $\binom{5}{2}(1/3)^2(2/3)^3 = 80/243$.

3 Expectation and variance

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$, $V(X) = E(X - E(X))^2$.

1. The density function of X is given by

$$f(x) = \begin{cases} 2x, 0 \le x \le 1 \\ 0, otherwise \end{cases}$$

Find $E(X)$. ($E(X) = \int_{\infty}^{\infty} xf(x)dx = 2/3$)

4 Change of variable

If X is continuous with distribution function F_X and density function f_x, find the density function of $Y = 2X$. $f_Y(a) = \frac{1}{2}f_X(a)$. Another way to determine

$$\epsilon f_Y(a) \approx P(a - \epsilon/2 < Y < a + \epsilon/2) = P(a/2 - \epsilon/4 < X < a/2 + \epsilon/4) \approx \epsilon/2 f_Y(a/2)$$

The density function of X is given by

$$f(x) = \begin{cases} 1, 0 < x < 1 \\ 0, o.w. \end{cases}$$

Find $E(e^X)$. Alternatively, we can evaluate function of $Y = e^X$ and then calculate EY .