Random Variables - Class 13

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1 Random variables

1.1 Examples

1. Suppose the # of accidents occurring on a highway each day is a Poisson r.v. with parameter $\lambda = 3$. Find the probability that 3 or more accidents occur today given that at least 1 accident occurs today. $P(X \geq 3 \mid X \geq 1) =$

2. Suppose a biased coin that lands on heads with probability p is flipped 10 times. Given that a total of $6 heads$ result, find the conditional probability that the rst 3 outcomes are H, T, T. (Let $X_1 0$ be the # of H's in the 10 experiments. Then $X_1 0 \sim B(10, p)$ We would like to calculate

$$\begin{aligned} P(HTT \mid 6H) &= \frac{P(HTT \cap 6H)}{P(6H)} \\ &= \frac{P(HTT \cap 5H\, in\, remaining\, 7\, flips)}{P(6H)} \\ &= \frac{p(1-p)^2 \binom{7}{5} p^5 (1-p)^2}{\binom{10}{6} p^6 (1-p)^4} \end{aligned}$$

3. Let X be a nonnegative integer-valued r.v.Then $E(X) = \sum_{i=1}^{\infty} P(X \geq i)$.

4. Useful in theoretical derivation

5. $RHS =$

$$\begin{aligned} \sum_{i=1}^{\infty} P(X \geq i) &= \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P(X = k) \\ \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(X = k) &= \sum_{k=1}^{\infty} kP(X = k) = E(X) \end{aligned}$$

6. An urn contain N white and M black balls. Balls are randomly selected, one at a time, until a black one is obtained. If we assume that each selected ball is replaced before the next one is drawn, what is the probability that

   (a) exactly n draws are needed;

(b) at least k draws are needed?

(If we let X denote the number of draws needed to select a black ball, then X is a geometric random variable with parameter $\frac{M}{(M+N)}$).

7. Two envelope problem: Let's say you are given two indistinguishable envelopes, each of which contains a positive sum of money. One envelope contains twice as much as the other. You may pick one envelope and keep whatever amount it contains. You pick one envelope at random but before you open it you're offered the possibility to take the other envelope instead. What will you do?

   (Lets denote by A the amount in your selected envelope. The probability that A is the smaller amount is 1/2, and that it's the larger also 1/2 The other envelope may contain either 2A or A/2 If A is the smaller amount the other envelope contains 2A If A is the larger amount the other envelope contains A/2 The other envelope contains 2A with probability 1/2 and A/2 with probability 1/2 So the expected value of the money in the other envelope is: $(2A \times 1/2 + A/2 \times 1/2)$.

   Since the expected value in the other envelop is larger than what is in the envelop you selected, you should switch. After the switch one can reason in exactly the same manner as above The most rational thing to do is to switch back again To be rational one will thus end up switching envelopes indefinitely As it seems more rational to open just any envelope than to switch indefinitely we have a contradiction!

2 Continuous Random Variables

Discrete random variables are those whose set of possible values is either finite or countably infinite. Continuous random variables are those whose set of possible values is uncountable, etc lifetime of a light bulb, amount of precipitation in a year.

2.1 Probability density function

1. If there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers

$$P(X \in B) = \int_B f(x)dx,$$

   then the function f is called the probability density function of the random variable X.

2. All prob statements about a continuous r.v.X can be answered in terms of its density function f.

3. $\int_{-\infty}^{\infty} f(x)dx = 1$.

4. The probability of a certain interval $[a, b]$ in $P(a \le X \le b) = \int_a^b f(x)dx$.

5. $P(X = a) = \int_{x=a}^{a} f(x)dx = 0$.

6. $P(X < a) = P(X \le a) = F(a) = \int_{-\infty}^{a} f(x)dx$.

7. In addition, since $F(x) = \int_{-\infty}^{x} f(t)dt$, so that $F'(x) = f(x)$.

8. Probability density function can be considered as the measure of how likely that the random variable will be near a. Meaning of Density: When ϵ is small

$$P(a - \epsilon/2 \le X \le a + \epsilon/2) = \int_{a-\epsilon/2}^{a+\epsilon/2} f(x)dx = \epsilon f(a).$$

So $f(a) = \lim_{\epsilon \to 0} \frac{P(|X-a|<\epsilon/2)}{\epsilon}$.

9. Find the probability density function for the random variable Y with cdf

$$F(y) = \begin{cases} 0, y < 0 \\ 1 - e^{-y^2}, y \ge 0 \end{cases}$$

2.2 Examples

1. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(4x - 2x^2), 0 < x < 2 \\ 0\, o.w. \end{cases}$$

What is the value of $C(3/8)$? and what is the value of $P(X > 1) = 1/2$

2. The amount of time, in hours, that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \lambda e^{-x/100}, x \ge 0 \\ 0\, o.w. \end{cases}$$

What is the probability that (a) a computer will function between 50 and 150 hours before breaking down $(e^{-1/2} - e^{-3/2}) \approx 0.384$
(b) it will function less than 100 hours? $(1 - e^{-1}) \approx 0.633$

3. The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} 0, x \leq 100 \\ 100/x^2, x > 100 \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events $E_i, i = 1, 2, 3, 4, 5$, that the i-th such tube will have to be replaced within this time, are independent. ($P(E_i) = \int_0^{150} f(x)dx = 100 \int_{x=100}^{150} 1/x^2 dx = 1/3$. $\binom{5}{2}(1/3)^2(2/3)^3 = 80/243$.