Random Variables - Class 14

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1 Random variables

1.1 Examples

1. Two envelope problem: Let's say you are given two indistinguishable envelopes, each of which contains a positive sum of money. One envelope contains twice as much as the other. You may pick one envelope and keep whatever amount it contains. You pick one envelope at random but before you open it you're offered the possibility to take the other envelope instead. What will you do?

   (Lets denote by A the amount in your selected envelope. The probability that A is the smaller amount is 1/2, and that it's the larger also 1/2. The other envelope may contain either 2A or A/2. If A is the smaller amount the other envelope contains 2A. If A is the larger amount the other envelope contains A/2. The other envelope contains 2A with probability 1/2 and A/2 with probability 1/2. So the expected value of the money in the other envelope is: $(2A \times 1/2 + A/2 \times 1/2)$.

   Since the expected value in the other envelop is larger than what is in the envelop you selected, you should switch. After the switch one can reason in exactly the same manner as above. The most rational thing to do is to switch back again. To be rational one will thus end up switching envelopes indefinitely. As it seems more rational to open just any envelope than to switch indefinitely we have a contradiction!

2 Continuous Random Variables

1. The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} 0, x \leq 100 \\ 100/x^2, x > 100 \end{cases}$$

   What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events $E_i, i = 1, 2, 3, 4, 5$, that the i-th such tube will have to be replaced within this time, are independent. ($P(E_i) = \int_0^{150} f(x)dx = 100 \int_{x=100}^{150} 1/x^2 dx = 1/3$. $\binom{5}{2}(1/3)^2(2/3)^3 = 80/243$.

3 Important rule

Leibnitz rule:

$$\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(t)dt = g(f_1(x))f_1'(x) - g(f_2(x))f_2'(x)$$

3.1 Problem

Suppose that if you are s minutes early for an appointment, then you incur the cost cs, and if you are s minutes late, then you incur the cost ks. Suppose that the travel time from where you presently are to the location of your appointment is a continuous random variable having probability density function f. Determine the time at which you should depart if you want to minimize your expected cost.

Observe that

$$C_t(X) = \begin{cases} c(t-X), if X \leq t \\ s(X-t), if X > t. \end{cases}$$

Differentiate $E(C_t(X))$ with respect to t and equate to 0 to get the solution.

4 Expectation and variance

$E(X) = \int_{-\infty}^{\infty} xf(x)dx$ $V(X) = E(X - E(X))^2$, $E(aX + b) = aE(X) + b, V(aX + b) = a^2V(X)$

1. The density function of X is given by

$$f(x) = \begin{cases} 2x, 0 \leq x \leq 1 \\ 0, otherwise \end{cases}$$

   Find $E(X)$. ($E(X) = \int_{\infty}^{\infty} xf(x)dx = 2/3$)

2. For a non-negative continuous random variable Y (which means $P(Y < 0) = 0$ or $f_Y(t) = 0$ for $t \leq 0$)

$$E(Y) = \int_0^{\infty} P(Y > y)dy \tag{1}$$

3. Example: A stick of length 1 is split at a point U that is uniformly distributed over (0, 1). Determine the expected length of the piece that contains the point a, $0 \leq a \leq 1$? Claim: greater than 1/2. Given U, let $L(U)$ denote the length of the substick that contains the point a. So

$$L(U) = \begin{cases} 1 - U, U < a \\ U, U \geq a \end{cases}$$

Hence

$$\begin{aligned} E(L(U)) &= \int_0^a (1-u)du + \int_a^1 udu \\ &= 1/2 + a(1-a) \end{aligned}$$

5 Change of Variable

If X is continuous with distribution function F_X and density function f_X, find the density function of $Y = 2X$. $f_Y(a) = \frac{1}{2} f_X(a)$. Another way to determine

$$\epsilon f_Y(a) \approx P(a - \epsilon/2 < Y < a + \epsilon/2) = P(a/2 - \epsilon/4 < X < a/2 + \epsilon/4) \approx \epsilon/2 f_Y(a/2)$$

The density function of X is given by

$$f(x) = \begin{cases} 1, 0 < x < 1 \\ 0, o.w. \end{cases}$$

Find $E(e^X)$. Alternatively, we can evaluate function of $Y = e^X$ and then calculate EY. $E(e^X) = \int_0^1 e^x dx = e - 1$ Try to find the density of Y. Take the cdf approach. Clearly

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, if y \leq 0 \\ P(X \leq \log y) = \log y, if y \in [1, e] \\ 1 if y > e \end{cases}$$

$$f_Y(y) = \begin{cases} 0, if y \leq 0 \\ 1/y, if y \in [1, e] \\ 0 if y > e \end{cases}$$

$E(Y) = \int_1^e y1/ydy = (e-1)$.