Continuous Random Variables - Class 16

October 24, 2013 Debdeep Pati

1 Change of Variable for monotone transformations

If X is continuous with distribution function F_X and density function f_X, find the density function of $Y = 2X$. $f_Y(a) = \frac{1}{2} f_X(a)$. Another way to determine

$$\epsilon f_Y(a) \approx P(a - \epsilon/2 < Y < a + \epsilon/2) = P(a/2 - \epsilon/4 < X < a/2 + \epsilon/4) \approx \epsilon/2 f_Y(a/2)$$

The density function of X is given by

$$f(x) = \begin{cases} 1, 0 < x < 1 \\ 0, o.w. \end{cases}$$

Find $E(e^X)$. Alternatively, we can evaluate function of $Y = e^X$ and then calculate EY. $E(e^X) = \int_0^1 e^x dx = e - 1$ Try to find the density of Y. Take the cdf approach. Clearly

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, if y \leq 1 \\ P(X \leq \log y) = \log y, if y \in [1, e] \\ 1 if y > e \end{cases}$$

$$f_Y(y) = \begin{cases} 0, if y \leq 0 \\ 1/y, if y \in [1, e] \\ 0 if y > e \end{cases}$$

$E(Y) = \int_1^e y 1/y dy = (e - 1)$.

1.1 A theorem for monotone transformations

In general, we have the following theorem to obtain the density function of $Y = g(X)$ given the density of X. X is a continuous r.v. with density $f_X(\cdot)$. Suppose $g(\cdot)$ is a strictly monotone and differentiable function. Then $Y = g(X)$ has a prob density function given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\frac{d}{dy} g^{-1}(y)| if\, y = g(x) \text{ for some } x \\ 0, otherwise \end{cases}$$

Since g is strictly monotone, we can define the inverse function $g^{-1}(y)$. The distribution of Y is $F_Y(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$ where $y = g(x)$ for some $x = g^{-1}(y)$. DIfferentiation yields $f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$. When $y \neq g(x)$for any x, $F_Y(y)$ is either 0 or 1 and thus $f_Y(y) = 0$.

If $Y = X^n$ for any nonnegative random variable X, $g(x) = x^n$, $g(y) = y^{1/n}$. From the theorem $f_Y(y) = \frac{1}{n}y^{1/n-1}f(y^{1/n})$ for $y > 0$ and $= 0$ if $y \leq 0$.

However, this theorem does not apply if g is not strictly monotone. (show that for $Y = X^2$).

Recommendation for general transformations: Follow the step by step approach:

1. Find the distribution function F_X of X from the density f_X of X.
2. Using the transformation, find the distribution function F_Y of Y.
3. Take derivative of F_Y to find the density f_Y of Y.

2 Normal distribution

We say that X is normally distributed with parameters μ and σ^2, denoted by $X \sim N(\mu, \sigma^2)$ if its density is given by

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$$

Why call it normal? Many random phenomena obey a normal distribution, e.g., the height of a man, the measurement error (due to the Central Limit Theorem introduced later) See Pages 207-208 of the Sheldon Ross book for some interesting historical notes!!

2.1 Some properties

1. $f(x)$ is a density function, i.e.,

$$\int_{\mathbb{R}} f(x)dx = 1$$

2. If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu, Var(X) = \sigma^2$.
3. A very important property for normal distribution. If $X \sim N(\mu, \sigma^2)$, then any linear function of X is also normally distributed as a normal distribution is specified by the mean μ and the variance σ^2 alone. More specifically, if $aX + b \sim N(a\mu + b, a^2\sigma^2)$. Hence $\frac{X-\mu}{\sigma} \sim N(0, 1)$ which is also called the standard normal distribution.

4. If $X \sim N(0,1)$, then the density function is denoted by $\phi(x)$ and the cdf is denoted by $\Phi(x) = P(X \leq x)$.

5. Table of $\Phi(x)$ or the area under the normal curve to the left of x is given in Table 5.1 in page 201 for different values of x. In the table you'll only see values of $\Phi(x)$ for positive values of x. This is because you can get Φ values for the negative values from the positive values using the simple identity

$$\Phi(-x) = 1 - \Phi(x)$$

which is a result of the symmetry of the standard normal curve.

6. Calculating percentiles of Z requires a "backwards" look-up in the table. That is, given a probability, p, find the z such that $\Phi(z) = p$, or equivalently, $z = z_p = \Phi^{-1}(p).z$ z is called the $100 \times p$ th percentile of a standard normal distribution. We probably won't find percentiles exactly using the table and so we need to approximate or interpolate to find the corresponding z_p

7. Percentiles Example: Find the 95th percentile of Z. (1.65 famous landmark on the standard normal, well worth remembering)

2.2 Examples

1. Dominos pizza knows that the average length of time from receiving an order to delivering to the customer is 20 minutes with a standard deviation 7 min 45 seconds. Treat these sample statistics as population parameters for now. Dominoes wants to guarantee a delivery time as part of a marketing campaign, Your pizza in ?? minutes or your money back! Dominoes is willing to refund 10% of their orders, what is the quickest delivery time they should set the guarantee at? ($\Phi^{-1}(0.9) \times 7.75 + 20 \approx 29.9$) So you guarantee delivery in 30 minutes or less and you'll get a refund on 10% of the pizzas. (From another perspective this is a "Buy ten to get one free program").