Random Variables - Class 14

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1 Change of Variable

If X is continuous with distribution function F_X and density function f_X, find the density function of $Y = 2X$. $f_Y(a) = \frac{1}{2} f_X(a)$. Another way to determine

$$\epsilon f_Y(a) \approx P(a - \epsilon/2 < Y < a + \epsilon/2) = P(a/2 - \epsilon/4 < X < a/2 + \epsilon/4) \approx \epsilon/2 f_Y(a/2)$$

The density function of X is given by

$$f(x) = \begin{cases} 1, 0 < x < 1 \\ 0, o.w. \end{cases}$$

Find $E(e^X)$. Alternatively, we can evaluate function of $Y = e^X$ and then calculate EY. $E(e^X) = \int_0^1 e^x dx = e - 1$ Try to find the density of Y. Take the cdf approach. Clearly

$$F_Y(y) = P(Y \leq y) = \begin{cases} 0, if y \leq 0 \\ P(X \leq \log y) = \log y, if y \in [1, e] \\ 1 if y > e \end{cases}$$

$$f_Y(y) = \begin{cases} 0, if y \leq 0 \\ 1/y, if y \in [1, e] \\ 0 if y > e \end{cases}$$

$E(Y) = \int_1^e y 1/y dy = (e - 1)$.

1.1 A theorem for monotone transformations

In general, we have the following theorem to obtain the density function of $Y = g(X)$ given the density of X. X is a continuous r.v. with density $f_X(\cdot)$. Suppose $g(\cdot)$ is a strictly monotone and differentiable function. Then $Y = g(X)$ has a prob density function given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) |\frac{d}{dy} g^{-1}(y)| if\, y = g(x) \text{ for some } x \\ 0, otherwise \end{cases}$$

Since g is strictly monotone, we can define the inverse function $g^{-1}(y)$. The distribution of Y is $F_Y(y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$ where $y = g(x)$ for some $x = g^{-1}(y)$. DIfferentiation yields $f_Y(y) = f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)$. When $y \neq g(x)$for any x, $F_Y(y)$ is either 0 or 1 and thus $f_Y(y) = 0$.

If $Y = X^n$ for any nonnegative random variable X, $g(x) = x^n$, $g(y) = y^{1/n}$. From the theorem $f_Y(y) = \frac{1}{n}y^{1/n-1}f(y^{1/n})$ for $y > 0$ and $= 0$ if $y \leq 0$.

However, this theorem does not apply if g is not strictly monotone. (show that for $Y = X^2$).

Recommendation: Always start with the distribution function.