1 Delta Method

Definition 1. For a sequence of rev’s $Y_n$, and sequences of real numbers $\mu_n$ and positive numbers $\sigma_n$, we define

$$Y_n \sim AN(\mu_n, \sigma^2_n) \text{ if } \frac{Y_n - \mu_n}{\sigma_n} \overset{d}{\to} N(0, 1).$$

AN stand for “asymptotically normal”. If $Y_n \sim AN(\mu_n, \sigma^2_n)$, then for large $n$ the distribution of $Y_n$ is well approximated by $N(\mu_n, \sigma^2_n)$. Note that, in the definition of AN, $\mu_n$ and $\sigma^2_n$ do not have to be mean and variance of $Y_n$.

CLT: If $X_1, X_2, X_3, \ldots$ are iid with mean $\mu$ and variance $\sigma^2$, and $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$, then

$$\bar{X}_n \sim AN(\mu, \sigma^2/n).$$

(Here we are taking $\mu_n = \mu$ and $\sigma^2_n = \sigma^2/n$ in the definition of AN.

Theorem 1. (Delta method). Suppose that $Y_n \sim AN(\mu, \sigma^2_n)$ with $\sigma_n \to 0$. Let $g$ be a real-valued function differentiable at $x = \mu$ with $g'(\mu) \neq 0$. Then

$$g(Y_n) \sim AN(g(\mu), [g'(\mu)]^2 \sigma^2_n).$$

Example: Suppose $X_1, X_2, \ldots, X_n$ is a random variable with $E\mu(X_1) = \mu \neq 0$. Suppose we want to estimate the function $g(\mu) = 1/\mu$. Consider the mean of a random sample $\bar{X}$. For $\mu \neq 0$, applying the Delta Theorem we have,

$$\sqrt{n} \left( \frac{1}{\bar{X}} - \frac{1}{\mu} \right) \sim N \left( 0, \left( \frac{1}{\mu} \right)^4 \text{Var}_\mu X_1 \right)$$

in distribution. If we do not know the variance of $X_1$, we need to estimate it using $S^2$. Also $\mu$ can be estimated using $\bar{X}$ so that

$$\text{Var} \left( \frac{1}{\bar{X}} \right) \approx \left( \frac{1}{\bar{X}} \right)^4 S^2$$

Furthermore, since both $\bar{X}$ and $S^2$ are consistent estimators, we can again apply Slutsky’s Theorem to have for $\mu \neq 0$,

$$\sqrt{n} \left( \frac{1}{\bar{X}} - \frac{1}{\mu} \right) \left( \frac{1}{\bar{X}} \right)^2 S \to N(0, 1).$$

in distribution. This is used to find confidence intervals for $1/\mu$. 
