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1 Delta Method

Definition 1. *For a sequence of rev's Y_n, and sequences of real numbers μ_n and positive numbers σ_n, we define*

$$Y_n \sim AN(\mu_n, \sigma_n^2) \quad \textit{if} \quad \frac{Y_n - \mu_n}{\sigma_n} \xrightarrow{d} N(0,1).$$

AN stand for "asymptotically normal". If $Y_n \sim \text{AN}(\mu_n, \sigma_n^2)$, then for large n the distribution of Y_n is well approximated by $\text{N}(\mu_n, \sigma_n^2)$. Note that, in the definition of AN, μ_n and σ_n^2 do not have to be mean and variance of Y_n.
CLT: If $X_1, X_2, X_3, \ldots$ are iid with mean μ and variance σ^2, and $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, then

$$\bar{X}_n \sim \text{AN}(\mu, \sigma^2/n).$$

(Here we are taking $\mu_n = \mu$ and $\sigma_n^2 = \sigma^2/n$ in the definition of AN.

Theorem 1. *(Delta method). Suppose that $Y_n \sim AN(\mu, \sigma_n^2)$ with $\sigma_n \to 0$. Let g be a real-valued function differentiable at $x = \mu$ with $g'(\mu) \neq 0$. Then*

$$g(Y_n) \sim AN(g(\mu), [g'(\mu)]^2\sigma_n^2).$$

Example: Suppose $X_1, X_2, \ldots, X_n$ is a random variable with $E_\mu(X_1) = \mu \neq 0$. Suppose we want to estimate the function $g(\mu) = 1/\mu$. Consider the mean of a random sample $\bar{X}$. For $\mu \neq 0$. applying the Delta Theorem we have,

$$\sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right) \sim \text{N}\left(0, \left(\frac{1}{\mu}\right)^4 \text{Var}_\mu X_1\right)$$

in distribution. If we do not know the variance of X_1, we need to estimate it using S^2. Also μ can be estimated using $\bar{X}$ so that

$$\hat{Var}\left(\frac{1}{\bar{X}}\right) \approx \left(\frac{1}{\bar{X}}\right)^4 S^2$$

Furthermore, since both $\bar{X}$ and S^2 are consistent estimators,we can again apply Slutsky's Theorem to have for $\mu \neq 0$,

$$\frac{\sqrt{n}\left(\frac{1}{\bar{X}} - \frac{1}{\mu}\right)}{\left(\frac{1}{\bar{X}}\right)^2 S} \to \text{N}(0,1).$$

in distribution. This is used to find confidence intervals for $1/\mu$.