STA 5327 Exam 1

February 18, 2016

Name:

FSUID:

Please sign the following pledge and read all instructions carefully before starting the exam.

Pledge: I have neither given nor received any unauthorized aid in completing this exam, and I have conducted myself within the guidelines of the University Honor Code.

Signature: ______________________________

INSTRUCTIONS:

- This is an closed-book, closed-notes exam. However, a formula page is provided at the back.
- Total time is 75 minutes (9:30 A.M to 10:45 A.M.)
- **Show all work**, clearly and in order, if you want to receive full credit. When you use your calculator, explain all relevant mathematics. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- **Answer all the questions in the space provided. You may attach additional sheets if necessary.**
- This test has 4 problems and is worth 80 points. It is your responsibility to make sure that you have all of the problems.
- **Good luck!**

Prob. No.	Max Points	Earned Pts.
1	20	
2	20	
3	20	
4	20	

TOTAL: __________

Question 1. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as Unif$(-\theta, \theta)$ for an unknown parameter $\theta > 0$. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the order statistics. Is the statistic $T = (X_{(1)}, X_{(n)})$ minimal sufficient for θ? If yes, explain your answer. If not, find (and also prove) a minimal sufficient statistic for θ.

$$f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{2\theta} \mathbb{1}[-\theta \le x_i \le \theta]$$

$$= \left[\frac{1}{2\theta}\right]^n \prod_{i=1}^{n} \mathbb{1}[|x_i| \le \theta]$$

$$= \left[\frac{1}{2\theta}\right]^n \mathbb{1}[T \le \theta]$$

where $T_1 = \max_{1 \le i \le n} |x_i|$

Use Lehman-Scheffe Theorem to show T is MSS

$$\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta)}{f_{\underset{\sim}{X}}(\underset{\sim}{y} \mid \theta)} = \frac{\mathbb{1}[\max_{1 \le i \le n} |x_i| \le \theta]}{\mathbb{1}[\max_{1 \le i \le n} |y_i| \le \theta]}$$

Clearly if $T_1(\underset{\sim}{x}) = T_1(\underset{\sim}{y})$, $\frac{f(\underset{\sim}{x} \mid \theta)}{f(\underset{\sim}{y} \mid \theta)} = 1$

If $f(\underset{\sim}{x}) = c(\underset{\sim}{x}, \underset{\sim}{y}) f(\underset{\sim}{y})$, apply "region trick"

$A(\underset{\sim}{x}) = \{\theta : f(\underset{\sim}{x} \mid \theta) > 0\}$, $A(y) = \{\theta : f(y \mid \theta) > 0\}$

$= [\max_{1 \le i \le n} |x_i|, \infty)$ $\qquad = [\max_{1 \le i \le n} |y_i|, \infty)$

Since $A(x)$ must be same as $A(y)$, we have $T_1(\underset{\sim}{x}) = T_1(\underset{\sim}{y})$

$T = (X_{(1)}, X_{(n)})$ is <u>NOT</u> a function of $T_1 = \max_{1 \le i \le n} |X_i|$

(say $n = 3$, $X_1 = -3$, $X_2 = 1$, $X_3 = 5$. $T_1 = 5$, $T = (-3, 5)$.
Cannot recover T from T_1)

So T is NOT MSS

Question 2. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be i.i.d discrete uniform distribution on $\{1, 2, \ldots, N\}$ where N is an unknown parameter taking values in the set of all positive integers, i.e. $\{1, 2, \ldots\}$. Recall that $P(X_1 = j \mid N) = 1/N$ for all $j = 1, \ldots, N$.

a) (8 points) Find a minimal sufficient statistic for N. No need to actually show that it is minimal sufficient.

b) (12 points) Show (using the definition of completeness) that the statistic obtained in a) is complete.

a) $p(x_1, \ldots, x_n \mid N) = \left(\frac{1}{N}\right)^n I[X_{(n)} \leq N]$

$X_{(n)}$ is MSS. $P_N(X_{(n)} = k) = \begin{cases} 1 & \text{if } N = 1, k = 1 \\ \left(\frac{K}{N}\right)^n - \left(\frac{K-1}{N}\right)^n & \text{if } N > 1, \\ & K = 1, 2, \ldots, N \end{cases}$

b) $\underline{X_{(n)} \text{ is CSS}}$

$E_N(g(X_{(n)})) = 0 \quad \forall N = 1, 2, \ldots$

$\underline{N=1}$ $\quad g(1) = 0$

$\underline{N=2}$ $\quad \frac{g(1)}{2^n} + g(2)\left(1 - \frac{1}{2^n}\right) = 0 \Rightarrow g(2) = 0$

$\underline{N=3}$ $\quad g(1) P(X_{(n)} = 1) + g(2) P(X_{(n)} = 2) + g(3) P(X_{(n)} = 3) = 0$

$\Rightarrow g(3) = 0 \quad [\because P(X_{(n)} = 3) \neq 0]$

$\vdots$

Continuing $\quad g(k) = 0 \quad \forall k = 1, 2, 3, \ldots$

$P_N(g(X_{(n)}) = 0) = 1 \quad \forall N \geq 1$

$\Rightarrow X_{(n)}$ is complete.

Question 3. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be i.i.d from a Pareto distribution, denoted $\mathrm{Pa}(\alpha, \beta)$, for unknown (α, β) satisfying $\alpha > 0, \beta > 2$ with pdf

$$f(x \mid \alpha, \beta) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}}, & x > \alpha. \\ 0, & x \leq \alpha. \end{cases}$$

Find method of moment estimators $(\hat{\alpha}, \hat{\beta})$ for (α, β) satisfying $\hat{\alpha} > 0$ and $\hat{\beta} > 2$. You may use the following facts:

$$\mathbb{E}(X_1) = \frac{\alpha\beta}{\beta - 1}, \quad \mathbb{E}(X_1^2) = \frac{\alpha^2\beta}{\beta - 2}.$$

$$\frac{\alpha\beta}{\beta-1} = \mu_1, \qquad \frac{\alpha^2\beta}{\beta-2} = \mu_2$$

$$\alpha = \left(\frac{\beta-1}{\beta}\right)\cdot\mu_1, \qquad \frac{\frac{(\beta-1)^2}{\beta^2}\mu_1^2\cdot\beta}{\beta-2} = \mu_2$$

$$\Rightarrow \frac{(\beta-1)^2}{\beta(\beta-2)} = \frac{\mu_2}{\mu_1^2}$$

$$\Rightarrow \frac{\beta^2 - 2\beta + 1}{\beta^2 - 2\beta} = \frac{\mu_2}{\mu_1^2}$$

$$\Rightarrow \frac{1}{\beta^2 - 2\beta} = \frac{\mu_2 - \mu_1^2}{\mu_1^2}$$

$$\Rightarrow \beta^2 - 2\beta - \frac{\mu_1^2}{\mu_2 - \mu_1^2} = 0$$

$$B^2 - 4AC = 4 + \frac{4\mu_1^2}{\mu_2 - \mu_1^2} > 0$$

$$\Rightarrow \beta = \frac{2 \pm \sqrt{4 + \frac{4\mu_1^2}{\mu_2 - \mu_1^2}}}{2}$$

$$\hat{\beta} = 1 + \sqrt{1 + \frac{m_1^2}{m_2 - m_1^2}} > 2$$

$$\hat{\alpha} = \left(\frac{\hat{\beta} - 1}{\hat{\beta}}\right) m_1 > 0$$

Question 4. (20 pts.) Suppose $X_i, i = 1, 2, \ldots, n$ are i.i.d. samples from Beta$(1, \alpha)$ where $\alpha > 0$ is an unknown parameter. Using Basu's Lemma (or otherwise), find the conditional expectation

$$\mathbb{E}\left[\log(1 - X_1) \,\Big|\, \prod_{i=1}^{n}(1 - X_i)\right].$$

Hint: Observe that the distribution of $-\log(1 - X_i)$ is familiar to us.

$$X_1, \ldots, X_n \overset{iid}{\sim} \text{Beta}(1, \alpha)$$

$$Y_i = -\log(1 - X_i) \overset{iid}{\sim} \text{Exp}[\alpha] \qquad [\, E(Y_1) = 1/\alpha \,] \Rightarrow \Sigma Y_i = \text{CSS for } \alpha$$

$$E\left[+\log(1 - X_1) \,\Big|\, \prod_{i=1}^{n}(1 - X_i)\right] \qquad \frac{Y_1}{\Sigma Y_i} \text{ is ancillary}$$

$$= -E\left[\frac{-\log(1-X_1)}{-\Sigma \log(1-X_i)} \cdot \left(-\sum_{i=1}^{n} \log(1-X_i)\right) \,\Big|\, \underbrace{\prod_{i=1}^{n}(1-X_i)}_{\text{1-1 function of } \Sigma Y_i}\right]$$

(By Basu)

$$= +\sum_{i=1}^{n} \log(1 - X_i) \quad E\left[\frac{Y_1}{\Sigma Y_i} \,\Big|\, \prod_{i=1}^{n}(1 - X_i)\right]$$

$$= +\sum_{i=1}^{n} \log(1 - X_i) \quad E\left[\frac{Y_1}{\Sigma Y_i}\right]$$

By Basu again, $E[\Sigma Y_i] \cdot E\left[\frac{Y_1}{\Sigma Y_i}\right] = E[Y_1]$

$$\Rightarrow E\left[\frac{Y_1}{\Sigma Y_i}\right] = \frac{E[Y_1]}{E[\Sigma Y_i]} = \frac{1/\alpha}{n/\alpha} = 1/n$$

Hence $$E\left[\log(1 - X_1) \,\Big|\, \prod_{i=1}^{n}(1 - X_i)\right] = \frac{\sum_{i=1}^{n} \log(1 - X_i)}{n}$$

$$= \frac{\log\left(\prod_{i=1}^{n}(1 - X_i)\right)}{n}$$