Question 1. (20 pts.) Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as Unif$(a\theta, b\theta)$ for known constants a, b satisfying $0 < a < b$ and $\theta > 0$ is an unknown parameter. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the order statistics.

a) (9 points) Find a minimal sufficient statistic for θ. No need to show that it is actually minimal sufficient.

b) (7 points) Is the statistic found in a) complete? Justify your answer.

a) $f(x_i|\theta) = \frac{1}{(b-a)\theta} \mathbb{1}_{(a\theta, b\theta)}(x_i)$

$$\Rightarrow f(x_1, \ldots, x_n|\theta) = \underbrace{\boxed{\frac{1}{(b-a)^n}}}_{h(x)} \overbrace{\frac{1}{\theta^n} \mathbb{1}_{(X_{(n)} < b\theta)} \mathbb{1}_{(X_{(1)} > a\theta)}}^{g(T(x), \theta)}$$

By $\underbrace{\text{Factorization Criterion}}_{F\text{-}C}$, $T = (X_{(1)}, X_{(n)})$ is a SS

One can use $\underbrace{\text{Lehmann-Scheffé}}_{L\text{-}S}$ Thm to show T is MSS

b) This is a scale family, so $T_1 = \frac{X_{(1)}}{X_{(n)}}$ is ancillary.

If T were complete, Basu's Lemma would have implied T independent of T_1. But $T_1 = (X_{(1)}/X_{(n)})$ is a function of $T = (X_{(1)}, X_{(n)})$. So T cannot be independent of T_1. Hence <u>T cannot be a complete statistic</u>

Question 2. (20 pts.) Suppose we observe a single observation X from Binomial$(2, p)$ distribution where p can take only two values 0.25 and 0.5.

a) (10 points) Find a minimal sufficient statistic for p. Call it T. No need to show that it is actually minimal sufficient.

b) (10 points) Show that the statistic T found in a) is not complete.

(Hint: Try to find a function g which is not identically zero but $E_{p=0.25}\{g(T)\} = E_{p=0.5}\{g(T)\} = 0$).

a) $f(x \mid p) = \binom{2}{x} p^x (1-p)^{2-x}, \quad x = 0, 1, 2$

We only have 1 observation X.

So jointdensity is same as above.

$$f(x \mid p) = \underbrace{\binom{2}{x}}_{h(x)} \underbrace{(1-p)^2 e^{x \log \frac{p}{1-p}}}_{g(T(x), p)}$$

$T = X$ is a sufficient statistic by F.C.

It can be shown to be minimal using L-S Theorem

b) Need to show for some g

$$E_{p=0.25}\{g(T)\} = g(0)\binom{2}{0}(0.25)^0(0.75)^{2-0} + g(1)\binom{2}{1}(0.25)^1(0.75)^1 + g(2)\binom{2}{2}(0.25)^2(0.75)^0 = 0 \longrightarrow (1)$$

&

$$E_{p=0.5}\{g(T)\} = g(0)\binom{2}{0}(0.5)^0(0.5)^{2-0} + g(1)\binom{2}{1}(0.5)^1(0.5)^1 + g(2)\binom{2}{2}(0.5)^2(0.5)^0 = 0 \longrightarrow (2)$$

(1) $\Rightarrow$ $g(0)\frac{9}{16} + 2g(1)\frac{3}{16} + g(2)\frac{1}{16} = 0 \Rightarrow 9g(0) + 6g(1) + g(2) = 0$

(2) $\Rightarrow$ $g(0)\frac{1}{4} + 2g(1)\frac{1}{4} + g(2)\frac{1}{4} = 0 \Rightarrow g(0) + 2g(1) + g(2) = 0$

Choose $g(0) = 1 \Rightarrow \left.\begin{array}{l} 6g(1) + g(2) = -9 \\ 2g(1) + g(2) = -1 \end{array}\right\} \Rightarrow 4g(1) = -8$

$g(1) = -2$

$\Rightarrow g(2) = 3$

$$\boxed{g(0) = 1,\ g(1) = -2,\ g(2) = 3}$$

Question 3. (15 pts.) Consider the family of distributions with pdf

$$f(x \mid \theta) = \frac{3x^2}{\theta^3}, \quad 0 \le x \le \theta.$$

and cdf

$$F(x \mid \theta) = \begin{cases} \frac{x^3}{\theta^3}, & 0 \le x \le \theta \\ 1, & x > \theta \\ 0, & x < 0. \end{cases}$$

Let $X_1, \ldots, X_n$ be iid from $f(x \mid \theta)$. Show that $T = \max\{X_i, i = 1, \ldots, n\}$ is a complete statistic.

First find distribution of T. For $0 \le t \le \theta$

$$P(T \le t) = P(X_{(n)} \le t) = P(X_1 < t)^n = \left[\frac{t^3}{\theta^3}\right]^n = \frac{t^{3n}}{\theta^{3n}}$$

Density of T is

$$f_T(t) = \frac{d}{dt} P(T \le t) = \frac{3n\, t^{3n-1}}{\theta^{3n}}, \quad 0 \le t \le \theta$$

Suppose $E_\theta[g(T)] = 0$ for some g and $\forall\ \theta > 0$.

$$\Rightarrow \int_0^\theta g(t) f_T(t)\, dt = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow \frac{3n}{\theta^{3n}} \int_0^\theta t^{3n-1} g(t) = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow \int_0^\theta t^{3n-1} g(t) = 0 \quad \forall\ \theta > 0$$

$\Rightarrow$ By differenting w.r.t θ and using fundamental Thm of Calculus

$$\theta^{3n-1} g(\theta) = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow g(\theta) = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow g(t) = 0 \quad \forall\ t > 0$$

$\Rightarrow$ $g \equiv 0$ $\Rightarrow$ T is complete.

Question 4. (25 pts.) Suppose X_i are i.i.d. samples from $f(x \mid \theta)$, for $i = 1, \ldots, n$, where $\theta > 0$ is unknown, and the family of distributions $f(x \mid \theta)$ is the Rayleigh family given by

$$f(x \mid \theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad x \geq 0.$$

a) (10 points) Show that this is an exponential as well as a scale family.

b) (7 points) Find a complete and sufficient statistic for this family. No need to prove this is actually complete and sufficient.

c) (8 points) Find the value of

$$E\left[\frac{(\sum X_i)^2}{\sum X_i^2}\right]$$

using Basu's theorem if we know $EX_i = \theta\sqrt{\pi/2}$ and $\mathrm{Var}(X_i) = (4 - \pi)\theta^2/2$.

a) $f(x_1, \ldots, x_n) = \underbrace{\frac{1}{\theta^{2n}}}_{c(\theta)} \underbrace{\prod_{i=1}^{n} x_i}_{h(x)} \underbrace{e^{-\frac{\sum x_i^2}{2\theta^2}}}_{\exp\{T(x)\,w(\theta)\}}$

$\Rightarrow$ It is exponential family.

where $T(x) = \sum x_i^2$

$w(\theta) = -\frac{1}{2\theta^2}$

$f(x \mid \theta) = \frac{1}{\theta} \psi\left(\frac{x}{\theta}\right)$ where $\psi(x) = x e^{-\frac{x^2}{2}}$, $x \geq 0$

$\Rightarrow$ It is scale family.

b) By Master Theorem, $T(x) = \sum_{i=1}^{n} X_i^2$ is CSS.

c) $S = \frac{(\sum X_i)^2}{\sum X_i^2}$ is ancillary ($\because$ it is scale invariant)

By Basu, S & T are independent

$\Rightarrow E(ST) = E(T) \cdot E(S)$

$$E(ST) = E\left(\left(\sum X_i\right)^2\right)$$

$$= Var\left(\sum X_i\right) + \left[E\left(\sum X_i\right)\right]^2$$

$$= n\, Var(X_1) + \left(n\theta\sqrt{\pi/2}\right)^2$$

$$= n\frac{(4-\pi)\theta^2}{2} + \frac{n^2\theta^2\pi}{2}$$

$$E(T) = E\left(\sum X_i^2\right) = \sum E(X_i^2)$$

$$= \sum\left[E^2(X_i) + Var(X_i)\right]$$

$$= n.\frac{\theta^2\pi}{2} + n.\frac{(4-\pi)\theta^2}{2}$$

So $$E(S) = \frac{E(ST)}{E(T)}$$

$$= \frac{\cancel{n}\frac{(4-\pi)}{\cancel{2}}\cancel{\theta^2} + \frac{n^{\cancel{2}}\cancel{\theta^2}\pi}{\cancel{2}}}{\cancel{n}\frac{\cancel{\theta^2}\pi}{\cancel{2}} + \cancel{n}\frac{(4-\pi)\cancel{\theta^2}}{\cancel{2}}}$$

$$= \frac{4-\pi+n\pi}{\cancel{\pi}+4-\cancel{\pi}}$$

$$\boxed{E(S) = 1 + \frac{\pi(n-1)}{4}}$$